Null Controllability in a Fluid - Solid Structure Model

J.-P. Raymond *† M. Vanninathan‡

Abstract

We consider a system coupling the Stokes equations in a two dimensional domain with a structure equation which is a system of ordinary differential equations corresponding to a finite dimensional approximation of equations modeling deformations of an elastic body or vibrations of a rigid body. For that system we establish a null controllability result for localized distributed controls acting only in the fluid equations and there is no control in the solid part. This controllability result follows from a Carleman inequality that we prove for the adjoint system.

1 Introduction

Controllability of fluid – structure models is a challenging problem. Very recently Imanuvilov and Takahashi [16] and Boulakia and Osses [1] have studied the null controllability, locally about zero, of a system coupling the Navier-Stokes equations with the motion of a rigid body. Their analysis is based on Carleman estimates for a linearized system. In the system coupling the Navier-Stokes equations with a rigid body, the domain occupied by the fluid depends on the position of the solid and therefore depends on the time variable. The linearized system may be stated either in a time dependent domain as in [1] or in a fixed domain as in [16]. In the present paper, we are going to establish Carleman inequalities for a linearized fluid – solid structure model, stated in a fixed domain Ω. In some aspects our system is simpler than the linearized model considered in [16] and it is more complicated in some other aspects. On the one hand the model is simpler because we do not allow the structure to rotate, only translations are allowed. On the other hand it is more complicated because our structure may be considered as a finite dimensional approximation of systems modeling deformations of an elastic body or vibrations of a rigid body (these elastic deformations and vibrations are additional sources of instabilities in the coupled system as explained below).

For instance we could consider a structure equation of the form

$$q'' + Aq = -\int_{\Gamma_i} M^T \sigma(y, \pi) n, \tag{1.1}$$

*Université de Toulouse, UPS, Institut de Mathématiques, 31062 Toulouse Cedex 9, France,

†CNRS, Institut de Mathématiques, UMR 5219, 31062 Toulouse Cedex 9, France.

‡TIFR-CAM, Post Bag 6503, Bangalore 560065, India. The two authors are supported by CEFIPRA, project 3701-1. The first author is also supported par ANR CISIFS 09-BLAN-0213-03

where Γ_i is a part of $\partial\Omega$ and is the common boundary of the structure and the fluid, Ω is the two dimensional domain occupied by the fluid,

$$\sigma(y,\pi) = 2\,Dy - \pi\, n = \big(\nabla y + (\nabla y)^T\big) - \pi\, n$$

is the Cauchy stress tensor of the fluid velocity vectorfield, $q \in \mathbb{R}^N$, $A \in \mathbb{R}^{N\times N}$, $M \in \mathbb{R}^{N\times 2}$, n is the unit normal to Γ_i outward Ω. The term $-\int_{\Gamma_i} M^T\sigma(y,\pi)n$ represents the force exerted by the fluid on the structure. The equality of the fluid velocity and the structure velocity on $\Gamma_i \times (0,\infty)$ corresponds to the equation

$$y = Mq' \quad \text{on} \quad \Gamma_i \times (0,\infty).$$

When Γ_i is a flat part of the boundary $\partial\Omega$, equation (1.1) may be viewed as a finite dimensional Galerkin approximation of a beam equation of the form

$$z'' + \mathcal{A}z := z'' - \beta z_{x_1x_1} + \alpha z_{x_1x_1x_1x_1} = -\sigma(y,\pi)n\cdot n \qquad \text{on} \quad \Gamma_i \times (0,T), \tag{1.2}$$

completed by some boundary conditions (clamped boundary conditions or periodic boundary conditions...). Indeed if $(\zeta_k)_{k\in\mathbb{N}^*}$ is an orthonormal basis in $L^2(\Gamma_i)$ constituted of eigenfunctions of the elliptic operator $\mathcal{A}$ with associated boundary conditions, the Galerkin approximation of equation (1.2) in $\text{span}\Big\{\zeta_1,\cdots,\zeta_N\Big\}$ leads to an equation of the form (1.1) if z is approximated by $\Sigma_{k=1}^N q_k\,\zeta_k$, and if we set

$$q = (q_1,\cdots,q_N)^T, \quad A = \left(\int_{\Gamma_i} \mathcal{A}\zeta_k\,\zeta_\ell\right)_{1\le k,\,\ell\le N} \quad \text{and} \quad M = (\zeta_1\, n,\cdots,\zeta_N\, n)\,.$$

