Exam

Exercise 1. [The Fourier transform] Let G be a finite group and $\widehat{G}$ the set of (isomorphism classes of) its irreducible representations. If $\rho : G \to GL(V_\rho)$ is a representation in $\widehat{G}$ and φ is a function on G, define the Fourier transform $\hat{\varphi} : \widehat{G} \to \text{End}(V_\rho)$ as:

$$\hat{\varphi}(\rho) = \sum_{g \in G} \varphi(g)\rho(g)$$

- The convolution $\varphi \star \psi$ of two functions $\varphi, \phi : G \to \mathbb{C}$ is given by:

$$(\varphi \star \psi)(g) = \sum_{h \in G} \varphi(h)\psi(h^{-1}g)$$

  Prove that $\widehat{\varphi \star \psi}(\rho) = \hat{\varphi}(\rho)\hat{\psi}(\rho)$

- For $\varphi : G \to \mathbb{C}$, prove the Fourier inversion formula:

$$\varphi(g) = \frac{1}{|G|} \sum_{\rho \in \widehat{G}} \dim(V_\rho) \operatorname{Tr}\left(\rho(g^{-1}) \cdot \hat{\varphi}(\rho)\right)$$

  which is indeed an expansion of φ over matrix coefficients of irreducible representations.

- For $\varphi, \psi : G \to \mathbb{C}$, prove the Plancherel formula:

$$\sum_{g \in G} \varphi(g^{-1})\psi(g) = \frac{1}{|G|} \sum_{\rho \in \widehat{G}} \dim(V_\rho) \operatorname{Tr}\left(\hat{\varphi}(\rho)\hat{\psi}(\rho)\right)$$

Solution of exercise 1.

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$$\begin{aligned}
\widehat{\varphi \star \psi}(\rho) &= \sum_{g \in G} \varphi \star \psi(g)\rho(g) \\
&= \sum_{g,h} \varphi(h)\psi(h^{-1}g)\rho(g) \\
&= \sum_{g,h} \varphi(h)\psi(h^{-1}g)\rho(h)\rho(h^{-1}g) \\
&= \hat{\varphi}(\rho)\hat{\psi}(\rho)
\end{aligned}$$

- Fix $g \in G$:

$$\begin{aligned}
&\frac{1}{|G|}\sum_{\rho\in\widehat{G}} \dim(V_\rho)\,\mathrm{Tr}\left(\rho(g^{-1})\cdot\hat{\varphi}(\rho)\right)\\
&= \frac{1}{|G|}\sum_{h\in G}\varphi(h)\sum_{\rho\in\widehat{G}} \dim(V_\rho)\,\mathrm{Tr}\left(\rho(g^{-1})\cdot\rho(h)\right)\\
&= \frac{1}{|G|}\sum_{h\in G}\varphi(gh)\sum_{\rho\in\widehat{G}} \dim(V_\rho)\,\mathrm{Tr}\left(\rho(h)\right)\\
&= \sum_{h\in G}\varphi(gh)\frac{1}{|G|}\sum_{\rho\in\widehat{G}} \overline{\chi_\rho(e)}\chi_\rho(h)\\
&= \sum_{h\in G}\varphi(gh)\delta_{h,e}\\
&= \varphi(g)
\end{aligned}$$

The next to last step being obtained by orthogonality of columns in the character table.

- Using the previous questions:

$$\begin{aligned}
\sum_{g\in G}\varphi(g^{-1})\psi(g) &= (\varphi\star\psi)(e)\\
&= \frac{1}{|G|}\sum_{\rho\in\widehat{G}} \dim(V_\rho)\,\mathrm{Tr}\left(\widehat{\varphi\star\psi}(\rho)\right)\\
&= \frac{1}{|G|}\sum_{\rho\in\widehat{G}} \dim(V_\rho)\,\mathrm{Tr}\left(\hat{\varphi}(\rho)\hat{\psi}(\rho)\right)
\end{aligned}$$

