Final examination (3h)

- Only exercise 2 is in continuous time.
- In exercises 3 and 4, the candidate will be asked at certain questions to provide both a mathematical answer and a financial answer. Typically, a mathematical answer consists in invoking tools from mathematical finance such as the risk neutral measure. A financial answer requires exhibiting an arbitrage or hedging strategy.
- Exercises can be done in any order. Therefore, it is advised that you read all the subject before starting.
- You will not get the correction if you leave earlier.
- You are specifically asked to write with a black or blue pen. No pencils or red pen.
- For identification purposes, please display your ID card on the corner of the table.

Good luck! - and happy Valentine's day...

Exercise 1 (Lesson question - 3 points)
Consider a finite market model i.e finite time horizon and finite Ω. Define equivalent martingale measures and then state as clearly as possible the two fundamental theorems of asset pricing.

Exercise 2 (On some stochastic processes - 8 points)
In this exercise, all stochastic integrals with respect to Brownian motion will be assumed to be real martingales. Let W_t be a real standard Brownian motion.

A martingale
Let $\lambda \in \mathbb{R}$ and

$$M_t := e^{-\frac{1}{2}\lambda^2 t} \cosh\left(\lambda W_t\right)$$

where *cosh* is the hyperbolic cosine:

$$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

Prove M is a martingale.

On a certain affine process
Let $X_0 \in \mathbb{R}$ be a deterministic value. Define the stochastic process X_t to be the unique process starting at X_0 and solution to the following SDE (stochastic differential equation):

$$dX_t = \left(a - bX_t\right) dt + \sigma\sqrt{X_t} dW_t$$

with $b > 0$ and $\sigma > 0$. Existence and uniqueness are assumed.

1. Let $f(t) = \mathbb{E}\left(X_t\right)$ be the expectation at time t of our process. Show that f is a smooth function solving the ODE (ordinary differential equation):

$$f'(t) = a - bf(t)$$

2. Compute $\mathbb{E}(X_t)$ as a function of X_0, a, b and time t.

3. Thanks to the Itô formula, give the decomposition of X_t^2 as an Itô process.

4. Let $g(t) = \text{Var}(X_t)$ be the variance of our process at time t. Prove that it solves the ODE:

$$g'(t) = -2b\ g(t) + \sigma^2 f(t)$$

5. Solve the ODE. One can look for a solution of the form:

$$g(t) = C_0 + C_1 e^{-bt} + C_2 e^{-2bt}$$

Remark 0.1. *For your own culture, such processes are part of the so-called "affine processes" and are often used in order to model short-term interest rates or default intensities.*
These are tailored to have a mean-reverting property. For example, in the exercise, you see that $\mathbb{E}(X_t)$ *and* $Var(X_t)$ *converge to a long term value, independent of* X_0.

Exercise 3 (Spread option - 5 points)
Consider a complete finite market model with absence of opportunity of arbitrage. It has a single stock S. Its natural filtration is:

$$\mathcal{F}_t = \sigma(S_0, S_1, S_2, \dots, S_t)$$

Time is discrete with finite time horizon $t \in \{0, 1, 2, \dots, T\}$. Interest rate is r.
The spread option on the stock S, with maturity T and strikes $K_1 < K_2$ has payoff:

$$\Phi_T = (S_T - K_1)^+ - (S_T - K_2)^+$$

1. Prove that the price of this option is bounded by $B_T^0(K_2 - K_1)$, where B_T^0 is a zero coupon with maturity T. Give both a mathematical and a financial argument. Recall that the zero coupon is the amount that insures a payoff of 1$ at maturity.

2. If $C(T, K)$ is the price of a call option with maturity T and strike K, prove that the price of the spread option is given by:
$$C(T, K_1) - C(T, K_2)$$
Again, give both a mathematical and financial derivation of this result.