Another model of the form (1.1), the simplest one, corresponds to the case when $N = 2$, and when A and M are equal to the identity matrix in $\mathbb{R}^2$. This choice leads to the control system

$$\begin{aligned}
& y' - \Delta y + \nabla\pi = u\,\chi_{\omega\times(0,T)} \ \text{ and } \ \operatorname{div} y = 0 && \text{in} \quad Q,\\
& y = 0 && \text{on} \quad \Sigma_e,\\
& y = q' && \text{on} \quad \Sigma_i,\\
& y(0) = y^0 && \text{in} \quad \Omega,\\
& q'' + q = -\int_{\Gamma_i} \sigma(y,\pi)\,n && \text{in} \quad (0,T),\\
& q(0) = q^0 \qquad \text{and} \qquad q'(0) = q^1 && \text{in} \quad \mathbb{R}^2.
\end{aligned} \tag{1.3}$$

It corresponds to models introduced in [2, 3]. In this setting $Q = \Omega\times(0,T)$, $T > 0$, $\Sigma_e = \Gamma_e\times(0,T)$, $\Sigma_i = \Gamma_i\times(0,T)$, $\Gamma = \partial\Omega = \Gamma_e\cup\Gamma_i$, and we also use the notation $\Sigma = \Gamma\times(0,T)$ below. The model described by (1.3) corresponds to the case when the domain S occupied by the structure is an open set in $\mathcal{O}$, $\mathcal{O}$ is a simply connected bounded domain in $\mathbb{R}^2$ with a regular boundary Γ_e. We suppose that $\bar{S}\subset\mathcal{O}$, and we set $\Omega = \mathcal{O}\setminus\bar{S}$. Thus $\Gamma = \Gamma_e\cup\Gamma_i$ is the boundary of Ω and $\Gamma_e\cap\Gamma_i = \emptyset$. In (1.3), the control u is located in $\omega\subset\subset\Omega$.

For simplicity, in this paper we shall only consider the model (1.3). But a more elaborate model with a structure equation of the form (1.1) could also be considered (see e.g. [20] where we consider a coupling between the heat equation and a structure equation of the form (1.1)).

The main result of the paper is the following theorem which is a null controllability result in time $T > 0$ for system (1.3).

Theorem 1.1. *For all $y^0 \in L^2(\Omega)$ with $\operatorname{div} y^0 = 0$, $q^0 \in \mathbb{R}^2$ and $q^1 \in \mathbb{R}^2$ satisfying the conditions $y^0 \cdot n = q^1 \cdot n$ on Γ_i and $y^0 \cdot n = 0$ on Γ_e, there exists a function $u \in L^2(Q)$ such that the solution of (1.3) obeys*

$$y(T) = 0, \quad q(T) = 0 \quad \textit{and} \quad q'(T) = 0.$$

The proof of Theorem 1.1 is based on a Carleman estimate for the adjoint system associated with (1.3). The adjoint system is a backward evolution equation over the time interval $(0, T)$, with a terminal condition at time T. By a time reversal operation, we see that the adjoint sytem is similar to the original one

$$\begin{aligned}
&\phi' - \Delta\phi + \nabla p = f \quad \text{and} \quad \operatorname{div}\phi = 0 && \text{in} \quad Q,\\
&\phi = 0 && \text{on} \quad \Sigma_e,\\
&\phi = r' && \text{on} \quad \Sigma_i,\\
&\phi(0) = \phi^0 && \text{in} \quad \Omega,\\
&r'' + r = -\int_{\Gamma_i} \sigma(\phi, p)\, n && \text{in} \quad (0, T),\\
&r(0) = r^0 \qquad \text{and} \qquad r'(0) = r^1 && \text{in} \quad \mathbb{R}^2.
\end{aligned} \tag{1.4}$$

A Carleman estimate for the above system (with $f = 0$) is required to prove Theorem 1.1 and is established in section 8. In the case when the matrix A in the structure equation (1.1) is equal to $\mathbf{0}$, that is to say if the structure equation in (1.4) is replaced by

$$r'' = -\int_{\Gamma_i} \sigma(\phi, p)\, n \qquad \text{in} \quad (0, T),$$

then the Carleman inequalities established in [16] and [1] may be used to prove Theorem 1.1. The case $A \equiv \mathbf{0}$ corresponds to a non-vibrating rigid body. Considering a model as in (1.1), where A is a positive definite symmetric matrix allows us to take into account finite dimensional approximations of elastic deformations and vibrations of the structure.

When $A \equiv \mathbf{0}$, the method used in [16] consists in proving a Carleman inequality for the Stokes equation by adapting to the case when the boundary condition is nonhomogeneous ($\phi = r'$) the strategy developped in [9]. Let us briefly recall the different steps used in [16, 1, 19] to establish Carleman inequalities. The first step consists in using the Carleman estimates already proved for the heat equation in [5]. But new terms appear because the boundary conditions in the fluid equation are nonhomogeneous. Next in the method introduced in [15, 9] a gradient estimate of the fluid pressure deduced from [15] is used, later trace estimates of the pressure are derived, and finally the local term of the pressure, appearing in the RHS of the Carleman inequality when we estimate the gradient of the pressure, is removed.

