Pitman's theorem, curvature and quantum SL_2

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Statement (Discrete version)

Theorem (Pitman (1975))

Let $(X_t; t \in \mathbb{N})$ be a standard simple random walk on $\mathbb{Z}$. Then:

$$\Lambda_t = X_t - 2 \inf_{0 \le s \le t} X_s$$

is a Markov chain on $\mathbb{N}$ with transitions:

$$Q(\lambda, \lambda+1) = \frac{1}{2}\frac{\lambda+2}{\lambda+1} \quad Q(\lambda, \lambda-1) = \frac{1}{2}\frac{\lambda}{\lambda+1}$$

Comments

- Strange as $-\inf_{0 \le s \le t} X_s$ is a typical example of non-Markovian behavior.
- Strong rigidity: only 2 and 1 work - and 0 obviously.
- Relationship to the representation theory of SL_2: $2 = \alpha_1(\alpha_1^\vee)$ and

$$V(\lambda) \otimes \mathbb{C}^2 = V(\lambda+1) \oplus V(\lambda-1) \ .$$

Statement (Continuous version)

Via Donsker's invariance principle, Brownian motion is nothing but a very long simple random walk.

Theorem (Pitman (1975))

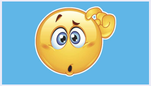
Let $(X_t; t \in \mathbb{R})$ be a Brownian motion on $\mathbb{R}$. Then:

$$\Lambda_t = X_t - 2 \inf_{0 \leq s \leq t} X_s$$

is Markov process. In law, it is a Bessel 3 process i.e it has the same statistical properties as

$$\left(\Lambda_t^0 := \sqrt{X_t^2 + Y_t^2 + Z_t^2}; t \geq 0 \right) ,$$

where (X_t, Y_t, Z_t) are three independent Brownian motions.



Zoology of proofs

There are many proofs:

- Pitman's original proof (1975) via combinatorial counting arguments.
- The Brownian proof of Rogers and Pitman (1981) using intertwinings of Markov kernels.
- The proof of Bougerol-Jeulin (2000) via curvature deformation inside the symmetric space $SL_2(\mathbb{C})/SU_2$. If r is the scalar curvature:

$$r : 0 \longleftrightarrow \infty$$

- After Biane worked on quantum walks with $\mathcal{U}_{q=1}(\mathfrak{sl}_2)$ (90s), Biane-Bougerol-O'Connell recognized in Pitman's theorem the (crystalline) rep. theory of $\mathcal{U}_{q=0}(\mathfrak{sl}_2)$ (2005).

$$q : 1 \longleftrightarrow 0$$

$\rightsquigarrow$ I would like to joint these two last proofs into a single global picture.

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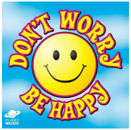
Setting

- Consider $G = SL_2(\mathbb{C})$, $K = SU_2$. The associated hyperbolic space is $\mathbb{H}^3 = G/K \approx NA$ via Gram-Schmidt.
- Rescale the Lie bracket of NA by $r \rightsquigarrow$ Rescales the curvature tensor by r^2.
- Bougerol and Jeulin consider $(g_t^r; t \geq 0)$, "a Brownian motion" on G/K with curvature $r > 0$. It is obtained by solving:

$$dg_t^r = \begin{pmatrix} \frac{1}{2} r dX_t & 0 \\ r(dY_t + iZ_t) & -\frac{1}{2} r dX_t \end{pmatrix} \circ g_t^r \ ,$$



  where (X, Y, Z) are independent Brownian motions, each on $\mathbb{R}$.
- Solving the differential equation yields:

$$g_t^r = \begin{pmatrix} e^{\frac{1}{2} r X_t} & 0 \\ r e^{\frac{1}{2} r X_t} \int_0^t e^{-r X_s} d(Y_s + iZ_s) & e^{-\frac{1}{2} r X_t} \end{pmatrix}$$

