Introduction to mathematical finance

Sheet 1 HS 2013

Hand-in your solutions until 24.09.2014, in Martina Dal Borgo's mailbox on K floor (or give the sheet in person on the 24.09 in class).

For the exercises keep in mind the *No- Arbitrage principle*:
Let X_t and Y_t, $t \in \mathbb{R} > 0$, denote the values of two financial instruments. Then if these two financial instruments will certainly have the same value at the future date T, then also in this moment the must have the same value i.e.

$$X_T = Y_T \quad \Rightarrow \quad X_t = Y_t,\ \forall\, t \leq T$$

• Forward and Futures contract

A *Forwards /Futures contract* is a commitment to buy or sell at a future date T called *maturity* a given amount of a commodity or an asset at a price agreed on today. Recall that the forward is an OTC contract, while futures are much more regulated. The price fixed now for future exchange is the *forward price* $f_{0,T}$, or the *future price* $F_{0,T}$, depending on the contract. The investor who agrees to buy the underlying at maturity is said to hold a *long position* in the contract, while the investor who agrees to sell it is said to hold a *short position* in the contract. For simplicity we assume that holding the asset neither provides income nor incurs costs in the period $[0, T]$ and that the annual risk-free interest rate is fixed $r \geq 0$.

A **forward contract** involves no cash flows up to date T and a single cash outflow (for the long-position investor) /inflow (for the short-position investor) equal to the forward price at maturity:

Day	Underlying asset price	Cash flow
0	S_0	0
T	S_T	$f_{0,T}$

Because no cash is exchanged up to date T, the value of the contract, the day we enter into, must be zero. If this is not the case, then for a zero-cash outlay, either the buyer or the seller receives a positive net present value, which would violate the no-arbitrage requirement.

However, over time, as the spot price of the underlying fluctuates the forward contract gets value. Note that because the underlying asset's price may fluctuate wildly, there is a huge risk that one of the parties will not be able to execute these obligations$^{(1)}$.

A **futures contract** is a forward-like contract which is marked to market daily, that is every single trading day t the two counterparties

- ∘ cancel the futures contract they entered into, replacing it by a new futures contract with the same delivery date but a new futures price $F_{t,T}$. These futures prices $F_{t,T}$ are changed every day to reflect market's evaluations and thus move in the same direction of the spot price of the asset. $F_{T,T}$ equals the spot price S_T at maturity.
- ∘ pay each other the difference between the new and the previous futures price $F_{t,T} - F_{t-1,T}$.

Day	Underlying asset price	Futures price	Cash flow
0	S_0	$F_{0,T}$	0
1	S_1	$F_{1,T}$	$F_{1,T} - F_{0,T}$
...	...	...	...
$T-1$	S_{T-1}	$F_{T-1,T}$	$F_{T-1,T} - F_{T-2,T}$
T	S_T	$F_{T,T}$	$F_{T,T} - F_{T-1,T}$

Marking to market eliminate (reduce) the counterparty risk because profits and losses on future positions are paid over every day at the end of the trading, nullifying any financial incentives for not making delivery[(1)].

Example[(1)] (*Counterparty risk)*
A tofu manufacturer enters a long position in a 3 month forward/futures (marked to market monthly) contract for 1.000 tons of soybeans at 165 CH/ton (the current price of soybeans is 160 CH/ton). Suppose that at the settlement date the price of soybeans has dropped to 100 CH/ton and that the futures prices evolve like

Month	1	2	3
Futures prices CH/ton	150	120	100

- (forward contract) at settlement date the tofu manufacturer is looking at a really big loss $100 - 165 = -65$ CH/ton and has interest in buying the asset at the spot market at 100 CH/ton rather than from the soybeans producer at 165 CH/ton. Having given no money yet, he will simply renege the contract.

- (futures contract) at settlement date the tofu manufacturer has already exchanged money according to the cash flows

Month	Futures price CH/ton	Buyer CH	Seller CH
1	150	-15	$+15$
2	120	-30	$+30$
3	100	-20	$+20$
Total cash flow		-65	$+65$

  So, at the end of the second moth, the buyer has already transferred 45 CH/ton to the account of the soybeans producer, making not really worthwhile running away from the contract.

In both cases we obtain the same cash flows, but the difference is that during the three months period, with the futures contract, both counterparties do not have to worry about other party reneging, because the amount of the cash that is owed to one party or the other has got so big that does not make it worthwhile to run away.