In the case when $A \equiv \mathbf{0}$, the above program can be followed as in [16]. When $A = A^* > \mathbf{0}$ some new difficulties appear. Firstly new terms of the form $\int_0^T e^{-2s\beta}|_{\Gamma_i}|r|^2$ appear in the RHS of the Carleman inequality. We are going to see that, contrary to what happens in the case of the heat–solid structure model studied in [19], this term cannot be estimated by an energy identity (because the energy estimate introduces again pressure terms, see section 6.1). The second difficulty comes from the fact that, when $A = A^* > \mathbf{0}$, the trace estimate of the pressure cannot be simply obtained as in [16] or [1]. Actually in [16, 1] the trace estimate of the pressure is similar to the one derived in [9]. In our case, because of the presence of r in the structure equation, we have to follow a completely new way. The method consists in decoupling the pressure term into two parts and in estimating them separately. One part corresponds to the pressure p_e associated with $P\phi$ (where P is the Leray projector) and the other part corresponds to the pressure p_s associated with $(I - P)\phi$. This is carried out in sections 4 and 5.

The contribution of the structure in the RHS of the Carleman inequality is eliminated in sections 6 and 7 via a combination of monotonicity and compactness arguments. The upshot of all these estimates is the Carleman inequality stated in Theorem 8.1 in which we have the presence of a local term of the pressure in the RHS. By duality, the above term gives rise to an additional (fictitious control) in the incompressibility equation as in [14]. To remove it, we require a regularity result stated and proved in section 9. The proof of Theorem 1.1 is completed in section 10.2.

The first version of our Carleman inequality, stated in Theorem 3.1, is very similar to the ones obtained in [1, Inequality (2.12)] and [16, Inequality (3.34)]. The difference comes from the fact that we obtain an estimate of $\int_0^T \left|\int_{\Gamma_i} D\psi\, n\right|^2$ (where ψ is related to ϕ by some weight function, see section 2). This is a new term which is not present in [16, 1]. It could have been dominated by the term involving the normal derivative of ψ because the tangential derivative of ψ vanishes on Γ_i. However we do not use this and proceed differently. Our treatment could be useful even in cases of [1, 16] in which rotation of rigid body is considered. That is why we have given a detailed proof of boundary estimates.

Throughout the paper, we use the usual summation convention with respect to repeated indices. Various constants independent of parameters (s, λ) and the solution are generically denoted by C, unless otherwise indicated.

2 Preliminaries

2.1 Well posedness of system (1.4)

Let V be the space defined by

$$V = \Big\{\phi \in H^1(\Omega;\mathbb{R}^2) \mid \operatorname{div}\phi = 0, \quad \phi = 0 \text{ on } \Gamma_e\Big\},$$

and denote by V' the topological dual of V. The space V will be equipped with the norm

$$\phi \mapsto \left(\int_\Omega |\nabla\phi|^2 dx\right)^{1/2}.$$

The norm V will be denoted by $\|\cdot\|_V$. The same kind of notation will be used for other Banach spaces. Let us remark that this norm is equivalent to the usual $H^1(\Omega;\mathbb{R}^2)$ norm on V. For simplicity, we shall write $H^1(\Omega)$ for $H^1(\Omega;\mathbb{R}^2)$, $L^2(\Omega)$ for $L^2(\Omega;\mathbb{R}^2)$, and the same abuse of notation will be done for other spaces like $H^{-1}(\Omega;\mathbb{R}^2)$ for example. This does not lead to confusion even if $L^2(\Omega)$ is used for $L^2(\Omega;\mathbb{R}^2)$ for velocity vectorfields while it can be used for $L^2(\Omega)$ itself for the pressure.

The norm in $\mathbb{R}^2$ will be simply denoted by $|\cdot|$. The inner product of $q \in \mathbb{R}^2$ and $r \in \mathbb{R}^2$ is denoted by $q \cdot r$.

We have to introduce the spaces

$$V^0(\Omega) = \Big\{y \in L^2(\Omega) \mid \operatorname{div} y = 0\Big\}, \quad V^0_n(\Omega) = \Big\{y \in V^0(\Omega) \mid y \cdot n = 0 \text{ on } \Gamma\Big\},$$
$$V^1_0(\Omega) = H^1_0(\Omega) \cap V^0_n(\Omega), \quad V^0(\Gamma) = \Big\{y \in L^2(\Gamma) \mid \int_\Gamma y \cdot n = 0\Big\}.$$

Let us recall that $L^2(\Omega;\mathbb{R}^2)$ is the orthogonal sum of $V^0_n(\Omega)$ and $\nabla(H^1(\Omega))$ (the space of functions which are gradients of functions belonging to $H^1(\Omega)$). The Leray projector P is the orthogonal projector in $L^2(\Omega;\mathbb{R}^2)$ onto $V^0_n(\Omega)$.