The result of Bougerol-Jeulin

Let Λ_t^r be the radial part of $g_t^r \in K \begin{pmatrix} e^{\frac{1}{2}\Lambda_t^r} & 0 \\ 0 & e^{-\frac{1}{2}\Lambda_t^r} \end{pmatrix} K$. With $\mathrm{Argch} := \cosh^{-1}_{|[0,\infty)}$, a simple computation shows that:

$$\Lambda_t^r = \frac{1}{r}\mathrm{Argch}\left[2r^2\left|e^{\frac{1}{2}rX_t}\int_0^t e^{-2\frac{1}{2}rX_s}(dY_s + idZ_s)\right|^2 + \cosh(rX_t)\right] .$$

Theorem

- *A norm process on* $\mathbb{R}^3$ *(Bessel 3):*

$$\Lambda_t^{r=0} = \sqrt{X_t^2 + Y_t^2 + Z_t^2} ,$$

- *The Pitman transform of* X*:*

$$\Lambda_t^{r=\infty} = X_t - 2 \inf_{0\leq s\leq t} X_s .$$

- *The distribution of* Λ^r *does not depend on* r*.*

Important: The Pitman transform shows up in infinite curvature, the norm of $\mathbb{R}^3$ in flat curvature.

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Some notations

Lie algebras / Invariant differential operators of order 1:

- $\mathfrak{su}_2 = T_e SU_2 = Span_{\mathbb{R}}(X, Y, Z)$
- (X, Y, Z) basis of anti-Hermitian matrices:

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \; ; Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \; ; Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; .$$

- $\mathfrak{su}_2$ is the compact form of $\mathfrak{sl}_2 = \mathbb{C} \otimes \mathfrak{su}_2 = T_e SL_2(\mathbb{C}) = Span_{\mathbb{C}}(H, E, F)$ where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; ; E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; ; F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \; .$$

Universal enveloping algebra / Invariant differential operators:

$$\mathcal{U}(\mathfrak{sl}_2) := T(\mathfrak{sl}_2) / \{x \otimes y - y \otimes x - [x, y]\} \; .$$

Biane's quantum walks

Imagine a particle moving in a non-commutative phase space. The algebra of observables is chosen to be $\mathcal{U}(\mathfrak{s}l_2)$. Not commutative in the spirit of quantum mechanics.

- At every time t, consider the representation $(\mathbb{C}^2)^{\otimes t} \equiv$ "Hilbert space of wave-functions".
- (X_t, Y_t, Z_t) are measuring operators in that representation.
- $\Lambda_t := \sqrt{\frac{1}{2} + X_t^2 + Y_t^2 + Z_t^2}$ Casimir element which acts as a constant on irreducible components of $(\mathbb{C}^2)^{\otimes t}$. $\rightsquigarrow$ "Measures which quantum sphere/irrep we are at, at time t".

  Important: Λ is nothing but "the Euclidean norm inside of quantified $\mathbb{R}^3$"
- The dynamic of Λ_t follows the Clebsch-Gordan rule:

$$V(\lambda) \otimes V(1) \approx V(\lambda+1) \oplus V(\lambda-1) \ .$$

Problem: $(\Lambda_t, X_t; t \in \mathbb{N})$ have separately the same dynamics as in Pitman's theorem. But there are not related via the Pitman transform. (Biane 90s)

Where is the Pitman transform?

- The Jimbo-Drinfeld quantum group is generally defined as the algebra:

$$\mathcal{U}_q(\mathfrak{sl}_2) := \langle K = q^H, K^{-1}, E, F \rangle / \mathcal{R} ,$$

where $q = e^h$, and $\mathcal{R}$ is the two-sided ideal of relations:

$$KEK^{-1} = q^2 E, \; KFK^{-1} = q^{-2} F, \; EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

- The relations $\mathcal{R}$ deform the relations induced by the classical commutator $[\cdot, \cdot]$ of $\mathfrak{sl}_2$:

$$\text{``}\mathcal{U}(\mathfrak{sl}_2) = \lim_{q \to 1} \mathcal{U}_q(\mathfrak{sl}_2)\text{''}$$

Message

We already knew that $\mathcal{U}_q(\mathfrak{sl}_2)$ is not a group. The goal of this talk is to show that it not a quantum deformation of $\mathcal{U}(\mathfrak{sl}_2)$ either!

