Correction of final examination

Exercise 1 (Lesson question - 3 points)
See course.

Exercise 2 (On some stochastic processes - 8 points)
In this exercise, all stochastic integrals with respect to Brownian motion will be assumed to be real martingales. Let W_t be a real standard Brownian motion.

A martingale
At least three proofs are possible. Either use the Ito formula to see that M is a stochastic integral with respect to Brownian motion:

$$dM_t = e^{-\frac{1}{2}\lambda^2 t} \sinh(\lambda W_t)\lambda dW_t$$

or notice that:

$$M_t = \frac{1}{2}\left(\mathcal{E}(\lambda W)_t + \mathcal{E}(-\lambda W)_t\right)$$

where

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}\langle X, X\rangle_t\right)$$

is the exponential martingale associated to the martingale X. A sum of two martingales is a martingale.
A third proof might consist in checking directly the martingale property, which basically tantamounts to reproving that the exponential martingale of Brownian motion is a martingale.

On a certain affine process
Define the stochastic process X_t to be the unique process starting at X_0 and solution to the following SDE (stochastic differential equation):

$$dX_t = (a - bX_t)\, dt + \sigma\sqrt{X_t} dW_t$$

with $b > 0$ and $\sigma > 0$. Existence and uniqueness are assumed.

1. Rewrite the SDE in integral form:

$$X_t = X_0 + \int_0^t (a - bX_s)\, ds + \sigma \int_0^t \sqrt{X_s} dW_s$$

   Taking the expectation and applying Fubini yields:

$$\begin{aligned} f(t) =& X_0 + at - b\mathbb{E}\left(\int_0^t X_s ds\right) \\ =& X_0 + at - b\int_0^t \mathbb{E}\left(X_s\right) ds \\ =& X_0 + at - b\int_0^t f(s) ds \end{aligned}$$

   This shows f is smooth. Moreover, it is the integral form of the ODE.

2. Such an ODE has solution:

$$f(t) = \left(X_0 - \frac{a}{b}\right) e^{-bt} + \frac{a}{b}$$

   Notice the asymptotic mean:

$$\mathbb{E}\left(X_t\right) \overset{t\to\infty}{\longrightarrow} \frac{a}{b}$$

3.
$$X_t^2 = X_0^2 + 2\int_0^t X_s\left(a - bX_s\right)ds + 2\sigma\int_0^t X_s\sqrt{X_s}dW_s + \sigma^2\int_0^t X_s ds$$

4. Again, take the expectation of the previous equation, and apply Fubini:

$$\begin{aligned}
g(t) =& \mathbb{E}\left(X_t^2\right) - f(t)^2\\
=& X_0^2 + \mathbb{E}\left(2\int_0^t X_s\left(a - bX_s\right)ds + \sigma^2\int_0^t X_s ds\right) - f(t)^2\\
=& X_0^2 + 2\int_0^t ds\left(af(s) - b\mathbb{E}(X_s^2)\right) + \sigma^2\int_0^t f(s)ds - f(t)^2\\
=& X_0^2 + 2\int_0^t ds\left(af(s) - bg(s) - bf(s)^2\right) + \sigma^2\int_0^t f(s)ds - f(t)^2
\end{aligned}$$

   Because of the ODE satisfied by f:

$$f(t)^2 = X_0^2 + 2\int_0^t f(s)\left(a - bf(s)\right)ds$$

   Hence the simplification:

$$g(t) = -2b\int_0^t g(s)ds + \sigma^2\int_0^t f(s)ds$$

5. The constant C_2 concerns the vector space of homogenous solutions and is determined by the initial condition 0. We can therefore just focus on when $C_0 + C_1e^{-bt}$ is particular solution of our ODE. Identifying terms shows that:

$$C_0 = \frac{a\sigma^2}{2b^2}$$

$$C_1 = \frac{\sigma^2\left(X_0 - \frac{a}{b}\right)}{b}$$

   Finally, using the initial condition:

$$C_2 = -C_1 - C_0$$

   Notice that the asymptotic variance is $C_0 = \frac{a\sigma^2}{2b^2}$.

