Chapter 1

Optimal Control of Elliptic Equations

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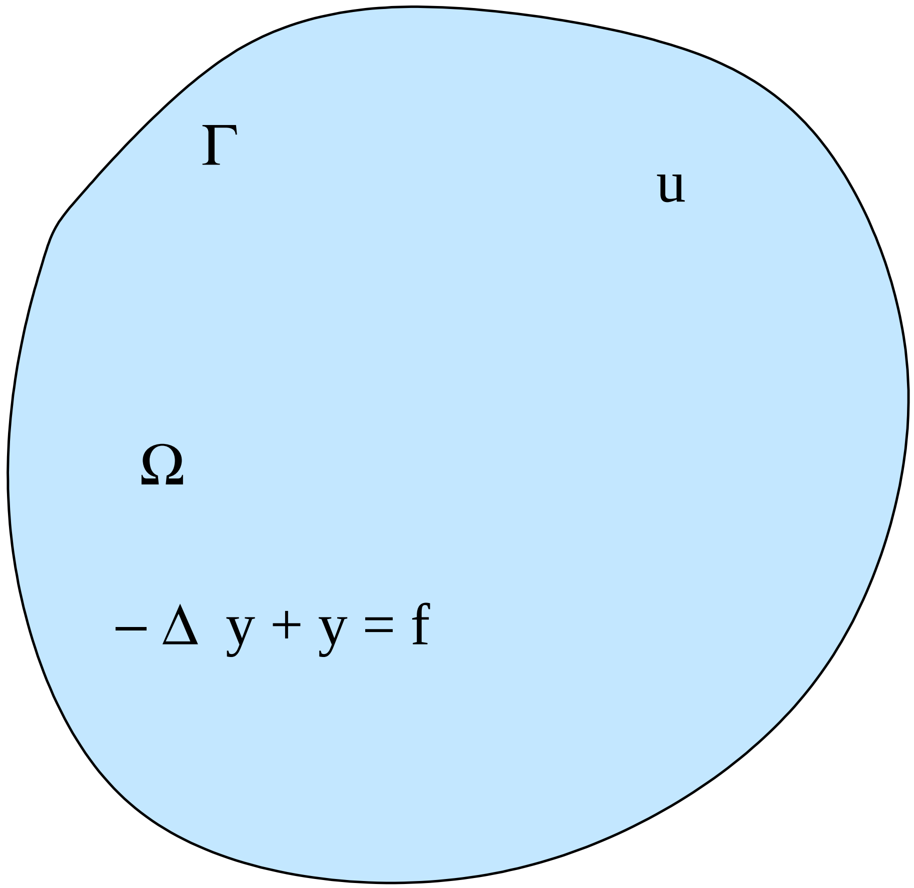
Part 1

The physical problem



- y may be an electrical potential
  and u a current density.
- y may be a temperature distribution
  and u a thermal flux.

Problem. Minimize the distance between y and a given distribution y_d

$$\int_{\Omega} |y - y_d|^2.$$

The consumed energy is

$$\int_{\Gamma} |u|^2.$$

The control problem

Minimize $J(y,u) = \frac{1}{2}\int_\Omega |y - y_d|^2 + \frac{\beta}{2}\int_\Gamma |u|^2$

$-\Delta y + y = f$ in Ω, $\qquad \frac{\partial y}{\partial n} = u$ on Γ.

$u \in L^2(\Gamma), \qquad f \in L^2(\Omega),$

$y_d \in L^2(\Omega), \qquad \beta > 0.$

Sobolev spaces

Prerequisite on Sobolev spaces

$$H^1(\Omega) = \left\{ y \in L^2(\Omega) \mid \frac{\partial y}{\partial x_i} \in L^2(\Omega) \right\}$$

$$\|y\|_{H^1(\Omega)} = \left(\|y\|^2_{L^2(\Omega)} + \Sigma_{i=1}^n \|\partial_{x_i} y\|^2_{L^2(\Omega)} \right)^{1/2},$$

$$H^2(\Omega) = \left\{ y \in H^1(\Omega) \mid \frac{\partial^2 y}{\partial x_i \partial x_j} \in L^2(\Omega) \right\}$$

$$\|y\|_{H^2(\Omega)} = \left(\|y\|^2_{H^1(\Omega)} + \Sigma_{i,j=1}^n \|\partial^2_{x_i\, x_j} y\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Intermediate spaces

$$H^2(\Omega) \subset H^s(\Omega) \subset H^1(\Omega) \subset H^\sigma(\Omega) \subset L^2(\Omega)$$

If $\Omega = \mathbb{R}^N$, $H^s(\Omega)$ can be characterized by Fourier transform.

$$I_\sigma(y) = \int_\Omega \int_\Omega \frac{|y(x) - y(\xi)|^2}{|x - \xi|^{n+2\sigma}} \, dx \, d\xi < \infty$$

$$\|y\|_{H^\sigma(\Omega)} = \Big(\|y\|^2_{L^2(\Omega)} + I_\sigma(y) \Big)^{1/2}, \quad 0 < \sigma < 1,$$

$$\|y\|_{H^s(\Omega)} = \Big(\|y\|^2_{H^1(\Omega)} + \Sigma_{i=1}^n I_{s-[s]}(\partial_{x_i} y) \Big)^{1/2}, \quad 1 < s < 2.$$

Trace theorems

$$H^{1/2}(\Gamma) = \left\{ y \in L^2(\Gamma) \mid |y|_{H^{1/2}(\Gamma)} < \infty \right\}$$

$$|y|_{H^{1/2}(\Gamma)} = \left(\int_\Gamma \int_\Gamma \frac{|y(x) - y(\xi)|^2}{|x - \xi|^{n-1+1}} \, dx \, d\xi \right)^{1/2},$$

$$\|y\|_{H^{1/2}(\Gamma)} = \left(\|y\|^2_{L^2(\Gamma)} + |y|^2_{H^{1/2}(\Gamma)} \right)^{1/2}.$$

$$\gamma_0 \ : \ y \longmapsto y|_\Gamma$$

$$\gamma_0 \ : \ H^1(\Omega) \longmapsto H^{1/2}(\Gamma)$$

$$\gamma_0 \ : \ H^s(\Omega) \longmapsto H^{s-1/2}(\Gamma) \qquad s > 1/2$$

γ_0 is a surjective op. from $H^s(\Omega)$ to $H^{s-1/2}(\Gamma)$.

$$\gamma_1 \ : \ y \longmapsto \frac{\partial y}{\partial n}$$

$$\gamma_1 \ : \ H^2(\Omega) \longmapsto H^{1/2}(\Gamma)$$

$$\gamma_1 \ : \ H^s(\Omega) \longmapsto H^{s-3/2}(\Gamma) \qquad s > 3/2$$

γ_1 is a surjective op. from $H^s(\Omega)$ to $H^{s-3/2}(\Gamma)$.

$$H_0^1(\Omega) = \{y \in H^1(\Omega) \mid \gamma_0 y = 0\},$$

$$H_0^1(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}$$

$$H^{-1}(\Omega) = (H_0^1(\Omega))', \qquad H^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))'.$$

Regularity results

Regularity result. If Ω is of class C^2 and if $f \in L^2(\Omega)$, then the solution y of the equation

$$-\Delta y + y = f \quad \text{in } \Omega, \qquad \frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma,$$

belongs to $H^2(\Omega)$ and

$$\|y\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Interpolation theorems

If

$$L \ : \ H^{1/2}(\Gamma) \longmapsto H^{2}(\Omega)$$

$$L \ : \ H^{-1/2}(\Gamma) \longmapsto H^{1}(\Omega)$$

that is if

$$\|Lu\|_{H^2(\Omega)} \leq C_1 \|u\|_{H^{1/2}(\Gamma)}$$

and

$$\|Lu\|_{H^1(\Omega)} \leq C_1 \|u\|_{H^{-1/2}(\Gamma)},$$

then

$$L \ : \ L^{2}(\Gamma) \longmapsto H^{3/2}(\Omega).$$

State equation

The state equation

$$-\Delta y + y = f \quad \text{in } \Omega, \qquad \frac{\partial y}{\partial n} = u \quad \text{on } \Gamma.$$

$$f \in L^2(\Omega), \qquad u \in L^2(\Gamma).$$

We set

$$a(y,z) = \int_\Omega \nabla y \cdot \nabla z + y\,z, \qquad \ell(z) = \int_\Omega f\,z + \int_\Gamma u\,z.$$

The variational formulation of the state equation is

$$\text{find } y \in H^1(\Omega) \text{ such that}$$

$$a(y,z) = \ell(z) \text{ for all } z \in H^1(\Omega).$$

By the Lax-Milgram theorem, the state equation admits a unique solution y in $H^1(\Omega)$.