Where is the Pitman transform? (II)

We need to consider $\mathcal{U}_{q=0}(\mathfrak{s}l_2)$ as the Pitman transform

$$\mathcal{P} : \pi \mapsto \pi(t) - 2 \inf_{0 \leq s \leq t} \pi(s)$$

has a special interpretation of rep. theory of $\mathcal{U}_q(\mathfrak{s}l_2)$ as $q \to 0$.

For the sake of simplicity: let $V_q(1) = \mathbb{C}^2$ be the standard representation of $\mathcal{U}_q(\mathfrak{sl}_2)$.

- Observing the simple random walk X_t corresponds to following the dynamic of a weight vector inside $V_q(1)^{\otimes t}$.
- At $q = 0$, there is a deterministic relation between X_t and the value of the Casimir.
- This relation is exactly the Pitman transform. Transition are indeed given by the Clebsch-Gordan rule:

$$V(\lambda) \otimes V(1) \approx V(\lambda + 1) \oplus V(\lambda - 1)$$

as structure constants do not change with q!

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The question

Message

The Pitman transform is understood to be intimately related to crystals, which appear at the representation theory of $\mathcal{U}_q(\mathfrak{sl}_2)$ at $q = 0$. Why would there be crystal-like phenomenons by taking curvature $r \to \infty$ in a symmetric space $SL_2(\mathbb{C})/SU_2 \approx NA$?

$\rightsquigarrow$ Single global picture? Interplay between both the representation of $\mathcal{U}_q(\mathfrak{sl}_2)$, as $q > 0$ varies, and the geometry of the symmetric space $SL_2(\mathbb{C})/SU_2$ with varying curvatures $r > 0$.

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A commutative diagram

Proposition (RC, F. Chapon)

Set $q = e^{-r}$. There exist a presentation of the Jimbo-Drinfeld quantum group $\mathcal{U}_q^\hbar(\mathfrak{sl}_2)$ such that the following diagram (between Hopf algebras) commutes:

$$\begin{array}{ccc} \mathcal{U}_q^\hbar(\mathfrak{sl}_2) & \xrightarrow{\hbar \to 0} & \mathbb{C}\left[(SU_2)_r^*\right] \\ \downarrow{\scriptstyle r \to 0} & & \downarrow{\scriptstyle r \to 0} \\ \mathcal{U}^\hbar(\mathfrak{sl}_2) & \xrightarrow{\hbar \to 0} & \mathbb{C}\left[\mathfrak{su}_2^*\right] \end{array}$$

Here $(SU_2)_r^*$ is the Poisson-Lie group dual to SU_2 and with curvature r:

$$(SU_2)_r^* := \left\{ \begin{pmatrix} \frac{1}{2}a & 0 \\ b & -\frac{1}{2}a \end{pmatrix} \mid a \in \mathbb{R},\ b \in \mathbb{C} \right\} ,$$

$$\forall X, Y \in (SU_2)_r^*,\ X *_r Y := \frac{1}{r} \log\left(e^{rX} e^{rY}\right) .$$

Fact: The curvature tensor of this Lie group is

$$R(X, Y, Z) = r^2 [X, [Y, Z]] .$$

An implementation of the orbit method

We also have a convergence of quantum observables to classical observables for all $r > 0$. In fact, as vector spaces:

$$\mathcal{U}_q^{\hbar}(\mathfrak{s}l_2) \approx \mathbb{C}\left[(SU_2)_r^*\right][[\hbar]] \ .$$

and:

$$\mathbb{C}\left[(SU_2)_r^*\right] \approx \mathcal{U}_q^{\hbar}(\mathfrak{s}l_2) \quad \text{mod } \hbar.$$

Theorem (RC, F. Chapon)

