Sheet 3

Standard facts

Exercise 1. [The unitary trick - (1.14) in [1]] Let V be an irreducible representation of the finite group G. Show that, up to scalars, there is a *unique* Hermitian inner product $\langle , \rangle$ on V invariant under G:

$$\forall g \in G, \forall (x, y) \in V^2, \langle g \cdot x, g \cdot y\rangle = \langle x, y\rangle$$

Solution of exercise 1.
The existence of an invariant scalar product $\langle , \rangle$ is given by the usual unitary trick, i.e by averaging any scalar product $\langle , \rangle_0$:

$$\forall (x, y) \in V^2, \langle x, y\rangle := \frac{1}{|G|} \sum_{g\in G} \langle g \cdot x, g \cdot y\rangle_0$$

Now for the uniqueness part: two invariant scalar products $\langle , \rangle$ and $(,)$ give rise to G-isomorphisms:

$$\begin{array}{rccc} \varphi : & V & \to & V^* \\ & x & \mapsto & \langle x, .\rangle \end{array}$$

$$\begin{array}{rccc} \psi : & V & \to & V^* \\ & x & \mapsto & (x, .) \end{array}$$

Therefore $\psi^{-1} \circ \varphi$ is a G-automorphism of V, which by Schur's lemma must be a multiple of the identity λid. This can be rewritten:

$$\forall (x, y) \in V^2, \langle x, y\rangle = \lambda (x, y)$$

Exercise 2. [(2.34) in [1]] Let V and W be irreducible representations of G and $L_0 : V \to W$ be any linear mapping. Define $L : V \to W$ by:

$$L(v) = \frac{1}{|G|} \sum_{g\in G} g^{-1} \cdot L_0(g \cdot v)$$

Show that $L = 0$ if V and W are not isomorphic, and that L is a multiplication by $\frac{trace(L_0)}{\dim V}$ if $W = V$.

Solution of exercise 2. It is easy to check that L is G-linear. Since V and W are irreducible, we can apply Schur's lemma.

- If V and W are not isomorphic, $L = 0$.
- If $V = W$, $L = \lambda id$ with:

$$\begin{aligned} \lambda &= \frac{Tr(L)}{\dim V} \\ &= \frac{1}{|G|} \frac{\sum_g Tr(gL_0 g^{-1})}{\dim V} \\ &= \frac{Tr(L_0)}{\dim V} \end{aligned}$$

On the dihedral group

Exercise 3. [(3.7) and (3.8) in [1]] The dihedral group $G = D_{2n}$ is the group of isometries of a regular n-gon in the plane. It is made of n rotations and n reflections. The purpose of this exercise is to understand the representation theory of this group.

- Method 1: Consider an arbitrary representation V. Notice that Γ, the subgroup of rotations is abelian. The space V breaks up into eigenspaces for the action of Γ. Analyze the action of reflections on these eigenspaces, and explain how to decompose the representation V.
- Method 2: Using character theory, give a character table.

Solution of exercise 3. Let $\omega = e^{\frac{2i\pi}{n}}$ and denote by r rotation of angle $\frac{2i\pi}{n}$ and s a reflection. Clearly:

$$D_{2n} = \langle r, s \rangle$$

More precisely, D_{2n} can be presented as the group generated by r and s subject to the relations:

$$r^n = 1$$

$$s^2 = 1$$

$$srs = r^{-1}$$

- Method 1: Let $\rho : G \to GL(V)$ be a representation. As Γ is abelian, the elements in $\varrho(\Gamma) \subset GL(V)$ can be simultaneously diagonalized, breaking V into a direct of eigenspaces for say r:

  $$V = \oplus_{k=0}^{n-1} E(\omega^k)$$

  Now notice that if $x \in E(\omega^k)$ meaning that $rx = \omega^k x$, then let $y = sx$ and:

  $$ry = rsx = sr^{-1}x = \omega^{-k} y$$

  Hence the two cases:

  - $\omega^{2k} \neq 1$: (x, sx) is free and generate an irreducible subrepresentation of V. In this representation, we can write:

    $$\rho(r) = \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{pmatrix}$$

    $$\rho(r) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  - $\omega^{2k} = 1$: If $\omega^k = 1$, either (x, sx) is free and generates an invariant subspace isomorphic to a sum of trivial and determinant representations; either (x, sx) is not free and generates already one of those representations depending on $sx = \pm x$.

    If $\omega^k = -1$, which is possible only when n is even, the space generated by (x, sx) if two dimensional breaks into one dimensional spaces. Two representations are possible, depending on whether s acts as 1 or -1.

- Method 2:

  - $n = 2p + 1$ odd: All reflections are conjugate and rotations with opposite angles are in the same conjugacy class. Hence $p + 2$ conjugacy classes.

    The irreducible representations are the trivial one, the determinant and the p two-dimensional representations described in "Method 1" where the character of r is $\omega^k + \omega^{-k}$.

– $n = 2p$ even: id and $-id$ form two conjugation classes. Reflections break into two conjugacy classes. Add to this $p - 1$ classes of two rotations with opposite (different) angles. Hence $p + 3$ conjugacy classes.

  The irreducible representations are four one dimensional ones, and $p - 1$ two dimensional ones (also described in in "Method 1").

That is sufficient to draw a character table, while using character theory to back-check our results.

References

[1] Fulton, Harris. Representation theory: A first course. Springer 1991.