Sheet 7

On one-parameter subgroups

Exercise 1. [(8.31) [1]] Show that if $\rho : \mathbb{R} \to G$ is a homomorphism from $(\mathbb{R}, +)$ to $(G, \cdot)$, then it is of the form φ_X with $X = \rho'(0)$

Exercise 2. Let $(X, Y) \in \mathfrak{g}^2$ be two elements in the Lie algebra of G, a Lie group.

- Show that:
$$\forall g \in G, \forall t \in \mathbb{R}, g\varphi_X(t)g^{-1} = \varphi_{Ad(g)(X)}(t)$$
- Prove that for every $Y \in \mathfrak{g}$:
$$t \mapsto Ad\left(\varphi_Y(t)\right)$$
is a one-parameter subgroup in $GL\left(\mathfrak{g}\right)$.
- Deduce that, if $[X, Y] = 0$, then the one-parameter subgroups φ_X and φ_Y commute:
$$\forall (s, t) \in \mathbb{R}^2, \varphi_X(s)\varphi_Y(t) = \varphi_Y(t)\varphi_X(s)$$

On the exponential map

Exercise 3. [(9.10) [1]] Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The subgroup of G generated by exponentiating the Lie subalgebra

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} | \forall Y \in \mathfrak{g}, [X, Y] = 0\}$$

is the connected component of the identity in the center $Z(G)$ of G.

- Prove that fact in the case where G is a linear group i.e a subgroup of $GL_n(\mathbb{R})$ using the Campbell-Hausdorff formula.
- Give an intrinsic proof, using the previous exercise.

References

[1] Fulton, Harris. Representation theory: A first course. Springer 1991.