Well-posedness of the system (1.4) is straightforward and it can be established using energy estimates, for instance. Indeed, if (ϕ, r) is a regular solution of system (1.4), multiplying (1.4) by (ϕ, r'), we get the energy identity:

$$\|\phi(t)\|^2_{L^2(\Omega)} + |r(t)|^2 + |r'(t)|^2 + 2\int_0^t\int_\Omega |\nabla\phi|^2 = 2\int_0^t\int_\Omega f\phi + \|\phi(0)\|^2_{L^2(\Omega)} + |r^0|^2 + |r^1|^2.$$

Existence of regular solutions to system (1.4) may be deduced from results in [21]. Using this, we can prove the following theorem.

Theorem 2.1. *Let $f \in L^2(0,T;L^2(\Omega))$, $\phi^0 \in V^0(\Omega)$, $r^0 \in \mathbb{R}^2$ and $r^1 \in \mathbb{R}^2$ satisfying the compatibility conditions $\phi^0 \cdot n = r^1 \cdot n$ on Γ_i and $\phi^0 \cdot n = 0$ on Γ_e. Then there is a unique solution $(\phi, r) \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V) \times C^1([0,T];\mathbb{R}^2)$ to the system* (1.4) *satisfying the energy inequality*

$$\|\phi\|_{C([0,T];L^2(\Omega))} + \|\phi\|_{L^2(0,T;V)} + \|\phi'\|_{L^2(0,T;H^{-1}(\Omega))} + \|r\|_{C^1([0,T];\mathbb{R}^2)} + \|r''\|_{L^2(0,T;\mathbb{R}^2)}$$
$$\leq C\Big\{\|f\|_{L^2(0,T;L^2(\Omega))} + \|\phi^0\|_{L^2(\Omega)} + |r^0| + |r^1|\Big\}. \quad ■$$

2.2 Transformed system

From now on we assume that $f = 0$ in (1.4). Carleman inequalities for the system (1.4) are stated in Theorems 8.1 and 9.1. Their proofs consist of several steps. In this section, we transform the system (1.4) to a new system satisfied by $(\psi, r) = (e^{-s\beta}\phi, r)$, where β is a weight function depending on a parameter λ. The Carleman inequalities are obtained for large values of parameters λ and s. In the next section we obtain a first Carleman inequality in Theorem 3.1. The goals of sections 4–8 is to eliminate the pressure p and the displacement of the structure r from the RHS of the inequality stated in Theorem 3.1. This is done only partially since a local term of the pressure is still remaining in Theorem 8.1. As explained in the introduction, we overcome this difficulty by using an additional control in the divergence condition, as in [14], which is subsequently removed in section 10.2 by using regularity results of section 9.

We begin by listing the properties of the test function η which is used in defining the change of variables. These properties are used at various stages of our computations below.

Lemma 2.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a nonempty open bounded set of annular type as defined in Section 1, and that ω_0 and ω are open subsets of Ω such that $\omega_0 \subset\subset \omega \subset\subset \Omega$. Then there exist a function $\eta \in C^4(\overline{\Omega})$ and positive constants C_{Γ_e} and C_{Γ_i} such that*

- $\eta(x) = C_{\Gamma_i} > 0, \quad \partial_n \eta = -1, \quad$ *and* $\quad \Delta\eta(x) = 0, \quad$ *for all* $x \in \Gamma_i$,
- $\eta(x) \geq C_{\Gamma_i} \quad$ *for all* $x \in \overline{\Omega}$,
- $\eta(x) = C_{\Gamma_e} \quad$ *and* $\quad \partial_n \eta \leq 0 \quad$ *for all* $x \in \Gamma_e$,
- $|\nabla\eta(x)| > 0 \quad$ *for all* $x \in \overline{\Omega \setminus \omega_0}$.

Proof. See [19, Lemma 3.1]. ∎

With a large parameter $\lambda \geq 1$, we introduce the functions

$$\begin{aligned} \xi(x,t) &= \frac{e^{\lambda(\eta + m\|\eta\|_\infty)}}{t^k(T-t)^k}, \quad m > 1, \\ \alpha(x) &= e^{\lambda m K_1} - e^{\lambda(\eta(x) + m\|\eta\|_\infty)} \qquad \forall x \in \overline{\Omega}, \end{aligned} \tag{2.1}$$

where $K_1 > 0$ is a constant, with $K_1 \geq 2\|\eta\|_\infty$ and η is the function obeying the conditions in Lemma 2.1. We set

$$\beta(x,t) = \frac{\alpha(x)}{t^k(T-t)^k}, \qquad \rho(x,t) = e^{\beta(x,t)},$$

where the constant k is chosen such that $k \geq 2$. In section 9, we shall have to set $k = 4$. Since η is constant on Γ_e and on Γ_i, the functions $\beta(\cdot,t)$ and $\rho(\cdot,t)$ are also constants there. In the following, we set

$$\rho_{\Gamma_i}(t) = \rho(\cdot,t)|_{\Gamma_i}.$$

With another large parameter $s \geq 1$, we also define the functions

$$f_s(x,t) = -\rho^{-s}(x,t)\nabla p(x,t), \quad g_s = f_s + s(\Delta\beta)\psi \quad \text{and} \quad \psi = \rho^{-s}\,\phi. \tag{2.2}$$