Let $\pi :=$ mod $\hbar$ *be the quotient map,* $\mathcal{O}_r(\lambda)$ *"the curved orbit" of* $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ *in* $(SU_2)_r^*$*. Then for all* $F \in \mathcal{U}_q^{\hbar}(\mathfrak{s}l_2)$*:*

$$\begin{aligned} \mathrm{Tr}_{V_q(\lambda/\hbar)}(F) &\stackrel{\hbar\to 0}{\longrightarrow} \int_{\mathcal{O}_r(\lambda)} \pi(F)(p)\omega(dp) \ , \\ \mathrm{Tr}_{V_q(\lambda/\hbar)\otimes V_q(\mu/\hbar)}(F) &\stackrel{\hbar\to 0}{\longrightarrow} \int_{\mathcal{O}_r(\lambda)\times\mathcal{O}_r(\mu)} \pi(F)(p *_r q)\omega(dp)\omega(dq) \ . \end{aligned}$$

Random walks / Convolution dynamics

Finally, Chapon and I have built tensor/convolution dynamics such that

$$\begin{array}{ccc}
\begin{array}{c}\Lambda_n \\ x_n \\ \text{Quantum random walks} \\ \text{on } \mathcal{U}_q^{\hbar}(\mathfrak{s}l_2)\end{array} & \xrightarrow{\hbar\to 0} & \begin{array}{c}\Lambda_t^r \\ \begin{pmatrix} e^{\frac{1}{2}rX_t} & 0 \\ re^{\frac{1}{2}rX_t}\int_0^t e^{-rX_s}d(Y_s+iZ_s) & e^{-\frac{1}{2}rX_t}\end{pmatrix} \\ \text{The convolution dynamic of} \\ \text{Bougerol-Jeulin}\end{array} \\
\downarrow{\scriptstyle r\to 0} & & \downarrow{\scriptstyle r\to 0} \\
\begin{array}{c}\Lambda_n = \sqrt{\frac{1}{2}+X_n^2+Y_n^2+Z_n^2} \\ (X_n, Y_n, Z_n) \\ \text{Biane's quantum random walks} \\ \text{on } \mathcal{U}^{\hbar}(\mathfrak{s}l_2)\end{array} & \xrightarrow{\hbar\to 0} & \begin{array}{c}\Lambda_t = \sqrt{X_t^2+Y_t^2+Z_t^2} \\ \begin{pmatrix} \frac{1}{2}X_t & 0 \\ Y_t+iZ_t & -\frac{1}{2}X_t\end{pmatrix} \\ \text{Convolution dynamic/} \\ \text{Flat BM on } \mathfrak{s}u_2^*\end{array}
\end{array}$$

the above convergences are in law.

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Summary

Starting point:

- Pitman's theorem (1975) is a result in probability theory with the rep. theory of "$\mathcal{U}_{q=0}(\mathfrak{sl}_2)$" (Crystals) lurking in the background.
- There is a proof of Bougerol-Jeulin (2000) by taking a curvature parameter r to $r = \infty$.

Result:

- There is a presentation of the quantum group $\mathcal{U}_q^{\hbar}(\mathfrak{sl}_2)$, which isolates the role of the Planck constant $\hbar$ and that of the parameter q.
- Its semi-classical limit is Poisson-Lie group with curvature r.
- Since $q = e^{-r}$, we have:

$$\text{Crystals } (q = 0) \longleftrightarrow \text{Infinite curvature } (r = \infty).$$

Message

$\mathcal{U}_q(\mathfrak{sl}_2)$ is quantum because $\mathcal{U}(\mathfrak{sl}_2)$ is already quantum (really)! Not a quantum deformation, but a deformation via curvature.

Progress

- (Done) The SL_2 case. Draft on the arxiv: 1st of April.
- (Close future) Higher rank case, finite type - substantial progress but not worked out completely.
- (Further down the road) Large span of litterature relating classical integrable systems and crystals (Gelfand-Tstetlin patterns by Guillemin-Sternberg, Harada, Kaveh). I would like to relate this work to integrable systems. Ingredient: Natural/explicit Ginzburg-Weinstein isomorphisms, in the spirit of the work of Alekseev-Meinrenken.

End

Thank you for your attention!