3. Describe situations where the spread option is desirable for an investor, and for the bank selling it.

Exercise 4 (Bullet option - 8 points)
Consider a binomial model with one stock S and a bond B.

$$S_t = S_0 \prod_{i=0}^{t} \xi_i$$

$$B_t = (1 + r)^t$$

where the ξ_i are independent and identically distributed. r is the interest rate. The natural filtration of S is denoted:

$$\mathcal{F}_t = \sigma(S_0, S_1, S_2, \dots, S_t)$$

Under the risk neutral measure $\mathbb{Q}$:

$$\mathbb{Q}(\xi_i = u) = p = 1 - \mathbb{Q}(\xi_i = d)$$

The “bullet option” with strikes $K_1 < K_2$ is an option with payoff at time t:

$$\Phi_t = \mathbb{1}_{\{K_1 \le S_t \le K_2\}}$$

General question
Describe the condition for absence of opportunity of arbitrage in terms of r, u and d. Give both a financial and mathematical argument for this inequality.
Pricing and hedging of the European option
The European option with payoff Φ_T at time T is called the European bullet option.

1. Prove that the price of the option at time t of the European bullet option is $P_t = P(t, S_t)$ where:
$$P(t,x) = \frac{1}{(1+r)^{T-t}} \mathbb{P}\left(\frac{\log\left(\frac{K_1}{x\ d^{T-t}}\right)}{\log(u/d)} \le Bin(T-t,p) \le \frac{\log\left(\frac{K_2}{x\ d^{T-t}}\right)}{\log(u/d)} \right)$$
with $Bin(n,p)$ being a binomial random variable with parameters $n \in \mathbb{N}$ and $0 < p < 1$.

2. Describe a replicating (or hedging) strategy in terms of a predictable process $\phi_t = (\alpha_t, \beta_t)\,, t = 1, 2, \dots, T$ such that:
$$\alpha_t = \alpha\,(t, S_{t-1})$$
$$\beta_t = \beta\,(t, S_{t-1})$$

The American option

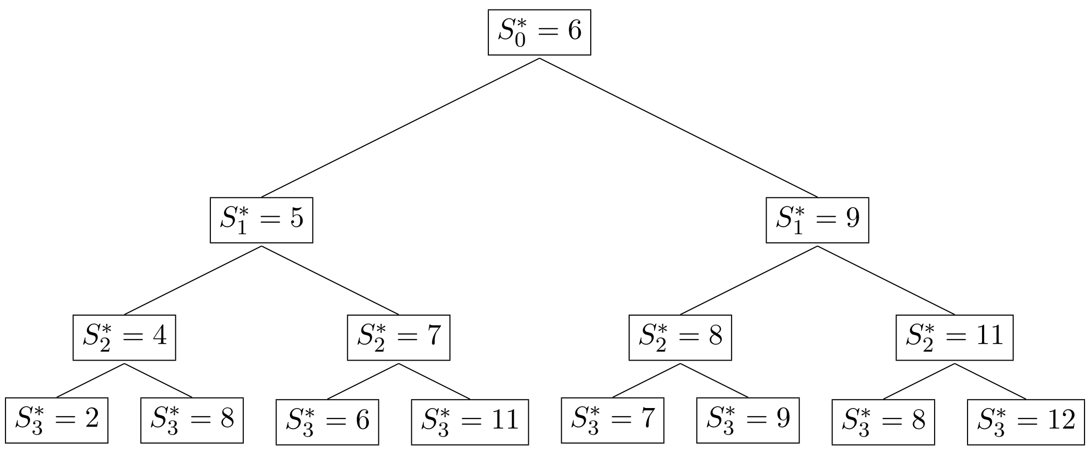
1. We assume $r > 0$. Let the price at time t of the American bullet option be $P_t^{am} = f(t, S_t)$. Prove that it satisfies the backward equation:
$$f(T,x) = \mathbb{1}_{\{K_1 \le x \le K_2\}}$$
$$f(t,x) = \mathbb{1}_{\{K_1 \le x \le K_2\}} + \mathbb{1}_{\{x<K_1 \text{ or } x>K_2\}} \left(\frac{p}{1+r} f(t+1, xu) + \frac{1-p}{1+r} f(t+1, xd) \right)$$

2. Describe the optimal stopping times for exercising the option, and discuss uniqueness.

Exercise 5 (4 points)
We consider a three-period arbitrage-free financial market model consisting of a bond and a stock with discounted price process $S^* = (S_0^*, S_1^*, S_2^*, S_3^*)$. We consider the call option with up and out barrier at 10$ and strike 7$:

$$\Phi^* = (S_3^* - 7)^+ \, \mathbb{1}_{\{\max_{t=0,1,2,3} S_t^* \le 10\}}$$

Assume that the processes evolve according to the following binomial tree, where each scenario has positive probability:



Compute the risk-neutral probability transitions and the price of the option.