Mid-term examination (2h)

Recall that the binomial model or CRR (Cox-Ross-Rubinstein) is a finite market model with finite time horizon $T < \infty$ and two assets: the bond B_t and a stock with spot S_t.

$$B_t = (1+r)^t$$

$$S_t = S_0 \prod_{i=1}^{t} \xi_i$$

The returns ξ_i take two values $d < u$. And the natural filtration generated by the spot process is:

$$\mathcal{F}_t = \sigma(S_0, S_1, \dots, S_t)$$

Exercise 1 (Lesson question - 3 points)

- In the binomial model, explain what is the risk neutral measure $\mathbb{Q}$ and give the formula for:

$$p = \mathbb{Q}(\xi_i = u)$$

- Let Φ_T be an $\mathcal{F}_T$-measurable random variable. What is the abstract formula giving the price P_t of the european option with maturity T and payoff Φ_T.

Exercise 2 (Binary option - 9 points)
For any time $t = 0, 1, \dots, T$, consider the payoff:

$$\Phi_t = \mathbb{1}_{S_t \geq K}$$

Pricing and hedging of the European option
The European option with payoff Φ_T at time T is called the European binary option.

1. Prove that the price of the option at time t of the European binary option is $P_t = P(t, S_t)$ where:
$$P(t,x) = \frac{1}{(1+r)^{T-t}} \mathbb{P}\left(Bin(T-t, p) \geq \frac{\log\left(\frac{K}{x\ d^{T-t}}\right)}{\log(u/d)} \right)$$
with $Bin(n,p)$ being a binomial random variable with parameters $n \in \mathbb{N}$ and $0 < p < 1$.

2. Describe a replicating (or hedging) strategy in terms of a predictable process $\phi_t = (\alpha_t, \beta_t), t = 1, 2, \dots, T$ such that:
$$\alpha_t = \alpha(t, S_{t-1})$$
$$\beta_t = \beta(t, S_{t-1})$$

The American option

1. Let the price of the option at time t of the American binary option be $P_t^{am} = f(t, S_t)$. Prove that it satisfies the backward equation:

$$f(T, x) = \mathbb{1}_{x \geq K}$$

$$f(t, x) = \mathbb{1}_{x \geq K} + \mathbb{1}_{x < K} \left(\frac{p}{1+r} f(t+1, xu) + \frac{1-p}{1+r} f(t+1, xd) \right)$$

2. Describe the optimal times for exercising the option. (Hint: The two situations to consider are $S_t \geq K$ and $S_t < K u^{T-t}$)

Exercise 3 (4 points)
Consider an American call option with maturity T and strike K. We assume $r \geq 0$.

1. Prove that the price of this option is $C_t = C(t, S_t)$ where

$$C(T, x) = (x - K)^+$$

$$C(t, x) = \max \left((x - K)^+ , \frac{p}{1+r} C(t+1, xu) + \frac{1-p}{1+r} C(t+1, xd) \right)$$

2. Prove that:

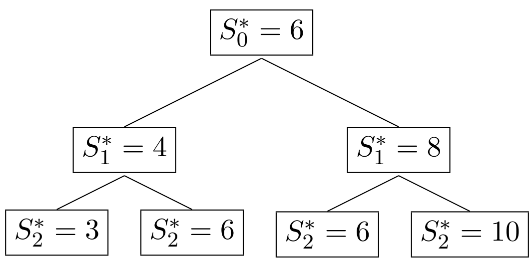
$$\frac{p}{1+r} C(t, xu) + \frac{1-p}{1+r} C(t, xd) \geq x - \frac{K}{1+r}$$

3. Deduce that the optimal exercising time is always the maturity T.

Exercise 4 (4 points)
We consider a two-period arbitrage-free financial market model consisting of a bond and a stock with discounted price process $S^* = (S_0^*, S_1^*, S_2^*)$. We consider the European option with discounted payoff:

$$\Phi^* = \max_{t=0,1,2} S_t^* - \min_{t=0,1,2} S_t^*$$

Assume that the processes evolve according to the following binomial tree, where each scenario has positive probability:



a) Determine the transition probabilities under $\mathbb{Q}$ such that S^* becomes a $\mathbb{Q}$-martingale.

b) Compute the risk-neutral pricing of the option.