# Sheet 6

## Warm up

**Exercise 1.** [(8.10) in [1]]
The skew-commutativity of the Lie bracket and the Jacobi identity follow from the naturality without using any embedding in $\mathfrak{gl}(V)$:

- Deduce the skew-commutativity $[X, X] = 0$ from the fact that any $X$ can be written the image of a vector by $d\rho_e$ for some homomorphism $\rho : \mathbb{R} \to G$ (a one-parameter subgroup).
- Given that the bracket is skew-commutative, verify that the Jacobi identity is equivalent to the assertion that:
$$ad = d\left(Ad\right)_e : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$
preserves the bracket. In particular, $ad$ is a map of Lie algebras.

## Description of some groups and their Lie algebra

**Exercise 2.** [(8.24) in [1]]

- With $Q$ a standard skew form, say $Q = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, describe the group $Sp_{2n}(\mathbb{R})$ and its Lie algebra $\mathfrak{sp}_{2n}(\mathbb{R})$ (as a subgroup of $GL_{2n}(\mathbb{R})$ and a subalgebra of $\mathfrak{gl}_{2n}(\mathbb{R})$. Give the dimension of the group.
- Same with $SO_{k,l}(\mathbb{R})$.
- Same with the complex group $SO_n(\mathbb{C})$.

## On the exponential map

**Exercise 3.** [(8.40) [1]] Show that exp is surjective for $G = GL_n(\mathbb{C})$ but not for $G = GL_n^+(\mathbb{R})$, $n > 1$. Note: $GL_n^+(\mathbb{R})$ is the group of invertible matrices with positive determinant.

**Exercise 4.** [(9.10) [1]] Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. The subgroup of $G$ generated by exponentiating the Lie subalgebra
$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} | \forall Y \in \mathfrak{g}, [X, Y] = 0\}$$
is the connected component of the identity in the center $Z(G)$ of $G$. Prove that fact in the case where $G$ is a linear group i.e a subgroup of $GL_n(\mathbb{R})$ using the Campbell-Hausdorff formula.

## References

[1] Fulton, Harris. Representation theory: A first course. Springer 1991.