The Whittaker process as weakly non-intersecting particles

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Sommaire

Generalities

There are two ways of doing mathematics:

- Exact computations in a world of rigid structures ("Algebra").
- Comparisons and variational approaches in a more fluid world ("Analysis").

Even probability theory does not escape such a dichotomy, and these two approaches can work hand in hand.

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Example: Lindberg's proof of CLT

- Proving the CLT for the "integrable case" of Gaussians is virtually trivial. Gaussian calculus is exact.
- Lindberg's swapping trick: Sums of i.i.d with matching moments will necessarily give the same result.

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Example 2: Wigner matrices

- Integrable case: Gaussian Unitary Ensemble (GUE). Using the rigid tool of determinantal point processes, one can prove the semi-circular law, sine-kernel at the edge and Tracy-Widom distribution for GUE.
- Tao and Vu's fourth moment theorem: Local statistics match with GUE if four first moments match.

In this talk, we will introduce an integrable process of weakly non-intersecting particles. The integrability finds its source in representation theory and will only be hinted to.

Sommaire

Analytic construction (1)

Consider two independent Brownian particles. The center of mass is a Brownian motion and is used for centering. Conditioning the particles to never intersect tantamounts to constructing Brownian motion conditioned to remain in $\mathbb{R}_+$. This conditioning is singular and gives the Bessel three process BES^3.

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Approach using regular conditioning: $W^{(\mu)}$ BM with drift $\mu > 0$ killed upon touching 0.

- Infinitesimal generator with Dirichlet boundary conditions:

$$\mathcal{L}^{(\mu)} = \frac{1}{2}\partial_x^2 + \mu\partial_x$$

- Special harmonic function for $\mathcal{L}^{(\mu)}$:

$$\mathbb{P}_x\left(W^{(\mu)} \text{ survives}\right)$$

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  Notice that this normalisation gives analytic extension and symmetry.

- The process conditioned to survive has generator:

$$\mathcal{G}^{(\mu)} = h_\mu(x)^{-1}\left(\frac{1}{2}\Delta - \frac{1}{2}\mu^2\right)h_\mu(x) = \frac{1}{2}\partial_x^2 + \mu \frac{cosh(\mu x)}{sinh(\mu x)}\partial_x$$

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By letting $\mu \to 0$, we recover the $BES^3 =$ BM conditioned to stay positive.

Geometric construction via RMT (Random Matrix Theory)

Consider the 2×2 Hermitian Brownian motion:

$$GUE_t := \begin{pmatrix} B_t^1 & B_t^2 + iB_t^3 \\ B_t^2 - iB_t^3 & -B_t^1 \end{pmatrix}$$

Its spectrum is $\{\Lambda_t, -\Lambda_t\}$ where $(\Lambda_t; t \geq 0) \stackrel{\mathcal{L}}{=} BES^3$:

$$\mathbb{P}\left(\Lambda_t \in dx\right) = \frac{1}{Z} x^2 e^{-\frac{x^2}{2t}}$$

Algebraic construction via Pitman (Rep. theory)

Theorem (Discrete Pitman(1975))

Let W a standard random walk on $\mathbb{Z}$. Then:

$$\Lambda_n := W_n - 2 \inf_{0 \le k \le n} W_k$$

is Markov with transition kernel on $\mathbb{N}$:

$$Q(x, x+1) = \frac{1}{2}\frac{x+2}{x+1} \quad Q(x, x-1) = \frac{1}{2}\frac{x}{x+1}$$

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After the diffusive scaling:

Theorem (Continuous Pitman(1975))

Let W a standard BM on $\mathbb{R}$. Then $\Lambda_t = W_t - 2 \inf_{0 \leq s \leq t} W_s$ is a BES^3.

Comments

- Very strong rigidity. No other coefficient but 2 works.
- The Pitman transform is of representation theoretic significance.
- The transition probabilities reflect structure constants of the representation theory of $\mathfrak{sl}_2$.

Representation theoretic explanation (1):

There is a representation-theoretic story to give here ($2 = \alpha(\alpha^\vee)$).
Consider the Lie algebra $\mathfrak{sl}_2$. For any $n \in \mathbb{N}$, highest weight, there is an irreducible representation $V(n)$ of dimension $n+1$.
$V(n) \rightsquigarrow \mathcal{B}(n)$ a crystal = a combinatorial object that can be realized as paths thanks to the Littelmann path model.