Exercise 2. [On the dicyclic group] Let $n \geq 1$. Define the dicyclic group as:

$$G := Dic_n = \langle x, a \mid a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1}\rangle$$

(Fact) Every element in G can be written in the form $a^k x^\varepsilon$, $0 \leq k < 2n$, $\varepsilon \in \{0, 1\}$.
The goal is to fill the following character table:

Conj. classes	$\mathcal{C}(id)$	$\mathcal{C}(a^k)$	$\mathcal{C}(a^n)$	$\mathcal{C}(x)$	$\mathcal{C}(ax)$
Cardinality					
$Triv$					
ε					
χ_{V_ℓ}					
U					
U'					

- G has an abelian subgroup:

$$\Gamma = \langle a\rangle \approx \mathbb{Z}/2n\mathbb{Z}$$

Prove that Γ is normal.

- Give the irreducible representations that are trivial on Γ. They will be denoted $Triv$ and ε.

- If we denote $\mathcal{C}(g)$ the conjugation class of an element $g \in G$, we have the following $n+3$ conjugation classes:

$$\mathcal{C}(a^k), 0 \leq k \leq n$$
$$\mathcal{C}(x), \mathcal{C}(ax)$$

Describe all the elements in each conjugation class and deduce its cardinality.

- Define what is an induced representation.
- Let χ_ℓ be the character of Γ such that:

$$\chi_\ell(a) = \omega^\ell$$

$$\omega = e^{\frac{i\pi}{n}}$$

Define the induced representation $V_\ell = \operatorname{Ind}_\Gamma^G \chi_\ell$, $0 \leq \ell < n$. Compute the characters χ_{V_ℓ} and show that:

$$\forall 1 \leq \ell \leq n, V_\ell \approx V_{2n-\ell}$$

- Prove that for $1 \leq \ell \leq n-1$, V_ℓ is irreducible, while V_0 and V_n are not.
- Prove that there are only two one dimensional representations U and U' left to find. Complete the character table - one has to distinguish between n even and n odd.

Solution of exercise 2.

- It suffices to check that conjugation by x stabilizes Γ, which is straightforward.
- G/Γ has two classes:

$$G/\Gamma = \{\Gamma, x\Gamma\}$$

Hence $G/\Gamma \approx \mathbb{Z}/2\mathbb{Z}$. The representations of G trivial on Γ are in correspondence with those of $\mathbb{Z}/2\mathbb{Z}$. There is the trivial one $Triv$ and ε such that:

$$\varepsilon(a^k) = 1$$

$$\varepsilon(a^k x) = -1$$

- $$\mathcal{C}(id) = \{e\}$$
$$\mathcal{C}(a^k) = \left\{a^k, a^{-k}\right\}, 1 \leq k \leq n-1$$
$$\mathcal{C}(a^n) = \{a^n\}$$
$$\mathcal{C}(x) = \{x, a^2x, \ldots, a^{2n-2}x\}$$
$$\mathcal{C}(ax) = \{ax, a^3x, \ldots, a^{2n-1}x\}$$

- Course definition.
- By definition, for the induced representation, there is a basis $(1, x1)$ - by choosing a representative of each element in G/Γ - such that:

$$V_\ell = \operatorname{Ind}_\Gamma^G \chi_\ell = \operatorname{Span}_\mathbb{C}(1, x1)$$

with action:

$$a^k \cdot 1 = \omega^{k\ell} 1$$
$$a^k \cdot x1 = \omega^{-k\ell} x1$$
$$\left(a^k x\right) \cdot 1 = \omega^{k\ell} x1$$
$$\left(a^k x\right) \cdot x1 = a^{k+n} 1 = (-1)^\ell \omega^{k\ell} 1$$

Hence the matrices:

$$\rho_{V_\ell}(a^k) = \begin{pmatrix} \omega^{k\ell} & 0 \\ 0 & \omega^{-k\ell} \end{pmatrix}$$

$$\rho_{V_\ell}(x) = \begin{pmatrix} 0 & (-1)^\ell \\ 1 & 0 \end{pmatrix}$$

And the characters:

$$\chi_{V_\ell}(a^k) = 2\cos\left(\frac{k\ell\pi}{n}\right)$$

$$\chi_{V_\ell}(a^k x) = 0$$

Since a character completely determines the representation:

$$\forall 1 \leq \ell \leq n, V_\ell \approx V_{2n-\ell}$$

We see that, for now, the interesting ones are the V_ℓ for $\ell = 0, \ldots, n$

- Compute character norms:

  Hence $n-1$ representations of dimension 2.