Taking $z = y$ in the equation we find

$$\int_\Omega \Big(|\nabla y|^2 + |y|^2 \Big) = \int_\Omega f\, y + \int_\Gamma u\, y$$

$$\leq \|f\|_{L^2(\Omega)} \|y\|_{L^2(\Omega)} + \|u\|_{L^2(\Gamma)} \|y\|_{L^2(\Gamma)}$$

$$\leq \|f\|^2_{L^2(\Omega)} + \frac{1}{4} \|y\|^2_{L^2(\Omega)} + C \|u\|_{L^2(\Gamma)} \|y\|_{H^1(\Omega)}$$

$$\leq \|f\|^2_{L^2(\Omega)} + \frac{1}{4} \|y\|^2_{L^2(\Omega)} + C^2 \|u\|_{L^2(\Gamma)} + \frac{1}{4} \|y\|^2_{H^1(\Omega)}.$$

We obtain

$$\int_\Omega \Big(|\nabla y|^2 + |y|^2 \Big) \leq 2 \|f\|^2_{L^2(\Omega)} + 2C^2 \|u\|_{L^2(\Gamma)}.$$

Conclusion

$$\|y\|_{H^1(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Gamma)} \big).$$

The result can be improved by using

$$\Big| \int_\Gamma u\, y \Big| \leq \|u\|_{H^{-1/2}(\Gamma)} \|y\|_{H^{1/2}(\Gamma)}.$$

We obtain

$$\|y\|_{H^1(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|_{H^{-1/2}(\Gamma)}\big).$$

Regularity when $u \in H^{1/2}(\Gamma)$

If $u \in H^{1/2}(\Gamma)$, since

$$\gamma_1 \ : \ H^2(\Omega) \longmapsto H^{1/2}(\Gamma)$$

is surjective, there exists $w_u \in H^2(\Omega)$ such that

$$\gamma_1 w_u = \frac{\partial w_u}{\partial n} = u.$$

Moreover w_u can be chosen so that

$$\|w_u\|_{H^2(\Omega)} \leq C\|u\|_{H^{1/2}(\Gamma)}.$$

We look for y of the form $y = w + z$.

Equation satisfied by z:

$$-\Delta z + z = f + \Delta w - w \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma.$$

Using the regularity result in $H^2(\Omega)$ we can write

$$\|z\|_{H^2(\Omega)} \leq C\|f + \Delta w - w\|_{L^2(\Omega)}$$

$$\leq C(\|u\|_{H^{1/2}(\Gamma)} + \|f\|_{L^2(\Omega)}).$$

Second conclusion

$$\|y\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^{1/2}(\Gamma)}).$$

Regularity if $u \in L^2(\Gamma)$

The solution y of the state equation

$$y = z + w_u,$$

where

$$-\Delta z + z = f \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma,$$

and

$$-\Delta w_u + w_u = 0 \quad \text{in } \Omega, \qquad \frac{\partial w_u}{\partial n} = u \quad \text{on } \Gamma.$$

We have proved

$$\|w_u\|_{H^1(\Omega)} \leq C\|u\|_{H^{-1/2}(\Gamma)}, \; \|w_u\|_{H^2(\Omega)} \leq C\|u\|_{H^{1/2}(\Gamma)}.$$

$$\|w_u\|_{H^{3/2}(\Omega)} \leq C\|u\|_{L^2(\Gamma)}, \; \|z\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Theorem. For every $f \in L^2(\Omega)$ and every $u \in L^2(\Gamma)$, the state equation admits a unique solution $y(u)$ in $H^1(\Omega)$, this solution belongs to $H^{3/2}(\Omega)$ and

$$\|y(u)\|_{H^{3/2}(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Gamma)}).$$

Existence of an optimal control

Theorem. ([Brézis, Chapter 3, Theorem 9])

Let E and F be two Banach spaces, and let T be a continuous linear operator from E into F. Then T is also continuous from $(E, \sigma(E, E'))$ into $(F, \sigma(F, F'))$.

Corollary. Let $(u_n)_n$ be a sequence converging to u for the weak topology of $L^2(\Gamma)$. Then the sequence $(y(u_n))_n$, where $y(u_n)$ is the solution to the state equation corresponding to the control function u_n, converges to $y(u)$ in $H^1(\Omega)$.

Proof. Set $y_n = y(u_n)$. We know that

$$\|y_n\|_{H^{3/2}(\Omega)} \leq C(\|f\|_{L^2} + \|u_n\|_{L^2(\Gamma)}).$$

To prove that $(y_n)_n$ converges to $y(u)$ in $H^1(\Omega)$, it is enough to prove that, from any subsequence extracted from $(y_n)_n$, we can extract an other subsequence converging to $y(u)$ in $H^1(\Omega)$.

Suppose that

$$y_k \longrightarrow \tilde{y} \quad \text{in } H^1(\Omega).$$

Passage to the limit

$$\int_\Omega \Big(\nabla y_k \cdot \nabla z + y_k\, z \Big) = \int_\Omega f\, z + \int_\Gamma u_k\, z.$$

$$\int_\Omega \Big(\nabla \tilde{y} \cdot \nabla z + \tilde{y}\, z \Big) = \int_\Omega f\, z + \int_\Gamma u\, z,$$

for all $z \in H^1(\Omega)$. Thus $\tilde{y} = y(u)$.

Theorem. ([Brézis, Chapter 3, Theorem 7])

Let E be a Banach space, and let $C \subset E$ be a convex subset. If C is closed in E, then C is also closed in $(E, \sigma(E, E'))$ (that is, closed in E endowed with its weak topology).

Corollary. ([Brézis, Chapter 3, Corollary 8]) Let E be a Banach space, and let $\varphi : E \mapsto]-\infty, \infty]$ be a lower semicontinuous convex function. Then φ is also lower semicontinuous for the weak topology $\sigma(E, E')$. In particular φ is sequentially lower semicontinuous.

Comments. Observe that

$$u \longmapsto \|u\|^2_{L^2(\Gamma)}$$

is convex and continuous in $L^2(\Gamma)$. Therefore, if

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(\Gamma),$$

then

$$\int_\Gamma u^2 \le \text{liminf}_{n\to\infty} \int_\Gamma u_n^2.$$

Theorem. Problem (P) admits a unique solution.