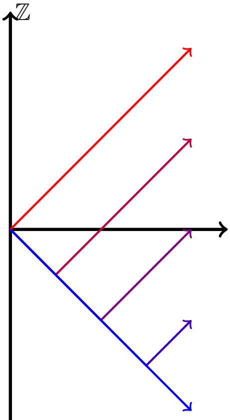

Figure: $\mathfrak{sl}_2$ path crystal with highest weight $n = 4$

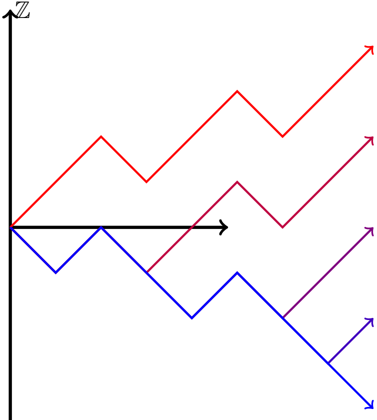

Figure: $\mathfrak{sl}_2$ path crystal with highest weight $n = 4$

Representation theoretic explanation (2)

The Pitman transform

$$\mathcal{P} : \pi \mapsto \pi(t) - 2 \inf_{0 \le s \le t} \pi(s)$$

has a special interpretation in the context of the Littelmann path model: It gives the dominant path in a crystal.

Let $V(1) = \mathbb{C}^2$ be the standard representation of $\mathfrak{sl}_2$.

- Looking at the standard random walk B_n can be seen as following a weight vector in $V(1)^{\otimes n}$.
- Looking at its Pitman transform X_n means following a highest weight in a decomposition of $V(1)^{\otimes n}$ into irreducibles. The transition probabilities are given by the Clebsch-Gordan rule:

$$V(n) \otimes V(1) \approx V(n+1) \oplus V(n-1)$$

Conclusion: Pitman's theorem is about the Markov property of **a highest weight process** and transition probabilities are expressed in terms of **structure constants**.

Sommaire

The exponential potential (1)

In order to have a weak repulsion from zero, an idea is to consider W a BM "slowly killed" when being negative. The framework of submarkovian generators fits the bill.

- Infinitesimal generator:

$$\mathcal{L}^{(\mu)} = \frac{1}{2}\partial_x^2 + \mu \partial_x - 2e^{-2x}$$

- Special harmonic function for $\mathcal{L}^{(\mu)}$:

$$\begin{aligned} & \mathbb{P}_x\left(W^{(\mu)} \text{ survives}\right) \\ = & \ \mathbb{E}_x\left(\exp\left(-2\int_0^\infty e^{-2W_s^{(\mu)}} ds\right)\right) \end{aligned}$$

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$$\mathcal{L}^{(\mu)} = \frac{1}{2}\partial_x^2 + \mu\partial_x - 2e^{-2x} = e^{-\mu x}\left(\frac{1}{2}\Delta - 2e^{-2x} - \frac{1}{2}\mu^2\right)e^{\mu x}$$

- Special harmonic function for $\frac{1}{2}\Delta - 2e^{-2x} - \frac{1}{2}\mu^2$:

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  This normalisation gives analytic extension and symmetry ($2x - t \longleftrightarrow t$ changes μ to $-\mu$).

- The process conditioned to survive has generator:

$$\mathcal{G}^{(\mu)} = \psi_\mu(x)^{-1}\left(\frac{1}{2}\Delta - 2e^{-2x} - \frac{1}{2}\mu^2\right)\psi_\mu(x) = \frac{1}{2}\partial_x^2 + \partial_x \log\psi_\mu(x)\partial_x$$

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The limit $\mu \to 0$ makes sense.