Notice that (since $\beta \to \infty$ as $t \to 0^+$ or as $t \to T^-$) $\psi(\cdot,0) = \psi(\cdot,T) = 0$ in Ω. With the definition

$$(\psi \otimes \phi)_{ij} = \psi_i\, \phi_j,$$

an easy calculation shows that

$$\begin{aligned}
\nabla\phi &= \nabla\big(e^{s\beta}\psi\big) = e^{s\beta}\big(\nabla\psi + s\psi\otimes\nabla\beta\big),\\
(\nabla\phi)^T &= e^{s\beta}\big((\nabla\psi)^T + s\nabla\beta\otimes\psi\big),\\
D\phi &= \frac{1}{2}(\nabla\phi + (\nabla\phi)^T) = e^{s\beta}\big(D\psi + \frac{s}{2}(\nabla\beta\otimes\psi + \psi\otimes\nabla\beta)\big)\\
\sigma(\phi,p)n &= 2D\phi\, n - pn = e^{s\beta}\big(2D\psi\, n + s(\nabla\beta\otimes\psi + \psi\otimes\nabla\beta)n\big) - pn\\
&= 2\rho_{\Gamma_i}^s D\psi\, n + s\,(\nabla\beta\otimes r' + r'\otimes\nabla\beta)\, n - pn \quad \text{on } \Sigma_i,
\end{aligned}$$

since $\psi = \rho_{\Gamma_i}^{-s}\, r'$ on Σ_i.

We set

$$M_1\psi = \psi' - 2s\nabla\psi\nabla\beta \quad \text{and} \quad M_2\psi = s\beta'\psi - \Delta\psi - s^2|\nabla\beta|^2\psi. \tag{2.3}$$

Thus the coupled system (1.4) can be rewritten in terms of (ψ, r) as follows:

$$\begin{aligned}
&M_1\psi + M_2\psi = f_s + s(\Delta\beta)\psi, \quad \operatorname{div}\psi = -s\nabla\beta\cdot\psi && \text{in} \quad Q,\\
&\psi = 0 && \text{on} \quad \Sigma_e,\\
&\psi = \rho_{\Gamma_i}^{-s}\, r' && \text{on} \quad \Sigma_i,\\
&\psi(0) = \psi(T) = 0 && \text{in} \quad \Omega,\\
&r'' + r = -2\rho_{\Gamma_i}^s \int_{\Gamma_i} D\psi\, n - s\int_{\Gamma_i}\big(r'\otimes\nabla\beta + \nabla\beta\otimes r'\big)\, n + \int_{\Gamma_i} pn && \text{in} \quad (0,T),\\
&r(0) = r^0 \text{ and } r'(0) = r^1.
\end{aligned} \tag{2.4}$$

3 Carleman inequality I

In this section, we prove the first version of the Carleman inequality for the transformed system (2.4). This is stated in Theorem 3.1. Writing the equation satisfied by ψ in the form $M_1\psi + M_2\psi = f_s + s(\Delta\beta)\psi$ is a crucial aspect of the proof. From the first equation of the system (2.4) it follows that

$$\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + 2(M_1\psi, M_2\psi)_{L^2(Q)} = \|f_s + s(\Delta\beta)\psi\|^2_{L^2(Q)}. \tag{3.1}$$

We begin by rewriting the cross term as follows

$$2(M_1\psi, M_2\psi)_{L^2(Q)} = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
&I_1 = 2\int_Q \big(s\beta'\psi - \Delta\psi - s^2|\nabla\beta|^2\psi\big)\cdot\psi', \quad I_2 = 4s\int_Q (\nabla\psi\nabla\beta)\cdot\Delta\psi,\\
&I_3 = 4s\int_Q \big(s^2|\nabla\beta|^2\psi - s\beta'\psi\big)\,(\nabla\psi\nabla\beta).
\end{aligned} \tag{3.2}$$

