Sheet 5

On Lie groups preserving certain structures

Exercise 1. [(7.3) in [1]]
Let $V \approx \mathbb{C}^n$ a complex vector space, and $H : V \times V \to \mathbb{C}$ a Hermitian form on V. Now, view $V \approx \mathbb{R}^{2n}$ as a real vector space. The multiplication by $i = \sqrt{-1}$ is then seen as an involutive automorphism on $\mathbb{R}^{2n}$, and elements in $GL_n(\mathbb{C})$ are seen as linear maps on $\mathbb{R}^{2n}$ commuting with the complex conjugation.

- Show that $Re(H)$, the real part of H is a symmetric form on the underlying real vector space, and that $Im(H)$ the imaginary part of H is skew-symmetric. They are related by:

$$C(v, w) = R(iv, w)$$

- Prove the invariance under i:

$$R(v, w) = R(iv, iw)$$
$$C(v, w) = C(iv, iw)$$

- Conversely, prove that every such R is the real part of a unique Hermitian form H.
- If H is the standard Hermitian form:

$$\forall (v, w) \in (\mathbb{C}^n)^2, H(v, w) = \sum_i \bar{v}_i w_i$$

Then R is standard, $R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

- Finally, deduce that the unitary group is a subgroup of $GL_{2n}(\mathbb{R})$:

$$U(n) = O_{2n}(\mathbb{R}) \cap Sp_{2n}(\mathbb{R})$$

Exercise 2. [(7.4) in [1] - Optional]
Let $\mathbb{H}$ be the field of quaternions. One can consider the group of $\mathbb{H}$-linear automorphisms of $\mathbb{H}^n \approx \mathbb{C}^{2n}$ (as $\mathbb{C}$-vector spaces), giving rise to $GL_n(\mathbb{H}) \subset GL_{2n}(\mathbb{C})$.
Then, one can ask the question of describing the subgroup of transformations leaving invariant a quaternionic Hermitian form.

On converings

Exercise 3. [(7.16) [1]] Let $M_2(\mathbb{C}) = \mathbb{C}^4$ be the space of 2×2 matrices, with symmetric form $Q(A, B) = \frac{1}{2} Trace(AB^\natural)$, where $B^\natural = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is the adjoint of the matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In fact, the quadratic form associated to Q is simply the determinant. Define the mapping:

$$\begin{array}{rccc} \varphi : & SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) & \to & SO_4(\mathbb{C}) \\ & (g, h) & \mapsto & \left(A \mapsto gAh^{-1}\right) \end{array}$$

Prove that φ is a $2:1$ covering and therefore realizes the universal cover of $SO_4(\mathbb{C})$, since $SL_2(\mathbb{C})$ is simply connected.

Exercise 4. [(7.17) [1]] Identify $\mathbb{C}^3$ with the space of traceless matrices in $M_2(\mathbb{C})$, and endow it with the non-degenerate symmetric bilinear form $Q(A,B) = Trace(AB)$. Define the mapping:

$$\begin{array}{cccc} \varphi: & SL_2(\mathbb{C}) & \to & SO_3(\mathbb{C}) \\ & g & \mapsto & \left(A \mapsto gAg^{-1}\right) \end{array}$$

Same question as before.

References

[1] Fulton, Harris. Representation theory: A first course. Springer 1991.