- Using the formula:

$$|G| = \sum_{\rho \in \hat{G}} |\dim V_\rho|^2$$

  we find that the two representations left have dimensions n_1 and n_2 such that:

$$n_1^2 + n_2^2 = 1$$

  Therefore both have dimension 1.

  Let χ be a character. Necessarily:

$$\chi(x)^4 = 1$$

$$\chi(a)^2 = 1$$

$$\chi(a)^n = \chi(x)^2$$

  Depending on the parity of n, one finds two more characters.

Exercise 3. [On $\mathfrak{sl}_2$] Recall that

$$\mathfrak{sl}_2 = \{x \in M_2(\mathbb{C}) \mid \operatorname{Tr} x = 0\}$$

It is a Lie algebra generated by $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We write $H = [X, Y] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The Lie bracket is given by:

$$[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H$$

Define the differential operators, acting on polynomials in two variables x and y:

$$P = y\frac{\partial}{\partial x}$$

$$Q = x\frac{\partial}{\partial y}$$

And let $\mathcal{L}$ be the Lie algebra generated by P and Q - with the commutator bracket.

- Prove that $\mathfrak{sl}_2$ and $\mathcal{L}$ are isomorphic as Lie algebras. In order to do so, exhibit a Lie algebra isomorphism between the two and specify the images of X, Y and H.
- Consider the space of homogenous polynomials in two variables of degree n:

$$V_n = \text{Span}_{\mathbb{C}} \left(x^k y^{n-k}, 0 \leq k \leq n \right)$$

  Prove that V_n is an irreducible representation of $\mathcal{L}$ and isomorphic to a highest weight representation of $\mathfrak{sl}_2$.
- For which $(r, s, t) \in \mathbb{N}^3$, do we have:

$$P^r [P, Q]^s Q^t x^k y^{n-k} = 0$$

  It is strongly advised to use the fact that V_n is a representation of $\mathcal{L} \approx \mathfrak{sl}_2$.

Solution of exercise 3.

- Define the linear map φ such that:

$$\begin{array}{rccc} \varphi: & X & \mapsto & P \\ & Y & \mapsto & Q \\ & H & \mapsto & [P, Q] = y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} \end{array}$$

  In order to prove it is a Lie algebra morphism, check that:

$$\forall a, b \in \mathfrak{sl}_2 \times \mathfrak{sl}_2, \varphi\left([a, b]\right) = [\varphi(a), \varphi(b)]$$

  By bilinearity, it suffices to do it for $\{(X, Y), (H, X), (H, Y)\}$, hence checking:

$$[[P, Q], P] = 2P$$

$$[[P, Q], Q] = -2Q$$

  The fact it is an isomorphism comes from the fact it is surjective between spaces of same dimension.
- First, notice that V_n is a representation because the operators P and Q stabilize homogenous polynomials of given degree.
  Seen as a representation of $\mathfrak{sl}_2$, we recognize the highest weight representation with highest weight n. The highest weight vector is y^n.
- Use the fact that the operators X, Y (P, Q) change weight spaces, while H ($[P, Q]$) does not.

Exercise 4. [On $\mathfrak{sl}_3$]

- Let $V = \mathbb{C}^3$ be the canonical representation of $\mathfrak{sl}_3$. V^* is its dual. Give the highest weight, the weight diagram and a weight space decomposition for V and V^*.
- Same question for $Sym^2 V$ and $Sym^2 V^*$.
- Decompose the tensor product $Sym^2 V \otimes Sym^2 V^*$ into irreducibles.

Solution of exercise 4. Done in class.