Proof. Existence

Set $0 \leq m = \inf(P) < \infty$. Set $F(u) = J(y(u), u)$.
Let $(u_n)_n$ be a minimizing sequence:

$$\lim_{n\to\infty} F(u_n) = m.$$

We can suppose that

$$\frac{\beta}{2}\int_\Gamma u_n^2 \leq F(u_n) \leq F(0).$$

Thus $(u_n)_n$ is bounded in $L^2(\Gamma)$. Suppose that

$$u_n \rightharpoonup \hat{u} \qquad \text{weakly in } \; L^2(\Gamma).$$

Thus

$$y(u_n) \longrightarrow y(\hat{u}) \qquad \text{in} \ \ L^2(\Omega).$$

With the continuity of $\|\cdot\|^2_{L^2(\Omega)}$ and the weakly lower semicontinuity of $\|\cdot\|^2_{L^2(\Gamma)}$, we obtain

$$\int_\Gamma \hat{u}^2 \leq \text{liminf}_{n\to\infty} \int_\Gamma u_n^2,$$

$$\int_\Omega (y(\hat{u}) - y_d)^2 = \lim_{\ n\to\infty} \int_\Omega (y(u_n) - y_d)^2.$$

Combining these results, we have

$$F(\hat{u}) \leq \text{liminf}_{n\to\infty} F(u_n) = m = \text{inf}(P).$$

Thus $\hat{u}$ is a solution of (P).

Uniqueness

We argue by contradiction.

Let u_1 and u_2 be two solutions of (P).

$$y\Big(\frac{1}{2}u_1 + \frac{1}{2}u_2\Big) = \frac{1}{2}y(u_1) + \frac{1}{2}y(u_2).$$

Since J is stricly convex, F is stricly convex:

$$F\Big(\frac{1}{2}u_1 + \frac{1}{2}u_2\Big) = J\Big(y\Big(\frac{1}{2}u_1 + \frac{1}{2}u_2\Big), \frac{1}{2}u_1 + \frac{1}{2}u_2\Big)$$

$$< \frac{1}{2}J(y(u_1), u_1) + \frac{1}{2}J(y(u_2), u_2) = \mathrm{min}(P).$$

Optimality conditions

Theorem. Necessary optimality conditons

Let $F \ : \ U \longmapsto \mathbb{R}$. Suppose that F is Gâteaux-differentiable at $\bar{u}$ and that

$$F(\bar{u}) = \text{inf}\{F(u) \mid u \in U\}.$$

Then

$$F'(\bar{u}) = 0.$$

Sufficient optimality conditons

Let F be a differentiable mapping from a Banach space U into $\mathbb{R}$. Suppose that F is convex and $F'(\bar{u}) = 0$, then $F(\bar{u}) \leq F(u)$ for all $u \in U$.

Proof. It is a consequence of the convexity inequality

$$F(u) - F(\bar{u}) \geq F'(\bar{u})(u - \bar{u}) = 0,$$

for all $u \in U$.

Optimality condition for the control problem

We set

$$F(u) = J(y(u), u) = I(y(u)) + G(u),$$

where $y(u)$ is the solution to the state equation.

Directional derivative

$$I(y(u+\lambda v)) - I(y(u))$$

$$= \frac{1}{2}\int_\Omega |y(u+\lambda v) - y_d|^2 - \frac{1}{2}\int_\Omega |y(u) - y_d|^2$$

$$= \int_\Omega (y(u+\lambda v) - y(u))\Big(\frac{1}{2}\Big(y(u+\lambda v) + y(u)\Big) - y_d\Big).$$

Equation satisfied by $z_\lambda = y(u+\lambda v) - y(u)$

$$-\Delta z_\lambda + z_\lambda = 0 \quad \text{in } \Omega, \qquad \frac{\partial z_\lambda}{\partial n} = \lambda v \quad \text{on } \Gamma.$$

Thus $z = \frac{y(u+\lambda v)-y(u)}{\lambda}$ obeys

$$-\Delta z + z = 0 \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n} = v \quad \text{on } \Gamma.$$

And

$$\frac{1}{2}\Big(y(u+\lambda v) + y(u)\Big) - y_d \longrightarrow y(u) - y_d.$$

$$\text{lim}_{\lambda \to 0} \frac{I(y(u+\lambda v)) - I(y(u))}{\lambda}$$

$$= \int_\Omega (y(u) - y_d) z(v)$$

where

$$-\Delta z + z = 0 \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n} = v \quad \text{on } \Gamma.$$

$$G(u+\lambda v) - G(u) = \frac{\beta}{2} \int_\Gamma \left(|u+\lambda v|^2 - |u|^2 \right)$$

$$= \frac{\beta}{2} \int_\Gamma (2u\,\lambda v + |\lambda v|^2).$$

$$\lim_{\lambda \to 0} \frac{G(u+\lambda v) - G(u)}{\lambda} = \beta \int_\Gamma u\, v.$$

Finally

$$F'(u;v) = \int_\Omega (y(u) - y_d)z + \frac{1}{2}\int_\Gamma u\,v,$$

where

$$-\Delta z + z = 0 \quad \text{in } \Omega, \qquad\qquad \frac{\partial z}{\partial n} = v.$$

Since

$$|F'(u;v)| \le C\|v\|_{L^2(\Gamma)},$$

F is G-differentiable and $F'(u)v = F'(u;v)$.

Identification of $F'(u) \; : \; L^2(\Gamma) \mapsto \mathbb{R}$

$$F'(u) \qquad \text{belongs to} \quad (L^2(\Gamma))' = L^2(\Gamma).$$

$$F'(u)v = \int_\Omega (y(u) - y_d) z + \int_\Gamma u\, v = \int_\Gamma \pi\, v.$$

We introduce

$$\Lambda \; : \; v \longmapsto z, \quad \text{from } L^2(\Gamma) \text{ to } L^2(\Omega).$$

$$F'(u)v = \Big((y(u) - y_d), \Lambda(v)\Big)_{L^2(\Omega)} + \Big(u, v\Big)_{L^2(\Gamma)}$$
$$= \Big(\Lambda^*(y(u) - y_d), v\Big)_{L^2(\Gamma)} + \Big(u, v\Big)_{L^2(\Gamma)}.$$

Green Formula between z and an other function p:

$$0 = \int_\Omega (-\Delta z + z)\, p$$

$$0 = \int_\Omega (-\Delta p + p)\, z - \int_\Gamma \frac{\partial z}{\partial n} p + \int_\Gamma \frac{\partial p}{\partial n} z.$$

$$0 = \int_\Omega (-\Delta p + p)\, z - \int_\Gamma v\, p + \int_\Gamma \frac{\partial p}{\partial n} z.$$

$$\int_\Omega (y(u) - y_d) z = \int_\Gamma q\, v$$

Thus if p is the solution to

$$-\Delta p + p = y(u) - y_d \quad \text{in } \Omega, \qquad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma.$$

then

$$\int_\Omega (y(u) - y_d) z = \int_\Gamma v\, p,$$

and

$$F'(u)v = \int_\Gamma (p + \beta u)\, v.$$

If $\bar{u}$ is the solution of (P) then

$$p + \beta \bar{u} = 0,$$

with

$$-\Delta p + p = y(\bar{u}) - y_d \quad \text{in } \Omega, \qquad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma.$$

Theorem. If $(\bar{y}, \bar{u})$ is the solution to (P) then $\bar{u} = -\frac{1}{\beta} p|_\Gamma$, where p is the solution of the equation

$$-\Delta p + p = \bar{y} - y_d \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma.$$

Conversely, if a pair $(\tilde{y}, \tilde{p}) \in H^1(\Omega) \times H^1(\Omega)$ obeys the system

$$\begin{aligned} -\Delta \tilde{y} + \tilde{y} &= f \quad \text{in } \Omega, & \frac{\partial \tilde{y}}{\partial n} &= -\tfrac{1}{\beta}\tilde{p} \quad \text{on } \Gamma, \\ -\Delta \tilde{p} + \tilde{p} &= \tilde{y} - y_d \quad \text{in } \Omega, & \frac{\partial \tilde{p}}{\partial n} &= 0 \quad \text{on } \Gamma, \end{aligned}$$

then the pair $(\tilde{y}, -\frac{1}{\beta}\tilde{p})$ is the solution of problem (P).