Exercise 3 (Spread option - 5 points)
For the sake of making things more interesting, we will consider a possibly random interest rate process $r_t, t \in \{0, 1, \ldots, T\}$. The filtration generated by both r and S is:

$$\mathcal{F}_t = \sigma\left(S_0, S_1, S_2, \ldots, S_t\right)\bigvee\sigma\left(r_0, r_1, r_2, \ldots, r_t\right)$$

Because we assumed completeness and absence of arbitrage in a finite market model, there is a unique risk neutral measure $\mathbb{Q}$. Hence the price of our option at time 0 is the discounted payoff conditionally to $\mathcal{F}_0$ under $\mathbb{Q}$:

$$P_0 = \mathbb{E}^{\mathbb{Q}}\left(\prod_{s=1}\left(1 + r_s\right)^{-1}\Phi_T|\mathcal{F}_0\right)$$

1. It is easy to see that the payoff is bounded $K_2 - K_1$. Then the result follows. The mathematical justification consists in invoking the positivity of conditional expectation:

$$P_0 \leq \mathbb{E}^{\mathbb{Q}}\left(\prod_{s=1}\left(1 + r_s\right)^{-1}|\mathcal{F}_0\right)\left(K_2 - K_1\right) = B_T^0(K_2 - K_1)$$

The last equality comes from the fact that the price of a zero coupon is obtained by discounting the value 1$.

The financial arguments consists in exhibiting an arbitrage if $P_0 > B_T^0(K_2 - K_1)$. Simply sell a spread and buy a nominal of $(K_2 - K_1)$ in zero coupons. The balance of this operation is positive. At maturity, you have to pay $\Phi_T \leq (K_2 - K_1)$. The net balance is:

$$(K_2 - K_1) - \Phi_T + \left(P_0 - B_T^0(K_2 - K_1)\right) B_T > 0$$

with B the bond.

2. The mathematical derivation of this result consists of invoking the linearity of conditional expectation. The financial justification consists in contructing a replicating portfolio made of a long position in a call with strike K_1 and a short position in a call with strike K_2.

3. A spread is desirable for an investor if he expects the stock S to increase beyond K_1, but not beyond K_2. Because it is cheaper than the call with strike K_1, it also allows a much bigger position with a smaller cash investment.

   This last argument is also valid from the point of view of the bank: it is a cheaper financial product than the call, and therefore can be sold more easily. Moreover, the downside risk is smaller: if the stock increases too much, the payoff remains bounded.

Exercise 4 (Bullet option - 8 points)
Consider a binomial model with one stock S and a bond B.

$$S_t = S_0 \prod_{i=0}^{t} \xi_i$$

$$B_t = (1+r)^t$$

where the ξ_i are independent and identically distributed. r is the interest rate. The natural filtration of S is denoted:

$$\mathcal{F}_t = \sigma\left(S_0, S_1, S_2, \ldots, S_t\right)$$

Under the risk neutral measure $\mathbb{Q}$:

$$\mathbb{Q}\left(\xi_i = u\right) = p = 1 - \mathbb{Q}\left(\xi_i = d\right)$$

The "bullet option" with strikes $K_1 < K_2$ is an option with payoff at time t:

$$\Phi_t = \mathbb{1}_{\{K_1 \leq S_t \leq K_2\}}$$

General question:

$$d < 1 + r < u$$

This condition is equivalent to:

$$0 < p = \frac{1 + r - d}{u - d} < 1$$

which is the necessary condition for the existence of an equivalent martingale measure. The financial meaning of such an inequality is clear: it is financially absurd to have an interest rate always more profitable or always less profitable than the stock.

Pricing and hedging of the European option:

The European option with payoff Φ_T at time T is called the European bullet option.