With calculations very similar to those in [19], we can transform I_1, I_2 and I_3 to arrive at the following identity

$$2(M_1\psi, M_2\psi)_{L^2(Q)} \quad = \quad J_1 + J_2 + J_3 + J_4 + J_5 + 2J_6, \tag{3.3}$$

where

$$J_1 = -4s^3 \int_Q \partial^2_{i,j}\beta\, \partial_j\beta\, \partial_i\beta\, |\psi|^2, \quad J_2 = 2s \int_\Sigma \partial_n\beta\, |\partial_n\psi|^2,$$

$$J_3 = 2s^2 \int_Q \beta'\, \Delta\beta\, |\psi|^2 - s \int_Q \beta''\, |\psi|^2 + 4s^2 \int_Q \nabla\beta' \cdot \nabla\beta |\psi|^2,$$

$$J_4 = 8 \int_0^T \Big| \int_{\Gamma_i} D\psi\, n \Big|^2 + 4 \int_0^T \left(\left(s\,\beta'\, r' + s \int_{\Gamma_i} \left(r' \otimes \nabla\beta + \nabla\beta \otimes r'\right) n - \int_{\Gamma_i} pn + r \right) \rho_{\Gamma_i}^{-s} \cdot \int_{\Gamma_i} D\psi\, n \right)$$
$$+2s^3 \int_{\Sigma_i} \left(\partial_n\beta\right)^3 |\psi|^2 - 2s^2 \int_{\Sigma_i} \beta'\, \partial_n\beta |\psi|^2 - 2s \int_{\Sigma_i} (\nabla\beta \cdot \psi)(\psi' \cdot n),$$

$$J_5 = -4s \int_Q \partial^2_{i,j}\beta\, \partial_j\psi_k\, \partial_i\psi_k\,, \quad J_6 = \int_Q \Big(s\Delta\beta\, |\nabla\psi|^2 - s^3 \Delta\beta\, |\nabla\beta|^2\, |\psi|^2 \Big).$$

The estimates of J_1, J_2, J_3 and J_5 can be performed as in [19]. With obvious minor adaptations we obtain

$$J_1 + J_3 \geq \frac{1}{2} C_1 s^3 \lambda^4 \int_{\Omega\times(0,T)} \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}} |\psi|^2 - Cs^3\lambda^4 \int_{\omega_0\times(0,T)} \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}} |\psi|^2,$$

$$J_2 = 2s \int_\Sigma \partial_n\beta\, |\partial_n\psi|^2 \geq 2s \int_{\Sigma_i} \frac{\lambda\, e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\, |\partial_n\psi|^2,$$

$$J_5 \geq -\frac{1}{2} \|M_2\,\psi\|^2_{L^2(Q)} - Cs^2\lambda^2 \int_Q \frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}}{t^{2k}\,(T-t)^{2k}}\, |\psi|^2 - Cs^2\lambda \int_Q \frac{e^{\lambda m K_1}}{t^{2k+1}\,(T-t)^{2k+1}}\, |\psi|^2$$
$$-Cs^3\lambda^3 \int_Q \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}} |\nabla\eta|^2\, |\psi|^2 - Cs\lambda \int_0^T \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^k(T-t)^k} \rho_{\Gamma_i}^{-s} r' \cdot \int_{\Gamma_i} \partial_n\psi,$$

for λ large and s large (depending on λ).

For J_6, following the calculations in [19], we can write that

$$J_6 \;\; \geq \;\; -Cs\lambda^4 \int_Q \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k} |\psi|^2 - Cs^2\lambda^2 \int_Q \frac{e^{\lambda m K_1}}{t^{2k+1}(T-t)^{2k+1}}\, |\psi|^2 - Cs^2\lambda^4 \int_Q \frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}}{t^{2k}(T-t)^{2k}}\, |\psi|^2$$
$$-\frac{1}{4} \int_Q |f_s|^2 - \frac{1}{4} \int_Q |M_1\psi|^2 + T_1 + T_2,$$

with

$$T_1 = s \int_0^T \left(\rho_{\Gamma_i}^{-s}\, \Delta\beta|_{\Gamma_i} \left(\int_{\Gamma_i} \partial_n\psi \right) \cdot r' \right) \quad \text{and} \quad T_2 = -\frac{s}{2} \int_{\Sigma_i} \partial_n(\Delta\beta)\, |\psi|^2.$$