Proof. Suppose that (y, p) is a solution to the optimality system

$$-\Delta y + y = f \quad \text{in } \Omega, \quad \frac{\partial y}{\partial n} = -\frac{1}{\beta} p \quad \text{on } \Gamma,$$
$$-\Delta p + p = y - y_d \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma,$$

Then

$$F'\Big(-\frac{1}{\beta} p\Big) = 0,$$

because

$$F'\Big(\frac{1}{\beta} p\Big) v = \int_\Gamma (p + \beta u)\, v = \int_\Gamma \Big(p - \beta \frac{1}{\beta} p\Big)\, v = 0.$$

Comments.

- The optimality system

$$-\Delta y + y = f \quad \text{in } \Omega, \quad \frac{\partial y}{\partial n} = -p \quad \text{on } \Gamma,$$
$$-\Delta p + p = y - y_d \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma,$$

can be approximated by a finite element method, and next solved by a conjugate gradient method.

- The optimal control is obtained by taking

$$u = -\frac{1}{\beta} p|_\Gamma.$$

Part 2

Control problem with control constraints

(P_2)

Minimize $J(y,u) = \frac{1}{2}\int_\Omega |y-y_d|^2 + \frac{\beta}{2}\int_\Gamma |u|^2$

$-\Delta y + y = f$ in Ω, $\qquad \frac{\partial y}{\partial n} = u$ on Γ.

$u \in U_{ad},$ a closed convex subset in $L^2(\Gamma)$.

$$U_{ad} = \{u \in L^2(\Gamma) \mid u_a \le u \le u_b\}.$$

Theorem. Problem (P_2) admits a unique solution.

We find the existence of a minimizing sequence

$$(u_n)_n \subset U_{ad},$$

and

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(\Gamma).$$

Since U_{ad} is a closed convex subset in $L^2(\Gamma)$,

$$u \in U_{ad}.$$

Thus u is a solution of problem (P_2).

Optimality conditions

Theorem. Necessary optimality conditons

Let $F : U \longmapsto \mathbb{R}$. Suppose that F is Gâteaux-differentiable at $\bar{u}$ and that

$$F(\bar{u}) = \inf\{F(u) \mid u \in U_{ad}\}.$$

Then

$$F'(\bar{u})(u - \bar{u}) \geq 0 \qquad \text{for all } u \in U_{ad}.$$

Sufficient optimality conditons

Let F be a differentiable mapping from a Banach space U into $\mathbb{R}$. Suppose that F is convex and $F'(\bar{u})(u-\bar{u}) \geq 0$ for all $u \in U_{ad}$, then $F(\bar{u}) \leq F(u)$ for all $u \in U_{ad}$.

Theorem. If $(\bar{y}, \bar{u})$ is the solution to (P_2) then

$$\int_\Gamma (\beta\bar{u} + p)(u - \bar{u}) \geq 0 \quad \text{for all } u \in U_{ad},$$

where p is the solution of the equation

$$-\Delta p + p = \bar{y} - y_d \quad \text{in } \Omega, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma.$$

Proof.

$$F'(\bar{u})v = \int_\Gamma (\beta\bar{u} + p)\, v,$$

where

$$-\Delta p + p = \bar{y} - y_d \quad \text{in } \Omega, \qquad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma.$$

Thus writing

$$F'(\bar{u})(u - \bar{u}) \geq 0 \qquad \text{for all } u \in U_{ad},$$

we obtain the necessary optimality condition of the theorem.

Theorem. Conversely, if a triplet $(\tilde{y}, \tilde{p}, \tilde{u}) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Gamma)$ obeys the system

$$
\begin{aligned}
&-\Delta \tilde{y} + \tilde{y} = f \quad \text{in } \Omega, \quad \frac{\partial \tilde{y}}{\partial n} = \tilde{u} \quad \text{on } \Gamma,\\
&-\Delta \tilde{p} + \tilde{p} = \tilde{y} - y_d \quad \text{in } \Omega, \quad \frac{\partial \tilde{p}}{\partial n} = 0 \quad \text{on } \Gamma,\\
&\int_\Gamma (\beta \tilde{u} + \tilde{p})(u - \tilde{u}) \geq 0 \quad \text{for all } u \in U_{ad},
\end{aligned}
$$

then the pair $(\tilde{y}, \tilde{u})$ is the solution of problem (P_2).

Proof. The theorem follows from the sufficient optimality condition stated before.

Case of bound constraints

$$u_a \le u \le u_b, \quad u_a,\, u_b \in L^2(\Gamma).$$

The necessary optimality condition

$$\int_\Gamma (\beta \bar{u} + p)(u - \bar{u}) \ge 0 \quad \text{for all } u \in U_{ad},$$

is equivalent to the pointwise relation

$$(\beta \bar{u}(x) + p(x))(u - \bar{u}(x)) \ge 0 \quad \text{for all } u \in [u_a(x), u_b(x)],$$

for almost every $x \in \Gamma$.

Recall that a Lebesgue point for $g \in L^1(\Gamma)$:

$$x_0 \in \Gamma \quad \text{s. t. } \lim_\varepsilon \frac{1}{|\Gamma \cap B(x_0,\varepsilon)|} \int_{\Gamma \cap B(x_0,\varepsilon)} g = g(x_0),$$

and

$$\lim_\varepsilon \frac{1}{|\Gamma \cap B(x_0,\varepsilon)|} \int_{\Gamma \cap B(x_0,\varepsilon)} |g(x) - g(x_0)| dx = 0.$$

We denote by Γ_0 the set of Lebesgue points of the functions

$$(\beta \bar{u} + p)(u_a - \bar{u}), \qquad u_a,$$
$$(\beta \bar{u} + p)(u_b - \bar{u}), \qquad u_b.$$

Let $x_0 \in \Gamma_0$, we choose

$$u = u_a \chi_{\Gamma \cap B(x_0,\varepsilon)} + \bar{u} \chi_{\Gamma \setminus B(x_0,\varepsilon)}.$$

We substitute u in the integral relation,

$$\int_\Gamma (\beta \bar{u} + p)(u - \bar{u}) \geq 0 \quad \text{for all } u \in U_{ad},$$

we divide by $|\Gamma \cap B(x_0, \varepsilon)|$, and by passing to the limit, we obtain

$$(\beta \bar{u}(x_0) + p(x_0))(u_a(x_0) - \bar{u}(x_0)) \geq 0.$$

Similarly

$$(\beta \bar{u}(x_0) + p(x_0))(u_b(x_0) - \bar{u}(x_0)) \geq 0.$$

Finally

$$(\beta\bar{u}(x_0) + p(x_0))(u - \bar{u}(x_0)) \geq 0,$$

for all $u \in [u_a(x_0), u_b(x_0)]$.