1. The price of the option at time t of the European bullet option is:

$$P_t = \frac{1}{(1+r)^{T-t}} \mathbb{E}^{\mathbb{Q}} \left(\mathbb{1}_{\{K_1 \leq S_T \leq K_2\}} | \mathcal{F}_t \right)$$

   Moreover:

$$S_T = S_t \prod_{i=t+1}^{T} \xi_i$$

   Because the random variables ξ_i for $i \geq t+1$ are independent from $\mathcal{F}_t$, we have $P_t = P(t,x)$ with:

$$P(t,x) = \frac{1}{(1+r)^{T-t}} \mathbb{Q} \left(K_1 \leq x \prod_{i=t+1}^{T} \xi_i \leq K_2 \right)$$

   Then the result follows by using the equality in law under $\mathbb{Q}$:

$$\prod_{i=t+1}^{T} \xi_i = u^{Bin(T-t,p)} d^{T-t-Bin(T-t,p)}$$

2. As done in the class, a replicating (or hedging) $\phi_t = (\alpha_t, \beta_t), t = 1, 2, \ldots, T$ should satisfy:

$$\alpha_t S_t + \beta_t B_t = P(t, S_{t-1}\xi_t)$$

   whether $\xi_t = u$ or $\xi_t = d$. Hence the system of linear equations in α and β:

$$\alpha_t S_{t-1} u + \beta_t B_t = P(t, S_{t-1} u)$$

$$\alpha_t S_{t-1} d + \beta_t B_t = P(t, S_{t-1} d)$$

   Solving it leads to:

$$\alpha_t = \frac{P(t, S_{t-1}u) - P(t, S_{t-1}d)}{S_{t-1}u - S_{t-1}d}$$

The American option:

1. The price of the American option is obtained by computing the Snell envelope of Φ_t, which is given by:

$$P_T^{am} = \mathbb{1}_{\{K_1 \leq S_T \leq K_2\}}$$

$$P_t^{am} = \max \left(\mathbb{1}_{\{K_1 \leq S_t \leq K_2\}}, \frac{1}{1+r} \mathbb{E} \left(P_{t+1}^{am} | \mathcal{F}_t \right) \right)$$

   By backward recurrence, one sees that there is a function f such that $P_t^{am} = f(t, S_t)$. That is a consequence of the Markovian behavior of our stock process. This function must satisfy the discrete Hamilton-Jacobi-Bellman equation:

$$f(T,x) = \mathbb{1}_{\{K_1 \leq x \leq K_2\}}$$

$$f(t,x) = \max \left(\mathbb{1}_{\{K_1 \leq x \leq K_2\}}, \frac{p}{1+r} f(t+1, xu) + \frac{1-p}{1+r} f(t+1, xd) \right)$$

   In order to put the final equation under the required form, we notice that (by backward recurrence or financial common sense):

$$0 \leq f(t,x) \leq 1$$

   With $r > 0$, the maximum concerns the right hand side only if $x < K_1$ or $x > K_2$.

2. Notice that if the spot at t is below K_1/u^{T-t} or higher than K_2/d^{T-t}, the payoff will always be zero. Therefore, any stopping time is an optimal stopping time. The earliest is right away and latest is at maturity. Let us call this the degenerate case.

   However, as intuition suggests, one should exercise the option as soon as the spot is between K_1 and K_2. In any other case, it is better to wait. That is validated by the theorem seen in class the smallest optimal stopping time τ is the first time when $P_t^{am} = \Phi_t$. There is no other one, unless we fall in the degenerate case.

Exercise 5 (4 points)

Notice that there are only two paths giving a non-zero payoff:

$$6, 5, 4, 8$$

$$6, 9, 8, 9$$

If the risk neutral probabilities of these two paths are p_1 and p_2, then the price is:

$$P = p_1 + 2p_2$$

The computation of these probabilities gives:

$$p_1 = \frac{3}{4}\frac{2}{3}\frac{1}{3} = \frac{1}{6}$$

$$p_2 = \frac{1}{4}\frac{2}{3}\frac{1}{2} = \frac{1}{12}$$

Then:

$$P = \frac{1}{3}$$