Pitman-type construction of Whittaker process

Theorem (Matsumoto-Yor(2000))

Let $W^{(\mu)}$ a Brownian motion with drift μ. Then:

$$\Lambda_t^{(\mu)} = W_t^{(\mu)} + \log\left(\int_0^t e^{-2W_s^{(\mu)}} ds\right)$$

is Markov with inf. generator

$$\psi_\mu^{-1}\left(\frac{1}{2}\frac{d^2}{dx^2} - 2e^{-2x} - \frac{\mu^2}{2}\right)\psi_\mu$$

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By Brownian rescaling and the Laplace method:

$$W_t^{(\mu)} + h\log\left(\int_0^t e^{-2\frac{W_s^{(\mu)}}{h}} ds\right) \overset{h\to 0}{\longrightarrow} W_t^{(\mu)} - 2\inf_{0\le s\le t} W_s^{(\mu)}$$

is Markov with inf. generator

$$\psi_{h,\mu}^{-1}\left(\frac{1}{2}\frac{d^2}{dx^2} - 2e^{-2\frac{x}{h}} - \frac{\mu^2}{2}\right)\psi_{h,\mu} \overset{h\to 0}{\longrightarrow} h_\mu^{-1}\left(\frac{1}{2}\frac{d^2}{dx^2} - \frac{\mu^2}{2}\right)h_\mu$$

Geometric construction

Such process appears in a curved version of the Hermitian Brownian motion. Consider, a left-invariant SDE on the lower triangular 2×2 matrices driven by W:

$$dB_t(W^{(\mu)}) = B_t(W^{(\mu)}) \circ \begin{pmatrix} dW_t^{(\mu)} & 0 \\ 2dt & -dW_t^{(\mu)} \end{pmatrix}$$

where $\circ$ stands for the Stratononich integral.

Its solution is:

$$\begin{aligned} B_t(W^{(\mu)}) &= \begin{pmatrix} e^{W_t^{(\mu)}} & 0 \\ e^{W_t^{(\mu)}} \int_0^t 2e^{-2W_s^{(\mu)}} \, ds & e^{-W_t^{(\mu)}} \end{pmatrix} \\ &= \begin{pmatrix} e^{W_t^{(\mu)}} & 0 \\ 2e^{\Lambda_t^{(\mu)}} & e^{-W_t^{(\mu)}} \end{pmatrix} \end{aligned}$$

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Consider the Weyl chamber $C = \{x \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_n\}$ and let $W^{(\mu)}$ be a BM with drift $\mu \in C$ killed upon touching ∂C.

- Infinitesimal generator with Dirichlet boundary conditions:

$$\mathcal{L}^{(\mu)} = \frac{1}{2}\Delta + \langle \mu, \nabla \rangle$$

- Special harmonic function for $\mathcal{L}^{(\mu)}$:

$$\mathbb{P}_x \left(W^{(\mu)} \text{survives} \right)$$

Analytic construction (2)

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- Special harmonic function for $\frac{1}{2}\Delta - \frac{1}{2}\|\mu\|^2$:

$$h_\mu(x) = \frac{e^{\langle \mu, x \rangle}}{\prod_{i<j}(\mu_i - \mu_j)} \mathbb{P}_x \left(W^{(\mu)} \text{survives} \right) \overset{\text{thm}}{=} \frac{\det\left(e^{\mu_i x_j}\right)_{i,j=1}^n}{\prod_{i<j}(\mu_i - \mu_j)}$$

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  Notice that we have analytic extension to $\mu \in \mathbb{C}^n$ and symmetry in the variables $(\mu_1, \ldots, \mu_n)$.

- Process conditioned to survive has generator:

$$\mathcal{G}^{(\mu)} = h_\mu(x)^{-1} \left(\frac{1}{2}\Delta - \frac{1}{2}\|\mu\|^2 \right) h_\mu(x) = \frac{1}{2}\Delta + \langle \nabla \log h_\mu, \nabla \rangle$$

As $\mu \to 0$, $h_\mu(x) \to \Delta(x) := \prod_{i<j}(x_i - x_j)$ (Not obvious!). And $\mathcal{G}^{(\mu=0)}$ is the generator of Dyson's Brownian motion.