From the pointwise relation, we deduce that

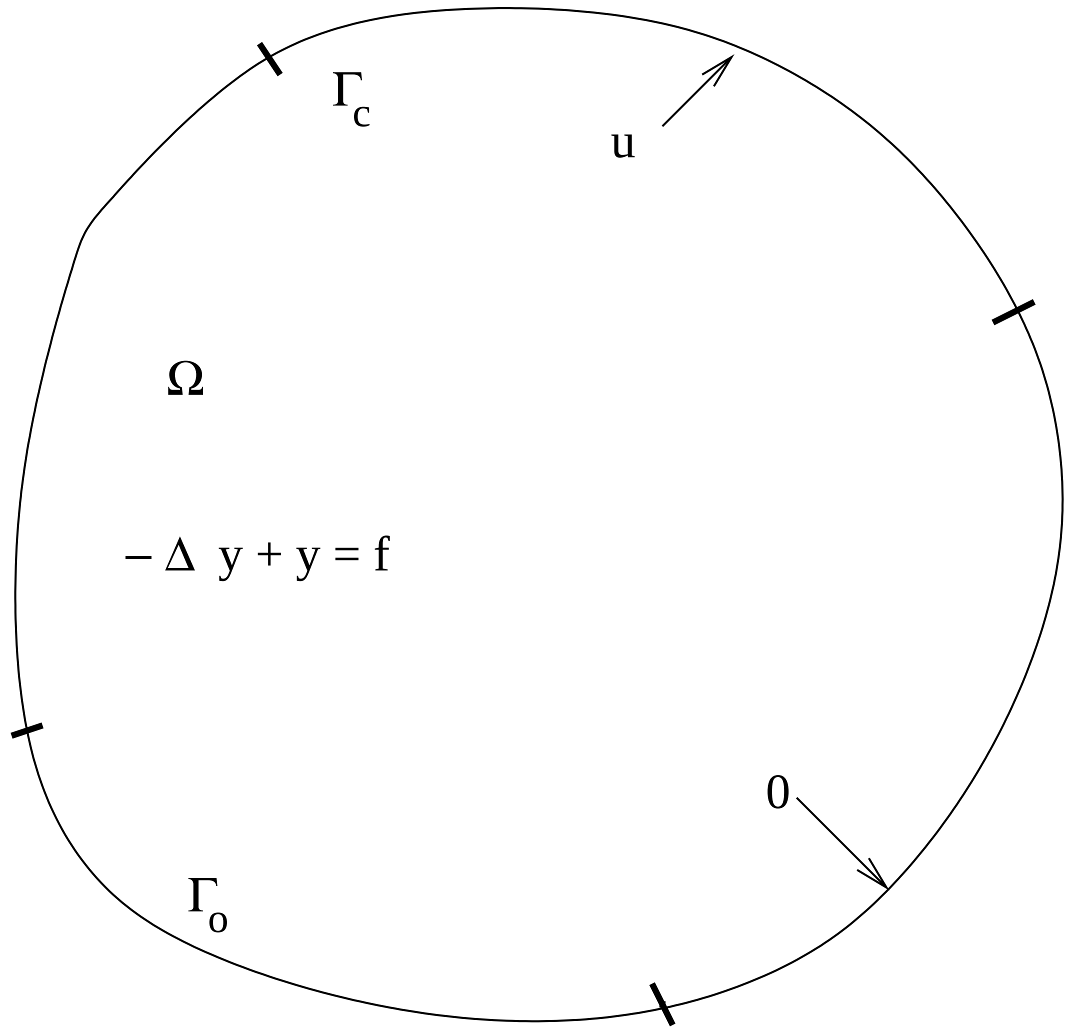
$$\bar{u}(x) = u_a(x) \qquad \text{if } \beta\bar{u}(x) + p(x) > 0,$$
$$u_a(x) \leq \bar{u}(x) \leq u_b(x) \qquad \text{if } \beta\bar{u}(x) + p(x) = 0,$$
$$\bar{u}(x) = u_b(x) \qquad \text{if } \beta\bar{u}(x) + p(x) < 0.$$

We can summarize these results by writing

$$\bar{u}(x) = \mathrm{Proj}_{[u_a(x), u_b(x)]}\Big(-\frac{1}{\beta}p(x)\Big) \qquad \text{for a. e. } x \in \Gamma.$$

Part 3

Exercise 1. Observation on the boundary



(P_3)

Minimize $J(y,u) = \dfrac{1}{2}\displaystyle\int_{\Gamma_o} |y - y_d|^2 + \dfrac{\beta}{2}\displaystyle\int_{\Gamma_c} |u|^2$

$-\Delta y + y = f$ in Ω,

$\dfrac{\partial y}{\partial n} = u$ on Γ_c, $\dfrac{\partial y}{\partial n} = 0$ on $\Gamma \setminus \Gamma_c$.

$u \in L^2(\Gamma)$, $f \in L^2(\Omega)$,

$y_d \in L^2(\Gamma_o)$, $\beta > 0$.

Questions.

1. Prove the existence of a unique optimal control u.

2. Write the first order optimality conditions for u.

Existence of a solution to the state equation

Lax-Milgram theorem in $H^1(\Omega)$ with

$$a(y,z) = \int_\Omega \Big(\nabla y \cdot \nabla z + y\, z \Big) \;,$$

$$\ell(z) = \int_\Omega f\, z + \int_{\Gamma_c} u\, z.$$

The variational formulation of the state equation is

find $y \in H^1(\Omega)$ such that

$a(y,z) = \ell(z)$ for all $z \in H^1(\Omega)$.

1. For every $u \in L^2(\Gamma)$, the state equation admits a unique solution in $H^1(\Omega)$. This solution belongs to $H^{3/2}(\Omega)$. Moreover

$$\|y\|_{H^{3/2}(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Gamma_c)}).$$

2. If $(u_n)_n$ is a sequence in $L^2(\Gamma_c)$ converging to u for the weak topology of $L^2(\Gamma_c)$, then the sequence $(y(u_n))_n$ converges to $y(u)$ in $H^1(\Omega)$.

Thus

$$y(u_n)|_{\Gamma_o} \longrightarrow y(u)|_{\Gamma_o} \quad \text{in } L^2(\Gamma_o).$$

3. The control problem admits a unique solution.

Optimality conditions

Setting $F(u) = J(y(u), u)$ we have

$$F'(u)v = \int_{\Gamma_o} (y(u) - y_d) z(v) + \beta \int_{\Gamma_c} u\, v,$$

where $z(v)$ is the solution of

$$-\Delta z + z = 0 \quad \text{in } \Omega,$$

$$\frac{\partial z}{\partial n} = v \quad \text{on } \Gamma_c, \qquad \frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma \setminus \Gamma_c.$$

What is the adjoint equation ? With an integration by parts between $z = z(v)$ and an other function p we have

$$0 = \int_\Omega (-\Delta p + p)\, z - \int_\Gamma \frac{\partial z}{\partial n} p + \int_\Gamma \frac{\partial p}{\partial n} z$$

$$0 = \int_\Omega (-\Delta p + p)\, z - \int_{\Gamma_c} v\, p + \int_\Gamma \frac{\partial p}{\partial n} z.$$

$$\int_{\Gamma_o} (y(u) - y_d) z = \int_{\Gamma_c} v\, p.$$

If we set

$$-\Delta p + p = 0,$$

and

$$\frac{\partial p}{\partial n} = y(u) - y_d \quad \text{on } \Gamma_o, \qquad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma \setminus \Gamma_o$$

