

André's reflection principle and functional equations in number theory

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- 2 Higher rank (Non-Archimedean)
- 3 The probabilistic proof of the Shintani-Casselman-Shalika formula
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Introduction

Rank 1 example: Let us explain the relationship between

- André's reflection principle:

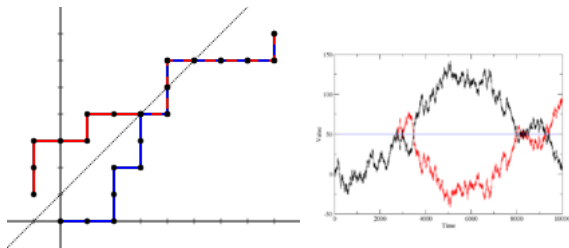


Figure: André's reflection principle (src:Wikipedia)

- The functional equation in number theory

$$\forall s \in \mathbb{C}, \Lambda(s) = \Lambda(1 - s)$$

where

$$\begin{cases} \Lambda(s) &= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \\ \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \end{cases}$$

(Variant of) André's reflection principle

Let $z > 1$ and $W^{(z)}$ be SRW such that:

$$\forall t, \mathbb{P} \left(W_{t+1}^{(z)} - W_t^{(z)} = 1 \right) = 1 - \mathbb{P} \left(W_{t+1}^{(z)} - W_t^{(z)} = -1 \right) = \frac{z}{z + z^{-1}} > \frac{1}{2}$$

and **killed** upon exiting \mathbb{N} .

Lemma

$$\mathbb{P}_x \left(W^{(z)} \text{ survives} \right) = 1 - z^{-2(x+1)}$$

Corollary (Probabilistic representation of SL_2 -characters)

$$s_{(x)} \left(z, z^{-1} \right) := \frac{z^{x+1} - z^{-(x+1)}}{z - z^{-1}} = \frac{z^{x+1}}{z - z^{-1}} \mathbb{P}_x \left(W^{(z)} \text{ survives} \right)$$

where s_λ is the Schur function associated to the shape λ .

(Variant of) André's reflection principle - Geometric case

Let $\mu > 0$ and $W^{(\mu)}$ be the sub-Markovian process on \mathbb{R} with generator:

$$\mathcal{L}^{(\mu)} = \frac{1}{2}\partial_x^2 + \mu\partial_x - 2e^{-2x}$$

i.e Brownian motion with drift μ and **killing measure** by a potential $V(x) = 2e^{-2x}$.

Proposition (Probabilistic representation of the K -Bessel/Whittaker function)

If

$$K_\mu(y) = \int_0^\infty \frac{dt}{t} t^\mu e^{-y(t+t^{-1})}$$

then

$$K_\mu(e^{-2x}) = \Gamma(\mu) e^{\mu x} \mathbb{P}_x \left(W^{(\mu)} \text{ survives} \right)$$

Remark: $K_\mu(e^{-2x})$ plays the role of character associated to a SL_2 -geometric crystals. In all cases, symmetric and analytic extension of survival probabilities.

Functional equation I: The higher peak of Eisenstein series

$\mathbb{H} = \{\Im z > 0\}$; $g \cdot z = \frac{az+b}{cz+d}$ usual action of SL_2 on \mathbb{H} .

Non-holomorphic Eisenstein series defined for $\Re(s) > 2$ and $z = x + iy \in \mathbb{H}$:

$$E(z; s) = \frac{1}{2} \sum_{(m,n) \neq 0} \frac{y^s}{(mz + n)^s}$$

$$E^*(z; s) = \pi^{-s} \Gamma(s) \zeta(2s) E(z; s) \quad (\text{Normalization})$$

Properties:

- Automorphic: $\forall g \in SL_2(\mathbb{Z}), E^*(g \cdot z; s) = E^*(z; s)$.
- 1-periodicity in $x \rightsquigarrow$ Expansion in **Fourier-Whittaker** coefficients:

$$E^*(z = x + iy; s) = \sum_{n \in \mathbb{Z}} A_n(y; s) e^{i2\pi xn} .$$

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$$A_n(y; s) = \begin{cases} * & \text{if } n = 0 \\ y^{\frac{1}{2}} \sigma_{s-\frac{1}{2}}(n) K_{s-\frac{1}{2}}(\pi|n|y) & \text{if } n \neq 0 \end{cases}$$

where

$$K_s(y) = \int_0^\infty \frac{dt}{t} t^s e^{-y(t+t^{-1})} \quad (\text{Bessel K-function})$$

$$\sigma_s(n) = \sum_{d|n} \left(\frac{n}{d^2}\right)^s \quad (\text{Normalized divisor function})$$

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Symmetries of A_1 -type:

$$\forall s \in \mathbb{C}, K_s(y) = K_{-s}(y), \sigma_s(n) = \sigma_{-s}(n)$$

$$\rightsquigarrow (\text{Functional equation}) \quad E^*(s; z) - E^*(1-s; z) = 0$$

Relationship

An important remark: σ_s is weakly multiplicative

$$\sigma_s(n) = \prod_{p|n} \sigma_s(p^{\nu_p(n)})$$
$$\sigma_s(p^k) = s_k(p^s, p^{-s})$$

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As such, the functional equations:

$$E^*(s; z) = E^*(1 - s; z); \quad \Lambda(s) = \Lambda(1 - s) \quad (\text{Constant term } *)$$

are implied by the A_1 -symmetry in:

$K_s(\cdot)$ The Archimedean Whittaker function

$s_k(p^s, p^{-s})$ The non-Archimedean Whittaker function at the prime p

Taking the problem backwards: What if these functions were expressed as survival probabilities from the start? Then

Functional equations + Analytic continuation \Leftrightarrow André's reflection principle

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Notations

We consider the case of GL_n for concreteness and not other (split) Chevalley-Steinberg groups.

