Mid-term examination (2h)

Recall that the binomial model or CRR (Cox-Ross-Rubinstein) is a finite market model with finite time horizon $T < \infty$ and two assets: the bond B_t and a stock with spot S_t. The interest free rate over a period is r.

$$B_t = (1+r)^t$$

$$S_t = S_0 \prod_{i=1}^{t} \xi_i$$

The returns ξ_i take two values $d < u$. And the natural filtration generated by the spot process is:

$$\mathcal{F}_t = \sigma\left(S_0, S_1, \ldots, S_t\right)$$

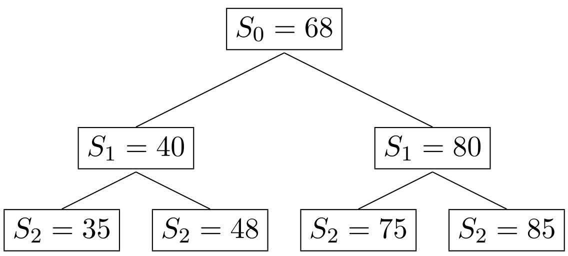
Presentation and readability (1 point)

Exercise 1 (Lesson question - 3 points)

- Explain in your own words what is the risk neutral probability $\mathbb{Q}$.
- On the practical side, how is it useful for pricing? Give a pricing formula for European options.
- On the theoretical side, how is it useful to prove absence of opportunity of arbitrage?

Exercise 2 (3 points)

We consider a two-period financial market model consisting of a bond and a stock with price process $S = (S_0, S_1, S_2)$. The interest rate over one period is 10%.
Assume that the processes evolve according to the following binomial tree, where each scenario has positive probability:



Is the model arbitrage-free? If not, exhibit an arbitrage opportunity. You are asked to formulate the arbitrage opportunity in the formalism seen in class.

Exercise 3 (4 points)

Consider a two period (one period = 1 year) binomial model with the following parameters:

$$u = 1.2,\ d = 0.9,\ 1 + r = 1.06$$

The initial spot value is $S_0 = 100$.

1. Is the model arbitrage free?

2. For the European option with payoff ($T = 2$):

$$\Phi_T = (S_T - S_0)^+$$

Give the value of the replicating portfolio and compute the number of shares to hold in the replicating strategy. You are encouraged to present your answer in the form of a tree.

3. Same question if:

$$\Phi_T = \left(\max_{t \leq T}(S_t) - S_0\right)^+$$

In this second case, the option is "path-dependent".

Exercise 4 (Asian call option - 9 points)
An Asian option is an option on the average value of a stock. Here, we consider for a maturity T and a strike K, option with payoff:

$$\Phi_T = \left(\frac{\sum_{t=0}^T S_t}{T+1} - K\right)^+$$

For convenience, we will denote the average process for $t > 0$ by:

$$A_t := \frac{\sum_{s=0}^t S_s}{t+1}$$

and the risk neutral probability is $q = p^*$.

1. Prove that the price of the option at time t of the Asian option is $C_t = C(t, A_t, S_t)$ where $(t, a, x) \mapsto C(t, a, x)$ is a deterministic function on $\{0, 1, \dots, T\} \times \mathbb{R}^+ \times \mathbb{R}^+$.

   Hint: One can first prove that $A_T = \frac{t+1}{T+1} A_t + \frac{S_t \sum_{s=t+1}^T \frac{S_s}{S_t}}{T+1}$ *and invoke independence of increments.*

2. Prove that it satisfies the backward equation:

$$C(T, a, x) = (a - K)^+$$

$$C(t, a, x) = \frac{1}{1+r}\left[qC\left(t+1, \frac{t+1}{t+2}a + \frac{xu}{t+2}, xu\right) + (1-q)\, C\left(t+1, \frac{t+1}{t+2}a + \frac{xd}{t+2}, xd\right)\right]$$

   Hint: One could prove the similar recurrence $A_{t+1} = \frac{t+1}{t+2} A_t + \frac{S_{t+1}}{t+2}$ *and use it.*

3. Describe a replicating (or hedging) strategy in terms of a predictable process $\varphi_t = (\alpha_t, \beta_t), t = 1, 2, \dots, T$. This process can be written $\varphi_t = \Delta(t, A_{t-1}, S_{t-1})$, i.e:

$$\alpha_t = \alpha(t, A_{t-1}, S_{t-1})$$

$$\beta_t = \beta(t, A_{t-1}, S_{t-1})$$

   Express Δ as a function of C.