we have

$$\int_{\Gamma_o} (y(u) - y_d) z = \int_{\Gamma_c} v\, p.$$

Conclusion

$$F'(u)v = \int_{\Gamma_c} (p\, v + \beta u\, v).$$

The optimal control $\bar{u}$ is characterized by

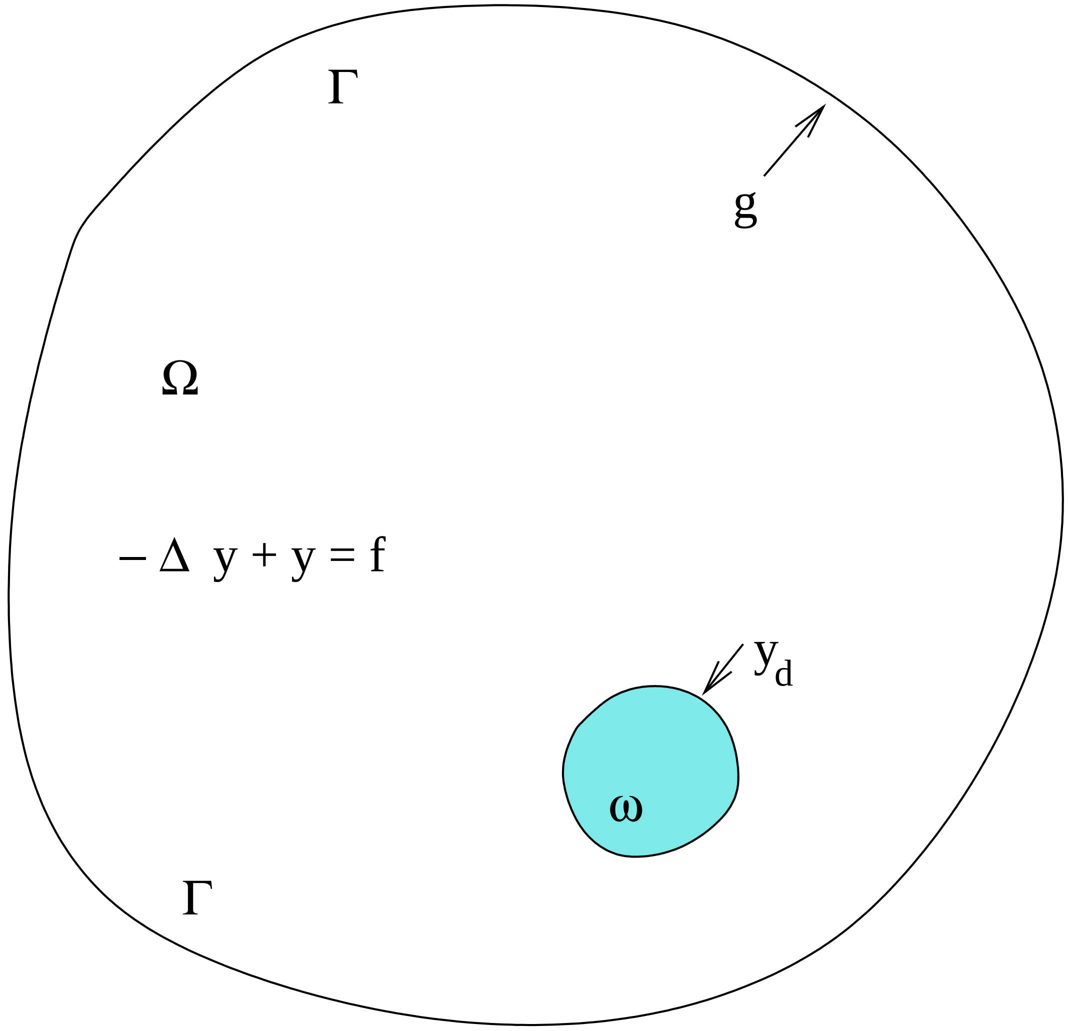
$$\bar{u} = -\frac{1}{\beta} p|_{\Gamma_c},$$

and

$$-\Delta p + p = 0 \quad \text{in } \Omega, \qquad \frac{\partial p}{\partial n} = \chi_{\Gamma_0}(y(\bar{u}) - y_d) \quad \text{on } \Gamma.$$

Part 4

Exercise 2. Identification of a boundary coefficient



(P_4)

Minimize $J(y,u) = \dfrac{1}{2}\displaystyle\int_\omega |y-y_d|^2 + \dfrac{\beta}{2}\int_\Gamma |u-h|^2$

$$-\Delta y + y = f \quad \text{in } \Omega, \qquad \frac{\partial y}{\partial n} + uy = g \quad \text{on } \Gamma.$$

$$u \in U_{ad} = \Big\{ u \in L^2(\Gamma) \mid u_a \le u \le u_b \Big\},$$

$$0 < u_a < u_b \in \mathbb{R}, \qquad \omega \subset \Omega.$$

We suppose that

$$y_d \in L^2(\omega),\ f \in L^2(\Omega), \quad h,\ g \in L^2(\Gamma), \quad \beta > 0.$$

Questions.

1. Prove the existence of an optimal control u.

2. Write the first order optimality conditions for u.

Existence of a solution to the state equation

Lax-Milgram theorem in $H^1(\Omega)$ with

$$a(y,z) = \int_\Omega \Big(\nabla y \cdot \nabla z + y\, z \Big) + \int_\Gamma u\, y\, z\ ,$$

$$\ell(z) = \int_\Omega f\, z + \int_\Gamma g\, z.$$

The variational formulation of the state equation is

find $y \in H^1(\Omega)$ such that

$a(y,z) = \ell(z)$ for all $z \in H^1(\Omega)$.

1. For every $u \in U_{ad}$, the state equation admits a unique solution in $H^1(\Omega)$. Moreover

$$\|y\|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}),$$

where C is independent of u.

Writing the boundary condition in the form

$$\frac{\partial y}{\partial n} = -uy + g \quad \text{on } \Gamma,$$

and using the estimates of part 1, we find

$$\|y\|_{H^{3/2}(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|-uy + g\|_{L^2(\Gamma)}).$$

Thus

$$\|y\|_{H^{3/2}(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)} + \|y\|_{H^1}\|u\|_{L^\infty}).$$

2. If $(u_n)_n$ is a sequence in U_{ad} converging to u for the weak-star topology of $L^\infty(\Gamma)$, then the sequence $(y(u_n))_n$ converges to $y(u)$ in $H^1(\Omega)$.

The mapping

$$u \longmapsto y(u)$$

is **nonlinear**.

Proof. Set $y_n = y(u_n)$. We know that

$$\|y_n\|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}).$$

and

$$\|y_n\|_{H^{3/2}(\Omega)} \leq C(\|f\|_{L^2} + \|g\|_{L^2} + \|y_n\|_{H^1}\|u_n\|_{L^\infty}).$$

To prove that $(y_n)_n$ converges to $y(u)$ in $H^1(\Omega)$, it is enough to prove that $y(u)$ is the unique cluster point of the sequence in $H^1(\Omega)$.

Let us denote by $(y_k)_k$ such a subsequence and suppose that $(y_k)_k$ converges to $\tilde{y}$ weakly in $H^{3/2}(\Omega)$, and strongly in $H^1(\Omega)$.

Passage to the limit

$$\int_\Omega \Big(\nabla y_k \cdot \nabla z + y_k\, z\Big) + \int_\Gamma u_k\, y_k\, z\ ,$$
$$= \int_\Omega f\, z + \int_\Gamma g\, z.$$

Since $(y_k)_k$ converges to $\tilde{y}$ in $H^1(\Omega)$,

$$\int_\Omega \Big(\nabla y_k \cdot \nabla z + y_k\, z\Big) \longrightarrow \int_\Omega \Big(\nabla \tilde{y} \cdot \nabla z + \tilde{y}\, z\Big)$$

$$y_k\big|_\Gamma \stackrel{L^2(\Gamma)}{\longrightarrow} \tilde{y}\big|_\Gamma,$$

$$\int_\Gamma u_k\, y_k\, z \longrightarrow \int_\Gamma u\, \tilde{y}\, z\ ,$$

for all $z \in H^1(\Omega)$. Thus $\tilde{y} = y(u)$.

3. The control problem admits at least one solution. In general the uniqueness cannot be proved because the mapping

$$u \longmapsto y(u)$$

is not affine.

Optimality conditions

4. Equation satisfied by $w_\lambda = \frac{y(u+\lambda v)-y(u)}{\lambda}$.

$$-\Delta w_\lambda + w_\lambda = 0 \quad \text{in } \Omega,$$

$$\frac{\partial w_\lambda}{\partial n} + (u+\lambda v)w_\lambda + vy(u) = 0 \quad \text{on } \Gamma.$$

For λ small enough ($|\lambda| \leq \lambda_0$), the bilinear form:

$$a(w,z) = \int_\Omega \Big(\nabla w \cdot \nabla z + w\, z\Big) + \int_\Gamma (u+\lambda v)\, w\, z$$

is coercive in $H^1(\Omega)$ and the coercivity constant is independent of λ. Thus

$$\|w_\lambda\|_{H^1(\Omega)} \leq C\|v\|_{L^\infty(\Gamma)}\|y(u)\|_{L^2(\Gamma)},$$

where C is independent of λ.