Exercise 1 Suppose that at time 0, a forward contract is signed with forward price $f_{0,T}$. Formally, one can define the value V_t at time t of this contract, as the fair price for buying or selling this already signed contract. Using the no-arbitrage principle prove that the value of a forward contract is given by

$$V_t = S_t - \frac{f_{0,T}}{(1+r)^{T-t}}, \quad t \in [0, T]. \tag{1}$$

Notice that the equality $V_0 = 0$ gives the forward price.

Exercise 2 Compute the value of a forward contract with forward price $f_{0,T} = 100$ CH and maturity $T = 6$ months, at date $t = 2$ months. The free-risk annual interest rate is $r = 10\%$ and the spot price two months from now is $S_t = 95$

Exercise 3 A tofu manufacturer needs 1.000 tons of soybeans in 3 months. To make sure that these will be available he enters a long position in a 3 month forward contract for 1.000 tons of soybeans at 165 CH/ton. The current price of soybeans is 160 CH/ton, $r = 3\%$. Suppose that a month from now the spot price of soybeans has fallen down to 155 CH/ton. At what price will you be able to sell the contract one month from now?

Exercise 4 Assume the current futures price for silver for delivery 5 days from today is 10.10 CH/ounce. Suppose that over the next 5 days, the futures price evolves as follows:

Day	0	1	2	3	4	5
Futures prices CH/ounce	10.1	10.2	10.25	10.18	10.18	10.21

If you have a long position of 25,000 ounces and maturity $T = 5$ days, what are your cash flows for the next five days?

Exercise 5 Suppose that the current spot price of a stock is 20 CH/share and that a bank is offering interest at the rate of 5% per annum on a 3-month deposit. Describe a strategy for creating an arbitrage profit and establish the amount of the profit in the two following cases:

i) a long position in a forward contract is available to buy 100 shares of the stock 3 months from now for 20.2 CH/share.

ii) a short position in a forward contract is available to buy 100 shares of the stock 3 months from now for 20.3 CH/share.

What is the only fair (i.e. that does not introduce arbitrage opportunities) forward price? Justify it using the no-arbitrage principle.

Exercise 6 (*Interest rate parity theorem*)
Consider two currencies: let francs (CH) be the local currency and dollars ($) be the foreign currency. Let S_0 denote the spot price today of the foreign currency, i.e. the current price in CH of one unit of the underlying asset, that is 1$. So S_0 refers to the today exchange rate CH/$. Consider a forward contract with maturity T and forward price $f_{0,T}$ (then in T years we pay $f_{0,T}$ CH and we receive 1$. Check that

$$\frac{f_{0,T}}{S_0} = \frac{(1 + r_l)^T}{(1 + r_f)^T}$$

where r_f, r_l are, respectively, the annual interest rates of the foreign currency and the local currency.

• Call and put option, strangle, straddle
Call and Put options can be combined to build new derivatives. For example:

- a *Call spread* is a combined option consisting of a long-position (we are the *buyer* the option) in an European Call option with strike K_1 and a short-position (we are the *writer* the option) in an European Call option with strike $K_2 > K_1$

- a *Put spread* is a combined option consisting of a long-position in an European Put option with strike K_1 and a short-position in an European Put option with strike $K_2 > K_1$
- a *Straddle* is a combined option consisting of a long-position in an European Call option and a long-position in an European Put option with strike the same strike K
- a *Strangle* is a combined option consisting in a long position in an European Call option with strike K_1 and in an European Put option with strike $K_2 < K_1$
- a *Butterfly spread* is a combined option consisting of a long-position in an European Call option with strike K_1, a long-position in an European Call option with strike $K_2 > K_1$ and short-position in two European Call options with strike $\frac{K_1+K_2}{2}$.

(all Call/Put options are assumed with the same maturity and written on the same underlying).

Exercise 1 (*Put-Call parity formula)*
Consider a Call and a Put option with S as underlying, both of European type with maturity T and strike K. Check that

$$C_t = P_t + S_t - K\,(1+r)^{-(T-t)}\,, \quad t \in [0,T]$$

where $r \geq 0$ is the annual free-risk interest rate.

Exercise 2 Consider a one period financial market (B,S)

Time	0	$T = 1$ (year)
Bond	B_0	$B_1 = (1+r)$
Risky asset	$S_0 \in \mathbb{R}_+$	S_1

where $r \geq 0$ is the risk-free interest rate over the period. We assume that for all $K \in \mathbb{R}_+$ the initial price of a call option on S, with maturity of 1 year and strike K is known and denoted by $C_0(K)$. For each one of the combined derivatives mentioned before

1) find the formula for the payoff and draw a graph of this payoff as a function of the spot price of the underlying asset. In which cases could be profitable to exercise the derivative?
2) using the Put-Call parity formula and the no-arbitrage principle, determine the price of the derivative at $t = 0$.