Sheet 2

Warm up

Exercise 1. [(2.5) in [1]]
Let G be a finite group acting on a set X and consider V the associated permutation representation. Prove that for $g \in G$ the character $\chi_V(g)$ is given by the number of fixed points of g acting on X.
Application: Give the character of the (left-)regular representation of G.

Solution of exercise 1. Let $g \in G$. In the basis $\{e_x, x \in X\}$, the matrix $\rho_V(g)$ has a one on the diagonal only when $g \cdot x = x$, which corresponds to a fixed point. Therefore, the character $\chi_V(g) = Tr(\rho_V(g))$ is exactly the number of fixed points for the action of g.
In the case of the regular representation, no element has fixed points except the identity $e \in G$, which fixes the entire group. Therefore:

$$\chi_R(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{otherwise} \end{cases}$$

On the symmetric group S_3

Recall that S_3, the symmetric group acting on 3 elements, has three irreducible representations known as:

a) $V(triv)$: The trivial representation.

b) $V(alt)$: The alternate representation.

c) $V(st)$: The standard representation.

Exercise 2. [(2.7) in [1]]
Decompose the representation $V(st)^{\otimes n}$ into irreducibles.

Solution of exercise 2. For shorter notations, write $V = V(st)^{\otimes n}$. If:

$$V = a_1 V(triv) \oplus a_2 V(alt) \oplus a_3 V(st)$$

Then, the character on the different conjugation classes (of the identity, transpositions and three-cycles) is given by:

$$\chi_V = (2^n, 0, (-1)^n)$$

And:

$$a_1 = \frac{1}{3}\left(2^{n-1} + (-1)^n\right)$$
$$a_2 = \frac{1}{3}\left(2^{n-1} + (-1)^n\right)$$
$$a_3 = \frac{1}{3}\left(2^n - (-1)^n\right)$$

Exercise 3.
Decompose the following representations into irreducibles:

- $V_1 = Sym^2(V(st))$
- $V_2 = \Lambda^2(V(st))$
- $V_3 = \big(Sym^2(V(st)) \oplus V(triv)\big) \otimes V(alt)$
- $V_4 = \big(Sym^2(V(st)) \oplus V(triv)\big) \otimes V(st)$

Solution of exercise 3. The characters are:

- $\chi_{V_1} = (3, 1, 0)$
- $\chi_{V_2} = (1, -1, 1)$
- $\chi_{V_3} = (4, -2, 1)$
- $\chi_{V_4} = (8, 0, -1)$

Hence, up to isomorphism:

- $V_1 \approx V(triv) \oplus V(st)$
- $V_2 \approx V(alt)$
- $V_3 \approx 2V(alt) \oplus V(st)$
- $V_4 \approx V(triv) \oplus V(alt) \oplus 3V(st)$

References

[1] Fulton, Harris. Representation theory: A first course. Springer 1991.