At least formally when $\lambda \to 0$

$$w_\lambda \longrightarrow z,$$

where z is the solution to

$$-\Delta z + z = 0 \quad \text{in } \Omega, \qquad \frac{\partial z}{\partial n} + u\, z + v y(u) = 0 \quad \text{on } \Gamma.$$

To prove that w_λ converges to z in $L^2(\Omega)$ or in $H^1(\Omega)$, we write the equation satisfied by $\zeta = w_\lambda - z$

$$-\Delta \zeta + \zeta = 0 \quad \text{in } \Omega, \qquad \frac{\partial \zeta}{\partial n} + u\zeta + \lambda v\, w_\lambda = 0 \quad \text{on } \Gamma.$$

With a classical estimate we get

$$\|\zeta\|_{H^1(\Omega)} \leq C|\lambda|\, \|v\|_{L^\infty(\Gamma)} \|w_\lambda\|_{L^2(\Gamma)}.$$

Conclusion

$$\lim_{\lambda\to 0}\left\|\frac{y(u+\lambda v)-y(u)}{\lambda}-z\right\|_{H^1(\Omega)}=0,$$

and setting $F(u)=J(y(u),u)$ we have

$$F'(u)v=\int_\omega (y(u)-y_d)z(v)+\beta\int_\Gamma (u-h)v.$$

What is the adjoint equation ? With an integration by parts between $z = z(v)$ and an other function p we have

$$0 = \int_\Omega (-\Delta p + p)\, z - \int_\Gamma \frac{\partial z}{\partial n} p + \int_\Gamma \frac{\partial p}{\partial n} z$$

$$0 = \int_\Omega (-\Delta p + p)\, z + \int_\Gamma (u\, z + v y(u)) p + \int_\Gamma \frac{\partial p}{\partial n} z.$$

$$\int_\omega (y(u) - y_d) z = \int_\Gamma q\, v.$$

If we set

$$-\Delta p + p = \chi_\omega (y(u) - y_d), \qquad \text{in } \Omega,$$

$$\frac{\partial p}{\partial n} + u\, p = 0, \qquad \text{on } \Gamma,$$

we have

$$\int_\omega (y(u) - y_d) z(v) = -\int_\Gamma v\, y(u)\, p.$$

Conclusion

$$F'(u)v = -\int_\Gamma v\, y(u)\, p + \beta \int_\Gamma (u - h) v.$$

If $(\bar{y}, \bar{u})$ is a solution of (P_4) then

$$\int_\Gamma \Big(-\bar{y}\, p + \beta(\bar{u} - h) \Big) (u - \bar{u}) \geq 0$$

for all $u_a \leq u \leq u_b$.

Pointwise conditions

$$\bar{u}(x) = u_a \qquad \text{if } (-\bar{y}\,p + \beta(\bar{u} - h))(x) > 0,$$

$$\bar{u}(x) = u_b \qquad \text{if } (-\bar{y}\,p + \beta(\bar{u} - h))(x) < 0,$$

$$u_a \leq \bar{u}(x) \leq u_b \qquad \text{if } (-\bar{y}\,p + \beta(\bar{u} - h))(x) = 0,$$

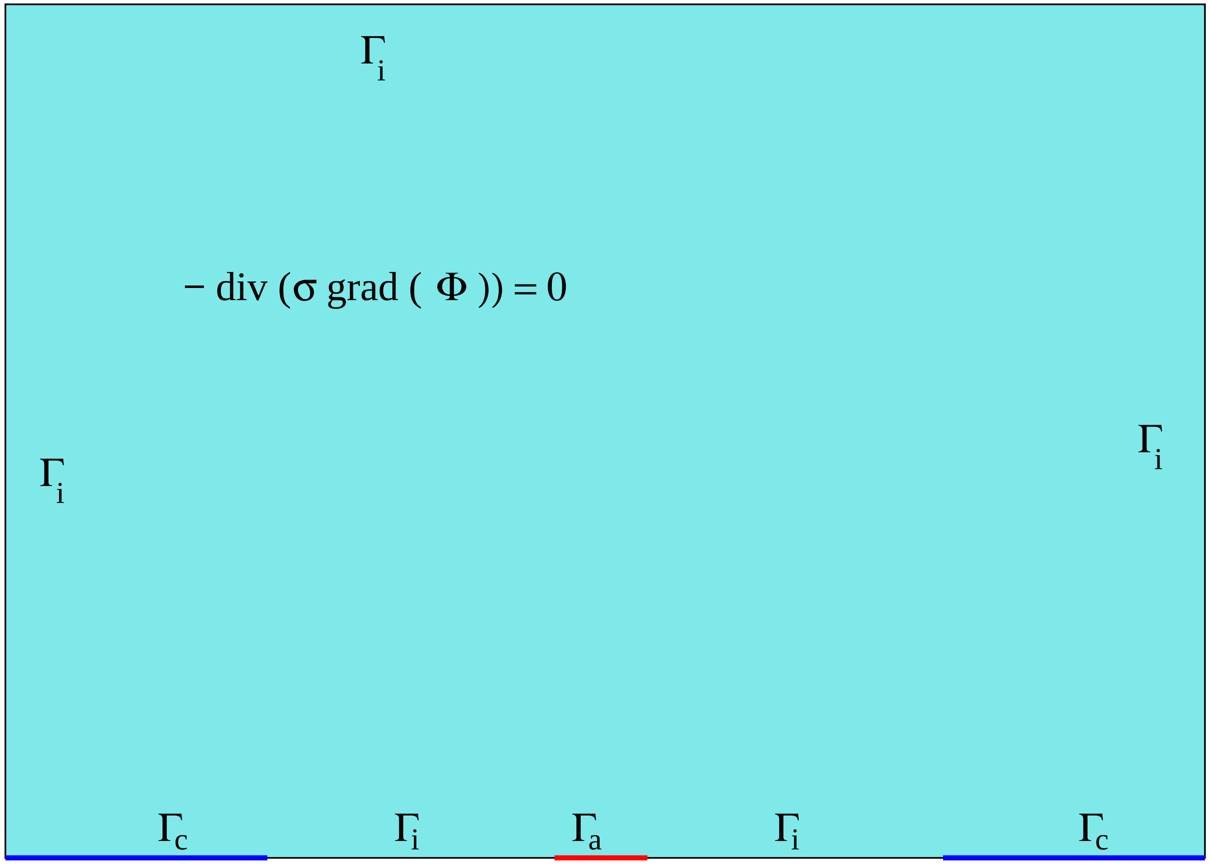
$$\bar{u}(x) = h(x) + \frac{1}{\beta}\bar{y}p(x).$$

We can summarize

$$\bar{u}(x) = \mathsf{Proj}_{[u_a,u_b]}\Big(h(x) + \frac{1}{\beta}\bar{y}p(x)\Big).$$

Part 5

Exercise 3. Control of an electrical potential



The electrical potential ϕ satisfies the elliptic equation

$$-\text{div}(\sigma\nabla\phi) = 0 \qquad \text{in } \Omega,$$

$$-\sigma\frac{\partial\phi}{\partial n} = u \quad \text{on } \Gamma_a, \qquad -\sigma\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \Gamma_i,$$

$$-\sigma\frac{\partial\phi}{\partial n} = f(\phi) \quad \text{on } \Gamma_c,$$

Γ_a corresponds to the **anode**, Γ_c corresponds to the **cathode**, Γ_i is the **rest of the boundary** Γ.

f is of class C^1, $f(0) = 0$, $0 < c_1 \leq f'(r) \leq c_2$ for all $r \in \mathbb{R}$,

the conductivity $\sigma > 0$.