- $\mathcal{K} = \mathbb{F}_q((T))$: field of Laurent series in the formal variable T with coefficients in the finite field \mathbb{F}_q . If $x \in \mathcal{K}$, then $\text{val}(x)$ is the index of the first non-zero monomial. If $k = \text{val}(x)$:

$$x = a_k T^k + a_{k+1} T^{k+1} + \dots$$

with $a_k \neq 0$. The ring of integers is made of elements of non-negative valuation and denoted $\mathcal{O} = \mathbb{F}_q[[T]]$.

- $G = GL_n(\mathcal{K})$: group of \mathcal{K} -points.
- $K = GL_n(\mathcal{O})$: maximal compact (open) subgroup.
- $A = \{ T^{-\mu} = \text{diag}(T^{\mu_1}, T^{\mu_2}, \dots, T^{\mu_n}) \mid \mu \in \mathbb{Z}^n \} \approx \mathbb{Z}^n$
- $A_+ = \{ T^{-\lambda} \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \} \approx \mathbb{N}^n$
- $N(\mathcal{K})$ is the unipotent subgroup of lower triangular matrices.

Facts:

$$G = NAK \quad (\text{Iwasawa or "Gram-Schmidt"})$$

$$G = KA_+K \quad (\text{Cartan or "Polar"})$$

Number theory facts

- (Langlands, Shahidi) Eisenstein series generalize from $\mathbb{H} = SL_2(\mathbb{R})/SO_2(\mathbb{R})$ to other Lie groups
 \rightsquigarrow Functions on G/K , general symmetric spaces.

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$$\Phi_z(g) := e^{\langle z, \mu \rangle} \quad \text{when } g \in NT^\mu K$$

- (Jacquet's thesis, 1967) Fourier-Whittaker coefficients of such generalized Eisenstein series are given by the **Whittaker function** \mathcal{W}_z . Let $\varphi_N : N \rightarrow (\mathbb{C}^*, \times)$ be a character trivial on integer points and w_0 is the permutation matrix $(n \dots 321)$.

$$\mathcal{W}_z(g) := \int_N \Phi_z(w_0 n g) \varphi_N(n)^{-1} dn$$

is convergent for $z \in \mathbb{C}$.

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Important message: As in rank 1, the functional equations for Eisenstein series is implied by the $W = S_n$ -symmetry after a normalization $\widetilde{\mathcal{W}}_z$

$$\forall \sigma \in S_n, \forall g \in G, \forall z \in \mathbb{R}^n, \widetilde{\mathcal{W}}_{\sigma z}(g) = \widetilde{\mathcal{W}}_z(g)$$

The Shintani-Casselman-Shalika formula

Recall:

$$\mathcal{W}_z(g) = \int_N \Phi_z(w_0ng) \varphi_N(n)^{-1} dn$$

The Whittaker function is essentially a function on $A \approx \mathbb{Z}^n$:

$$\forall (n, T^{-\lambda}, k) \in N \times A \times K, \mathcal{W}_z(nT^{-\lambda}k) = \varphi_N(n)\mathcal{W}_z(T^{-\lambda})$$

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Theorem (Shintani 76 for GL_n , Casselman-Shalika 81 for G general)

$$\mathcal{W}_z(T^{-\lambda}) = \begin{cases} e^{\frac{1}{2} \sum i \lambda_i} \prod_{i < j} (1 - q^{-1} e^{z_i - z_j}) s_\lambda(z) & \text{if } \lambda \text{ dominant,} \\ 0 & \text{otherwise.} \end{cases}$$

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Step 1: A crucial remark

There is a seemingly innocent remark in papers of Shintani and Lafforgue:

Remark (Shintani, L. Lafforgue)

If \mathcal{H} is a K bi-invariant measure on G , then:

$$\mathcal{W}_z \star \mathcal{H} = S(\mathcal{H})(z)\mathcal{W}_z, \quad \text{for a certain } S(\mathcal{H})(z) \in \mathbb{C}$$

“The Whittaker function is a convolution eigenfunction” \rightsquigarrow Harmonicity for a probabilist!