Using the above estimates in (3.1), we obtain

$$
\begin{aligned}
&\|M_1\,\psi\|^2_{L^2(Q)} + \|M_2\;\psi\|^2_{L^2(Q)} + \frac{1}{2}C_1 s^3\lambda^4 \int_{(\Omega\setminus\omega_0)\times(0,T)} \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2 \\
&-Cs^3\lambda^4\int_{\omega_0\times(0,T)} \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2 - Cs\,\lambda^4\int_Q \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}|\psi|^2 - C\,s^2\lambda^2\int_Q \frac{e^{\lambda(K_1+\eta)}}{t^{2k+1}(T-t)^{2k+1}}\,|\psi|^2 \\
&-C\,s^2\,\lambda^4\int_Q \frac{e^{2\lambda\eta}}{t^{2k}(T-t)^{2k}}\,|\psi|^2 - \frac{1}{2}\,\|f_s\|^2_{L^2(Q)} - \frac{1}{2}\,\|M_1\psi\|^2_{L^2(Q)} \\
&+2\,T_1 + 2T_2 - \frac{1}{2}\,\|M_2\psi\|^2_{L^2(Q)} \;-C\;s^2\;\lambda^2\int_Q \frac{e^{2\lambda\eta}}{t^{2k}(T-t)^{2k}}\,|\psi|^2 \\
&-C\,s\,\lambda\int_0^T \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}\big|_{\Gamma_i}}{t^k(T-t)^k}\rho_{\Gamma_i}^{-s} r'\cdot\int_{\Gamma_i}(\partial_n\psi)n - C\,s\,\lambda^3\int_Q \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}|\nabla\eta|^2\;|\psi|^2 \\
&-C\,s^2\,\lambda\int_Q \frac{e^{\lambda(K_1+\eta)}}{t^{2k+1}(T-t)^{2k+1}}\,|\psi|^2 - \;C\,s^3\,\lambda^3\int_Q \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\nabla\eta|^2\,|\psi|^2 + J_4 \\
&\le\;2\|f_s\|^2_{L^2(Q)} \;+\;C\;s^2\;\lambda^4\int_Q \frac{e^{2\lambda\eta}}{t^{2k}(T-t)^{2k}}\;|\psi|^2.
\end{aligned}
$$

We decompose the integral $s^2\,\lambda^4\,\int_Q \frac{e^{2\lambda\eta}}{t^{2k}(T-t)^{2k}}\;|\psi|^2$ into two parts, one part over $(\Omega\setminus\omega_0)\times(0,T)$ and another one over $\omega_0\times(0,T)$. The integral over $(\Omega\setminus\omega_0)\times(0,T)$ can be absorbed in the most dominating term, namely

$$
\frac{1}{2}C_1 s^3\lambda^4\int_{(\Omega\setminus\omega_0)\times(0,T)} \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2,
$$

by choosing s large (depending on λ). The integral over $\omega_0\times(0,T)$ can be pushed to RHS and estimated from above by

$$
Cs^3\lambda^4\int_{\omega_0\times(0,T)} \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2.
$$

At the end of this process, we get the following estimate:

$$
\begin{aligned}
&\|M_1\,\psi\|^2_{L^2(Q)} \;+\;\|M_2\,\psi\|^2_{L^2(Q)} \;+\;s^3\lambda^4\int_Q \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\;|\psi|^2 + T_1 + T_2 + J_4 \\
&\le C\Big\{\|f_s\|^2_{L^2(Q)} + s^3\lambda^4\int_{\omega_0\times(0,T)} \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\;|\psi|^2 + s\lambda\int_0^T \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}\big|_{\Gamma_i}}{t^k(T-t)^k}\rho_{\Gamma_i}^{-s}\,r'\cdot\int_{\Gamma_i}\partial_n\,\psi\Big\}.
\end{aligned}
$$

3.1 Treatment of boundary terms

The effect of the fluid-solid interaction in our model is felt in the treatment of boundary terms which are different from the ones in other classical models. We will estimate these boundary terms in this section. This will make appear various quantities associated with the solid part (so far, we have been working in the fluid region). Let us begin by naming the different terms in J_4 as follows:

$$T_3 = 8\int_0^T \Big|\int_{\Gamma_i} D\psi\, n\Big|^2, \quad T_4 = 2s^3 \int_{\Sigma_i} \big(\partial_n\beta\big)^3\,|\psi|^2,$$

$$T_5 = 4\int_0^T \left(\left(s\,\beta'\,r' + s\int_{\Gamma_i}\big(r'\otimes\nabla\beta + \nabla\beta\otimes r'\big)\,n - \int_{\Gamma_i} pn + r\right)\,\rho_{\Gamma_i}^{-s}\cdot\int_{\Gamma_i} D\psi\, n\right),$$

$$T_6 = -2\,s^2\int_{\Sigma_i}\beta'\,\partial_n\beta|\psi|^2,$$

$$T_7 = -2s\int_{\Sigma_i}(\nabla\beta\cdot\psi)(\psi'\cdot n).$$

Estimate of T_4. First let us consider T_4 which can be expressed as (since $\psi = \rho_{\Gamma_i}^{-s} r'$ on Σ_i)

$$T_4 = 2s^3\lambda^3\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}\int_{\Gamma_i}|r'|^2 = 2s^3\lambda^3\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|\Gamma_i|\,|r'|^2.$$