The control problem (P_5)

$$\inf\{J(\phi,u) \mid (\phi,u) \in H^1(\Omega) \times L^2(\Gamma_a),\ u_a \le u \le u_b\},$$

where (ϕ, u) solves the state equation and

$$J(\phi,u) = \frac{1}{2}\int_{\Gamma_c} (\phi - \phi_d)^2 + \frac{\beta}{2}\int_{\Gamma_a} u^2,$$

$u_a \in L^2(\Gamma_a)$ and $u_b \in L^2(\Gamma_a)$ are some bounds on the current u, and β is a positive constant.

1. Prove that (P_5) has at least one solution.

2. Write the first order optimality condition for the solutions to (P_5).

The state equation

Theorem. (The Minty-Browder Theorem, [Brézis])
Let E be a reflexive Banach space, and $\mathcal{A}$ be a nonlinear continuous mapping from E into E'. Suppose that

$$\langle \mathcal{A}(\phi_1) - \mathcal{A}(\phi_2), \phi_1 - \phi_2 \rangle_{E',E} > 0$$

for all ϕ_1, ϕ_2 in E, with $\phi_1 \neq \phi_2$, and

$$\lim_{\|\phi\|_E \to \infty} \frac{\langle \mathcal{A}(\phi), \phi \rangle_{E',E}}{\|\phi\|_E} = \infty.$$

Then, for all $\ell \in E'$, there exists a unique $\phi \in E$ such that $\mathcal{A}(\phi) = \ell$.

To apply this theorem, we set $E = H^1(\Omega)$, and we define $\mathcal{A}$ by

$$\langle \mathcal{A}(\phi), z \rangle_{(H^1(\Omega))', H^1(\Omega)} = \int_\Omega \sigma \nabla \phi \cdot \nabla z + \int_{\Gamma_c} f(\phi) z,$$

and ℓ by

$$\ell(z) = -\int_{\Gamma_a} z\, u.$$

3. We can prove that the state equation

$$\langle \mathcal{A}(\phi), z \rangle = \ell(z) \qquad \forall z \in H^1(\Omega),$$

admits a unique solution $\phi \in H^1(\Omega)$.

Observe that

$$f(\phi(x)) = f(0) + \int_0^1 f'(\theta\phi)\, d\theta\, \phi\, \phi(x).$$

Writing the nonlinear boundary condition in the form

$$\sigma \frac{\partial \phi}{\partial n} + a(x)\phi = 0 \qquad \text{on } \Gamma_c,$$

with

$$0 < c_1 \leq a(x) = \int_0^1 f'(\theta\phi(x))\, d\theta.$$

We can prove that

$$\|\phi\|_{H^1(\Omega)} \le C\|u\|_{L^2(\Gamma_a)},$$

and

$$\|\phi\|_{H^{3/2}(\Omega)} \le C\big(c_2\|\phi\|_{L^2(\Gamma_c)} + \|u\|_{L^2(\Gamma_a)}\big).$$

Thus if $(u_n)_n$ is a sequence weakly converging to u in $L^2(\Gamma_a)$, the sequence $(\phi_n)_n$, where $\phi_n = \phi(u_n)$, is bounded in $H^{3/2}(\Omega)$.

We can pass to the limit in the variational equation

$$\langle \mathcal{A}(\phi_n), z \rangle_{(H^1(\Omega))', H^1(\Omega)} = \ell_n(z) = -\int_{\Gamma_a} u_n \, z.$$

Indeed

$$\phi_n \longrightarrow \phi \quad \text{in } H^1(\Omega).$$

Thus

$$\int_{\Gamma_c} f(\phi_n) z \longrightarrow \int_{\Gamma_c} f(\phi) z$$

for all $z \in H^1(\Omega)$.

Thanks to this result, we prove that (P_5) admits at least one solution.

Optimality conditions

Equation satisfied by $\psi_\lambda = \frac{\phi(u+\lambda v)-\phi(u)}{\lambda}$.

$$-\mathrm{div}(\sigma \nabla \psi_\lambda) = 0 \qquad \text{in } \Omega,$$

$$-\sigma \frac{\partial \psi_\lambda}{\partial n} = v \quad \text{on } \Gamma_a, \qquad -\sigma \frac{\partial \psi_\lambda}{\partial n} = 0 \quad \text{on } \Gamma_i,$$

$$-\sigma \frac{\partial \psi_\lambda}{\partial n} = \frac{1}{\lambda}\Big(f(\phi(u+\lambda v)) - f(\phi(u))\Big) \quad \text{on } \Gamma_c.$$

We have

$$\frac{1}{\lambda}\Big(f(\phi(u+\lambda v)) - f(\phi(u))\Big) = b(x)\psi_\lambda,$$

with

$$b(x) = \int_0^1 f'(\phi(u) + \theta\phi(u+\lambda v))\, d\theta \geq c_1 > 0.$$

The limit of ψ_λ in $H^1(\Omega)$, is the solution ψ of the equation

$$-\text{div}(\sigma\nabla\psi) = 0 \qquad \text{in } \Omega,$$

$$-\sigma\frac{\partial\psi}{\partial n} = v \quad \text{on } \Gamma_a, \qquad -\sigma\frac{\partial\psi}{\partial n} = 0 \quad \text{on } \Gamma_i,$$

$$-\sigma\frac{\partial\psi}{\partial n} = f'(\phi)\psi \quad \text{on } \Gamma_c.$$

Conclusion

Setting $F(u) = J(\phi(u), u)$ we have

$$F'(u)v = \int_{\Gamma_c} (\phi(u) - \phi_d)\psi(v) + \beta\int_{\Gamma_a} u\,v.$$

What is the adjoint equation ? With an integration by parts between $\psi = \psi(v)$ and an other function p we have

$$0 = -\int_\Omega \mathsf{div}(\sigma \nabla p)\, \psi - \int_\Gamma \sigma \frac{\partial \psi}{\partial n} p + \int_\Gamma \sigma \frac{\partial p}{\partial n} \psi$$

$$0 = \int_\Omega (-\mathsf{div}(\sigma \nabla p)\psi + \int_{\Gamma_a} vp + \int_{\Gamma_c} f'(\phi)\psi p$$

$$+ \int_\Gamma \sigma \frac{\partial p}{\partial n} \psi.$$

$$\int_{\Gamma_c} (\phi(u) - \phi_d)\psi(v) = \int_{\Gamma_a} q\, v.$$

If we set

$$
\begin{aligned}
&-\mathrm{div}(\sigma\nabla p) = 0, && \text{in } \Omega,\\
&\sigma\frac{\partial p}{\partial n} = 0, && \text{on } \Gamma_i \cup \Gamma_a,\\
&\sigma\frac{\partial p}{\partial n} + f'(\phi)p = \phi(u) - \phi_d, && \text{on } \Gamma_c,
\end{aligned}
$$

we have

$$
\int_{\Gamma_c} (\phi(u) - y_d)\psi(v) = -\int_{\Gamma_a} v\, p.
$$

Conclusion

$$
F'(u)v = \int_{\Gamma_a} (-p + \beta u)v.
$$

If $(\bar{\phi}, \bar{u})$ is a solution of (P_5) then

$$\int_{\Gamma_a} \Big(- p + \beta \bar{u} \Big)(u - \bar{u}) \geq 0$$

for all $u_a \leq u \leq u_b$.

That is

$$\bar{u}(x) = \mathsf{Proj}_{[u_a, u_b]} \Big(\frac{1}{\beta} p(x) \Big).$$

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