Introduction to mathematical finance

Sheet 10 HS 2014

Hand-in your solutions until ..., 2013, in Martina Dal Borgo's mailbox on K floor.

Exercise 1 Discuss whether or not the following market models are free of arbitrage and complete using the I and the II Fundamental Theorem of Asset Pricing.

i) A one period (annual) financial market model on $\Omega = \{\omega_1, \omega_2, \omega_3\}$ consisting of a bond with annual interest rate $r = 5\%$ and two risky assets (S^1, S^2)

$$S_0^1 = 10, \qquad S_1^1 = \begin{cases} 12, & \{\omega_1\} \\ 8, & \{\omega_2\} \\ 6, & \{\omega_3\} \end{cases}$$

$$S_0^2 = 5, \qquad S_1^2 = \begin{cases} 10, & \{\omega_1\} \\ 4, & \{\omega_2\} \\ 5, & \{\omega_3\} \end{cases}$$

ii) The market models (B, S^1, S^2) and (B, S^1) of Exercise 4-Sheet 3 (no need to rewrite the calculus).

In the previous exercise, you realised that a market is not complete because it is lacking enough tradable assets. We say that a market is completed by introducing new tradable assets (generally options), such that the newly formed secondary market is complete.

Exercise 2

Consider a one-period financial market model on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, consisting of a bond with interest rate $r = 0$ and a risky stock with

$$S_0 = 1, \quad 0 < S_1(\omega_1) < S_1(\omega_2) < S_1(\omega_3) < S_1(\omega_4), \quad \mathbb{P}(\{\omega_i\}) > 0, \ i = 1, \ldots, 4.$$

We assume that the market is free of arbitrage. Show that there exist 2 different Call options with payoffs C_1^1, C_1^2 and initial prices C_0^1, C_0^2 which complete the market and keep it free of arbitrage.

Note:

The same exercise holds true if you consider a one-period financial market model on $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}, N > 2$ consisting of a bond with interest rate $r = 0$ and a risky stock with

$$S_0 = 1, \quad 0 < S_1(\omega_1) < S_1(\omega_2) < \cdots < S_1(\omega_N), \quad \mathbb{P}(\{\omega_i\}) > 0, \ i = 1, \ldots, N.$$

In this case there exist $N - 2$ Call options which complete the market and keep it free of arbitrage.

Exercise 3

Consider a one-period financial market model on $\Omega = \{\omega_1, \omega_2, \omega_3\}$, consisting of a bond with interest rate $r = 0$ and a risky stock with

$$S_0 = 1, \quad 0 < S_1(\omega_1) < S_1(\omega_2) < S_1(\omega_3), \ \mathbb{P}(\{\omega_i\}) > 0, \ i = 1, 2, 3.$$

In this setup the set of all random variables on Ω can be identified with $\mathbb{R}^3$. The aim of the following questions is to give a geometric interpretation of the model. You are strongly advised to use the drawing made in class.

1. Describe the following objects in $\mathbb{R}^3$:

   i) The set $\mathcal{Q}$ of all equivalent martingale measures.

   ii) The set $\mathcal{X}^r$ of all replicable options.

2. What is the geometric relation between $\mathcal{Q}$ and $\mathcal{X}^r$?

3. Prove that under the assumptions $S_1(\omega_1) < 1,\ S_1(\omega_3) > 1$ the model is free of arbitrage.

4. Give an example of a non replicable option.