Introduction to mathematical finance

Sheet 11 HS 2014

Hand-in your solutions until 11.12.2014*, in Martina Dal Borgo's mailbox on K floor.*

Let M be a continuous integrable adapted stochastic process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$. We say that M is a continuous-time martingale with respect to $(\mathcal{F}_t)$ and to the measure $\mathbb{P}$ if

$$\mathbb{E}^{\mathbb{P}}[M_t|\mathcal{F}_s] = M_s, \quad \forall\, 0 \leq s \leq t.$$

Exercise 1
Let W be a real Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ and $\sigma \in \mathbb{R}$. Prove that

i) W_t,

ii) $W_t^2 - t$,

ii) $e^{\sigma W_t - \frac{\sigma^2}{2}t}$.

are continuous $(\mathcal{F}_t)$-martingale.

Exercise 2
We consider an arbitrage-free discrete binomial market (S, B) on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ and a path-independent European-style derivative, that is the payoff is $\mathcal{F}_T$-measurable random variable Φ on $(\Omega, \mathcal{F}, \mathbb{P})$, which can be expressed as $\Phi = f(S_T)$ for some function f. Therefore, as in Exercise 3 Sheet 5, the price process P_t of the derivative can be expressed as

$$P_t = p(t, S_t),$$

where $p : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$. Assume that f is a Lipschitz function with Lispchitz constant K:

$$|f(x) - f(y)| \leq K|x - y|, \quad x, y \in \mathbb{R}.$$

Prove that the amount oh shares $(\alpha_t)_t$ of the risky asset the seller of the option has to hold in order to hedge the option satisfies

i) $|\alpha_t| \leq K, \forall\, t$,

ii) if $f \uparrow$, then $\alpha_t \geq, \forall\, t$,

iii) if $f \downarrow$, then $\alpha_t \leq 0, \forall\, t$.

Note that in case of

- a European Call $K = 1$ and f is increasing, therefore $\alpha_t \in [0, 1], \forall\, t$;
- a European Put $K = 1$ and f is decreasing, therefore $\alpha_t \in [-1, 0], \forall\, t$.

Give a financial interpretation of the above comments.
Hint: Recall that a call option is equivalent to the promise of buying the stock for a strike price.

Exercise 3
Explain in why the 2nd fundamental theorem is in fact an equivalence between:

- Uniqueness of the risk neutral measure $\mathbb{Q}$.
- The representation of $\mathbb{Q}$-martingales as stochastic integrals. Formally, for every $\mathbb{Q}$-martingale $M = (M_s)_{1 \leq s \leq T}$, there exists a predictable process $(\eta_s)_{1 \leq s \leq T}$ such that:

$$M_t = M_0 + \sum_{s=1}^{t} \eta_s \cdot \left(\tilde{S}_s - \tilde{S}_{s-1} \right)$$