Geometric construction via RMT

Consider the $n \times n$ Hermitian Brownian motion - marginally distributed as $\sqrt{t}GUE$:

$$GUE_t := \begin{pmatrix} B_t^{11} & B_t^{12} + i\widetilde{B}_t^{12} & \dots & B_t^{1n} + i\widetilde{B}_t^{1n} \\ B_t^{12} - i\widetilde{B}_t^{12} & B_t^{22} & \dots & B_t^{2n} + i\widetilde{B}_t^{2n} \\ \dots & \dots & \dots & \dots \\ B_t^{1n} - i\widetilde{B}_t^{1n} & B_t^{2n} + i\widetilde{B}_t^{2n} & \dots & B_t^{nn} \end{pmatrix}$$

Its spectrum $\{\Lambda_t^1 > \Lambda_t^2 > \cdots > \Lambda_t^n\}$ is a Markovian diffusion called Dyson's Brownian motion with generator:

$$\mathcal{G} = \frac{1}{2}\Delta + \langle \nabla \log \Delta, \nabla \rangle = \frac{1}{2}\Delta + \sum_{i<j} \frac{\partial_{\Lambda_i}}{\Lambda_i - \Lambda_j}$$

Moreover (GUE density):

$$\mathbb{P}\left(\Lambda_t \in dx\right) = \frac{1}{Z_n}\Delta(x)^2 e^{-\frac{\|x\|^2}{2t}}$$

Via a Pitman-type construction

Partially due to O'Connell-Yor for type A, and to Biane, Bougerol and O'Connell for general Lie type. In type A:

- There is a (deterministic) Pitman transform that folds paths in $\mathbb{R}^n$ into the cone C:

$$\mathcal{P}_{w_0} : \mathcal{C}_0\left([0,t], \mathbb{R}^n\right) \to \mathcal{C}_0\left([0,t], C\right)$$

- Coincides with the original Pitman transform for $n = 2$.

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- Coincides with the original Pitman transform for $n = 2$.
- Such transform is of representation-theoretic significance: It is the highest weight transform in the continuous Littelmann path model.
- If $W^{(\mu)}$ is a Brownian motion in $\mathbb{R}^n$ with drift μ then:

$$\left(\mathcal{P}_{w_0}\left(W^{(\mu)}\right)(t); t \geq 0\right)$$

  is the Markovian diffusion given by Brownian motion conditioned to remain in C - as in the "Analytic" construction.
  $\rightsquigarrow$ "Algebraic" construction of non-intersecting particles.

Application to Last Passage Percolation

(Blackboard explanation of LPP in an $n \times M$ box)

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When expliciting the first coordinate of the Pitman transform on $\mathbb{R}^n$:

$$\left(\mathcal{P}_{w_0} W\right)_1(t) = \sup_{0=t_0<t_1<\cdots<t_n=t} \sum_{i=1}^{n} W^i\left(t_i, t_{i-1}\right)$$

which is interpreted as a semi-discrete LPP obtained from the diffusive rescaling as $M \to \infty$.
Since $\mathcal{P}_{w_0} W$ is distributed as Dyson's Brownian motion, the above quantity is distributed as $\sqrt{t}$ times the largest eigenvalue of a GUE matrix (Baryshnikov and Tracy-Widom).

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Random matrix theory gives the weak convergence:

$$\frac{\left(\mathcal{P}_{w_0} W\right)_1(t) - 2\sqrt{tn}}{\sqrt{t} n^{\frac{1}{6}}} \overset{n\to\infty}{\longrightarrow} TW_2$$

Sommaire

Analytic construction (1)

Following naively the one dimensional logic, we will add the exponential potential for each wall in the Weyl chamber

$$C = \{x \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_n\}$$

hence the Toda potential

$$V(x) = \sum_{i=1}^{n-1} e^{-(x_{i+1}-x_i)}$$

- Infinitesimal generator of $W^{(\mu)}$ "slowly killed" BM:

$$\mathcal{L}^{(\mu)} = \frac{1}{2}\Delta + \langle \mu, \nabla \rangle - V(x)$$

- Special harmonic function for $\mathcal{L}^{(\mu)}$:

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Analytic construction (2)

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- Infinitesimal generator of $W^{(\mu)}$ "slowly killed" BM:

$$\mathcal{L}^{(\mu)} = \frac{1}{2}\Delta + \langle \mu, \nabla \rangle - V(x) = e^{-\langle \mu, x \rangle} \left(\frac{1}{2}\Delta - V(x) - \frac{1}{2}\|\mu\|^2 \right) e^{\langle \mu, x \rangle}$$

- Special harmonic function for $\frac{1}{2}\Delta - V(x) - \frac{1}{2}\|\mu\|^2$:

$$\psi_\mu(x) = \prod_{i<j} \Gamma(\mu_i - \mu_j) e^{\langle \mu, x \rangle} \mathbb{P}_x \left(W^{(\mu)} \text{survives} \right)$$

  Following Jacquet, this is the Archimedean Whittaker function.

