Sheet 4

On induced representations

In the following G is a finite group and H a subgroup.
Exercise 1. [(3.18) and (3.19) in [1]]
Let W be a representation of H and $V = \mathrm{Ind}W$ be the induced representation.

- Explain why for $g \in G$
$$\chi_V(g) = \sum_{\sigma \in G/H : g\sigma = \sigma} \chi_W\left(s^{-1}gs\right)$$
with $s \in \sigma$ is arbitrary.

- If C is a conjugacy class of G and $C \cap H$ decomposes into conjugacy classes $D_1, \dots, D_r$ of H, then:
$$\chi_V(C) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi_W\left(D_i\right)$$

- If W is the trivial representation of H then:
$$\chi_V(C) = \frac{[G:H]}{|C|} \cdot |C \cap H|$$

Exercise 2. [(3.23) in [1]]
Find the representation of $G = S_4$, the symmetric group in 4 elements, induced by:

- the one dimensional representation of the group $H = \langle (1234) \rangle$, where (1234) acts by multiplication by $i = \sqrt{-1}$.

- the one dimensional representation of the group $H = \langle (123) \rangle$, where (123) acts by multiplication by $j = e^{i\frac{2\pi}{3}}$.

Exercise 3. [(3.26) in [1]]
There is a unique non abelian group of order 21, which can be realized as the group of affine transformations $x \mapsto \alpha x + \beta$ of the line over the field with seven elements $\mathbb{F}_7$, with α a root of unity in that field. Find the irreducible representations and character table of the group.
Recall that affine transformations over any field can be seen as upper triangular matrices with coefficients in that field.

On the Fourier transform for finite groups

Exercise 4. [(2.35) [1]] Using the exercises in the previous sheet, show that if the irreducible reprentations of G are represented by unitary matrices, the matrix entries of these representations form an orthogonal basis of $a$ll functions, with inner product given by:
$$\langle f, g \rangle = \frac{1}{|G|} \sum_{h \in G} f(h)\overline{g(h)}$$

In the next exercise, we see that the inverse Fourier transform on the group gives an expansion over this orthogonal basis.

Exercise 5. [(3.32) [1]] Let G be a finite group and $\widehat{G}$ the set of (isomorphism classes) its irreducible representations. If $\rho : G \to GL(V_\rho)$ is a representation in $\widehat{G}$ and φ is a function on G, define the Fourier transform $\hat{\varphi} : \widehat{G} \to \mathrm{End}(V_\rho)$ as:

$$\hat{\varphi}(\rho) = \sum_{g \in G} \varphi(g)\rho(g)$$

- The convolution $\varphi \star \psi$ of two functions $\varphi, \phi : G \to \mathbb{C}$ is given by:

  $$\varphi \star \psi(g) = \sum_{h \in G} \varphi(h)\psi(h^{-1}g)$$

  Prove that $\widehat{\varphi \star \psi}(\rho) = \hat{\varphi}(\rho)\hat{\psi}(\rho)$

- For $\varphi : G \to \mathbb{C}$, prove the Fourier inversion formula:

  $$\varphi(g) = \frac{1}{|G|} \sum_{\rho \in \widehat{G}} \dim(V_\rho) Tr\left(\rho(g^{-1}) \cdot \hat{\varphi}(\rho)\right)$$

  If just like the previous exercise, $\rho(g)$ is taken to be a unitary matrix, then the formula becomes:

  $$\varphi(g) = \frac{1}{|G|} \sum_{\rho \in \widehat{G}} \dim(V_\rho) \sum_{i,j=1}^{\dim V_\rho} \overline{\rho(g)_{i,j}}\hat{\varphi}(\rho)_{i,j}$$

  which is indeed an expansion of φ over matrix coefficients of irreducible representations.

- For $\varphi, \psi : G \to \mathbb{C}$, prove the Plancherel formula:

  $$\sum_{g \in G} \varphi(g^{-1})\psi(g) = \frac{1}{|G|} \sum_{\rho \in \widehat{G}} \dim(V_\rho) Tr\left(\hat{\varphi}(\rho)\hat{\psi}(\rho)\right)$$

References

[1] Fulton, Harris. Representation theory: A first course. Springer 1991.