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Proposition

There exists a left-invariant random walk $(B_t(W^{(z)}); t \geq 0)$ on NA such that

- (Harmonicity) A simple transform of \mathcal{W}_z is harmonic for $(B_t(W^{(z)}); t \geq 0)$.
- $B_t(W^{(z)}) = N_t(W^{(z)}) T^{-W_t^{(z)}}$ and $W^{(z)}$ is the lattice walk on \mathbb{N}^n with:

$$\mathbb{P}\left(W_{t+1}^{(z)} - W_t^{(z)} = e_i\right) = \frac{z_i}{\sum_j z_j}$$

- The N -part of the increments are Haar distributed on $N(\mathcal{O})$.
- (Exit law) $N_t(W^{(z)}) \xrightarrow{t \rightarrow \infty} N_\infty(W^{(z)})$

Step 1: GL_2 example

Consider (W^1, W^2) a simple random walk on \mathbb{N}^2 by steps $(1, 0)$ and $(0, 1)$.

$$W_t^{(z)} = W_t^1 - W_t^2, \quad \text{SRW}$$

Form the random walk on $GL_2(\mathcal{K})$:

$$B_{t+1}(W) = B_t(W) \begin{pmatrix} T^{-(W_{t+1}^1 - W_t^1)} & 0 \\ U_t & T^{-(W_{t+1}^2 - W_t^2)} \end{pmatrix}$$

where $(U_t; t \geq 0)$ are i.i.d Haar distributed on \mathcal{O} . A simple computation gives that:

$$B_t(W) = \begin{pmatrix} T^{-W_t^1} & 0 \\ T^{-W_t^1} \sum_{s=0}^{t-1} T^{-(W_s^2 - W_s^1)} U_s & T^{-W_t^2} \end{pmatrix},$$

hence

$$N_\infty(W^{(z)}) = \begin{pmatrix} 1 & 0 \\ \sum_{s=0}^{\infty} T^{W_s^{(z)}} U_s & 1 \end{pmatrix}.$$

Step 2: Trivial key lemma

Imperfect relation to tropical calculus:

$$\text{val}(x + y) \geq \min(\text{val}(x), \text{val}(y))$$

The perfect marriage of probability and non-Archimedean fields appears in:

Lemma (“The trivial key lemma”)

If U and U' are independent and Haar distributed on \mathcal{O} , then for any two integers a and b in \mathbb{Z} :

$$T^a U + T^b U' \stackrel{\mathcal{L}}{=} T^{\min(a,b)} U$$

In a sense, non-Archimedean Haar random variables are “tropical calculus friendly”. By this lemma:

$$N_\infty(W^{(z)}) = \begin{pmatrix} 1 & 0 \\ \sum_{s=0}^{\infty} T^{W_s^{(z)}} U_s & 1 \end{pmatrix} \stackrel{\mathcal{L}}{=} \begin{pmatrix} 1 & 0 \\ T^{\min_{0 \leq s} W_s^{(z)}} U & 1 \end{pmatrix}$$

and U Haar distributed on \mathcal{O} .

Step 3: Poisson formula

$$\begin{aligned}\mathcal{W}_z(T^{-\lambda}) &= e^{\frac{1}{2} \sum i \lambda_i} \prod_{i < j} \left(1 - q^{-1} e^{z_i - z_j}\right) e^{\langle z, \lambda \rangle} \mathbb{E} \left(\varphi_N \left(T^{-\lambda} N_\infty(W^{(z)}) T^\lambda \right) \right) \\ &\sim e^{\langle z, \lambda \rangle} \mathbb{E} \left(\varphi_N \left(T^{-\lambda} N_\infty(W^{(z)}) T^\lambda \right) \right) \\ &\stackrel{\text{Def of } \varphi_N}{=} e^{\langle z, \lambda \rangle} \mathbb{E} \left[\prod_{i=1}^{n-1} \psi \left(T^{\lambda_i - \lambda_{i+1} + \min_{0 \leq s} W_s^i - W_s^{i+1}} U_i \right) \right] \\ &\stackrel{\text{Average } U_i}{=} e^{\langle z, \lambda \rangle} \mathbb{E} \left[\prod_{i=1}^{n-1} \mathbb{1}_{\{\lambda_i - \lambda_{i+1} + \min_{0 \leq s} W_s^i - W_s^{i+1} \geq 0\}} \right] \\ &= e^{\langle z, \lambda \rangle} \mathbb{P}_\lambda \left(W^{(z)} \text{ remains in } C \right)\end{aligned}$$

Shintani-Casselmann-Shalika formula in short

Theorem

Non-Archimedean Whittaker functions are proportional to Schur functions.

Sketch of probabilistic proof.







- Non-Archimedean Whittaker functions are harmonic for certain random walks $(B_t(W^{(z)}), t \geq 0)$ on $GL_n(\mathcal{K})$, driven by a lattice walk $W^{(z)}$ on \mathbb{Z}^n (with drift).
- Poisson formula (not the summation one!): The Whittaker function is an expectation of the random walks' exit law.
- The stationary distribution is made of many sums which can be simplified thanks to the "Trivial key lemma", giving infinimas.
- Integrate out the Haar random variables. Get indicator functions. Whittaker function is proportional to the probability of a lattice walk W staying inside the Weyl chamber.
- The reflection principle gives a determinant. This determinant is the Schur function.



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References

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Acknowledgments

Thank you for your attention!