- Process conditioned to survive is the Whittaker process. Generator:

$$\mathcal{G}^{(\mu)} = \psi_\mu(x)^{-1} \left(\frac{1}{2}\Delta - \frac{1}{2}\|\mu\|^2 \right) \psi_\mu(x) = \frac{1}{2}\Delta + \langle \nabla \log \psi_\mu, \nabla \rangle$$

Geometric construction - "Hypoelliptic BM on a lower triangular matrices"

Let $W^{(\mu)}$ be a Brownian motion with drift μ on $\mathbb{R}^n$. For notational reasons, we drop the superscript (μ) and put indices as exponents. Consider the SDE on lower triangular matrices:

$$dB_t(W^{(\mu)}) = B_t(W^{(\mu)}) \circ \begin{pmatrix} dW_t^1 & 0 & 0 & \cdots & 0 \\ dt & dW_t^2 & 0 & \ddots & \vdots \\ 0 & dt & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & dW_t^{n-1} & 0 \\ 0 & \cdots & 0 & dt & dW_t^n \end{pmatrix}$$

and its solution $B_t(W^{(\mu)})$ is given by:

$$\begin{pmatrix} e^{W_t^1} & 0 & 0 & \cdots \\ e^{W_t^1} \int_0^t e^{W_s^2 - W_s^1} ds & e^{W_t^2} & 0 & \cdots \\ e^{W_t^1} \int_0^t e^{W_{s_1}^2 - W_{s_1}^1} ds_1 \int_0^{s_1} e^{W_{s_2}^2 - W_{s_2}^1} ds_2 & e^{W_t^2} \int_0^t e^{W_s^3 - W_s^2} ds & e^{W_t^3} & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Via a Pitman-type construction

- There is a geometric Pitman transform:

$$\mathcal{T}_{w_0} : \mathcal{C}_0\left([0,t], \mathbb{R}^n\right) \to \mathcal{C}_0\left([0,t], \mathbb{R}^n\right)$$

which degenerates to $\mathcal{P}_{w_0} = \lim_{h \to 0} h \mathcal{T}_{w0} h^{-1}$. In fact:

$$\left(\mathcal{T}_{w_0} W\right)_k (t) = \log \det \left(B_t \left(W\right)_{i=n,\ldots,n-k+1}^{j=1,\ldots,k} \right)$$

- Such transform is of representation-theoretic significance: It is the highest weight transform in the geometric Littelmann path model (constructed in chapter 4 of thesis).

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- (O'Connell 2009 in type A; chapter 6 of thesis for general Lie type)
  If $W^{(\mu)}$ is a Brownian motion in $\mathbb{R}^n$ with drift μ then:

$$\left(\mathcal{T}_{w_0}\left(W^{(\mu)}\right)(t); t \geq 0\right)$$

  is the Whittaker process - as in the "Analytic" construction.
  $\rightsquigarrow$ "Algebraic" construction of *weakly* non-intersecting particles.

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Theorem (Borodin-Corwin-Ferrari)

There is a $\beta^* > 0$ and constants a_t b_t such that for all $0 \leq \beta < \beta^*$:

$$\frac{\frac{\beta}{n}\left(\mathcal{T}_{w_0} \frac{nW}{\beta}\right)_1(t) - a_t \sqrt{n}}{b_t n^{\frac{1}{6}}} \stackrel{n\to\infty}{\longrightarrow} TW_2$$

Sommaire

Conclusion

What we did:

- We presented the Whittaker process as n particles slowly killed with an exponential potential and conditioned to survive.
- It degenerates via rescaling to Dyson's Brownian motion, hence expected random-matrix behaviors.
- It has a representation-theoretic (or Pitman-type) construction.