Estimate of T_5. Next, we can estimate T_5 in the following way:

$$\begin{aligned}
|T_5| \le &\int_0^T \Big|\int_{\Gamma_i} D\psi\, n\Big|^2 + 4\int_0^T \Big|s\,\beta'\,r' + s\int_{\Gamma_i}\big(r'\otimes\nabla\beta+\nabla\beta\otimes r'\big)\,n - \int_{\Gamma_i} pn + r\Big|^2\,\rho_{\Gamma_i}^{-2s}\\
&\le \frac{1}{8}T_3 + 16s^2e^{2\lambda mK_1}T^2\int_0^T \frac{k^2}{t^{2k+2}(T-t)^{2k+2}}\rho_{\Gamma_i}^{-2s}|r'|^2\\
&\quad +16s^2\lambda^2\int_0^T \frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{2k}(T-t)^{2k}}\rho_{\Gamma_i}^{-2s}|\Gamma_i|^2|r'|^2 + 16\int_0^T \rho_{\Gamma_i}^{-2s}\,|r|^2 + 16\int_0^T \rho_{\Gamma_i}^{-2s}\,\Big|\int_{\Gamma_i} pn\Big|^2.
\end{aligned}$$

By choosing s large enough (depending on λ, $s \ge s_0(\lambda) = \lambda^{-3}e^{2mK_1}$) and choosing $k \ge 2$, we have

$$|T_5| \le \frac{1}{8}T_3 + \frac{1}{8}\,T_4 + 16\int_0^T \rho_{\Gamma_i}^{-2s}\,|r|^2 + 16\int_0^T \rho_{\Gamma_i}^{-2s}\,\Big|\int_{\Gamma_i} pn\Big|^2.$$

Estimate of T_1. Next, we can estimate T_1 as follows :

$$\begin{aligned}
|T_1| = s\Big|\int_0^T \rho_{\Gamma_i}^{-s}\,r'\,\Delta\beta|_{\Gamma_i}\cdot\int_{\Gamma_i}\partial_n\psi\Big| &\le \frac{1}{4}\int_0^T\Big|\int_{\Gamma_i}\partial_n\psi\Big|^2 + s^2\int_0^T \rho_{\Gamma_i}^{-2s}|\Delta\beta|_{\Gamma_i}|^2|r'|^2\\
&\le \frac{1}{8}J_2 + Cs^2\lambda^4\int_0^T \frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{2k}(T-t)^{2k}}\rho_{\Gamma_i}^{-2s}\,|r'|^2.
\end{aligned}$$

Once again we see that for large s (depending on λ, $s \ge 8\lambda$) we have $|T_1| \le \frac{1}{8}\,J_2 + \frac{1}{8}T_4$.

Estimate of T_2. To estimate T_2, we express it as

$$T_2 = -\frac{s}{2}\int_{\Sigma_i}\partial_n(\Delta\beta)\rho_{\Gamma_i}^{-2s}|r'|^2$$

in which we use the estimate (for λ large)

$$|\partial_n(\Delta\beta)| \le C\lambda^3\frac{e^{\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^k(T-t)^k} \text{ on } \Sigma_i.$$

This easily leads to $|T_2| \le \frac{1}{8}T_4$ for s large (depending on λ).

Estimate of T_6. Analogous arguments establish that

$$|T_6| \le Cs^2\lambda e^{\lambda m K_1} \int_0^T \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{2k+1}(T-t)^{2k+1}} \rho_{\Gamma_i}^{-2s} \int_{\Gamma_i} |r'|^2 \le \frac{1}{8}T_4.$$

for s large (depending on λ, $s \ge \lambda^{-3}e^{\lambda m K_1}$).

Assembling these estimates together, we obtain

$$|T_5| + |T_6| + |T_1| + |T_2| \le \frac{1}{8}T_3 + \frac{1}{2}T_4 + C\int_0^T \rho_{\Gamma_i}^{-2s}\,|r|^2 + C\int_0^T \rho_{\Gamma_i}^{-2s} \left|\int_{\Gamma_i} pn\right|^2 + \frac{1}{8}J_2.$$

Hence

$$T_1+T_2+J_2+J_4 \ge \frac{3}{4}T_3+\frac{1}{2}T_4+\frac{7s}{4}\int_{\Sigma_i} \frac{\lambda\, e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\,|\partial_n\psi|^2 - C\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 - C\int_0^T \rho_{\Gamma_i}^{-2s}\left|\int_{\Gamma_i} pn\right|^2 + T_7.$$

Our next task is to estimate T_3 from below. To this end, we use (2.4) and write

$$\int_{\Gamma_i} D\psi\, n = -\frac{1}{2}\rho_{\Gamma_i}^{-s}(r''+r) - \frac{s}{2}\rho_{\Gamma_i}^{-s}\int_{\Gamma_i}\left(r'\otimes\nabla\beta + \nabla\beta\otimes r'\right) n + \frac{1}{2}\rho_{\Gamma_i}^{-s}\int_{\Gamma_i} pn.$$

Hence

$$\left|\int_{\Gamma_i} D\psi\, n\right|^2 \ge \frac{1}{8}\rho_{\Gamma_i}^{-2s}|r''|^2 - \frac{3}{4}\rho_{\Gamma_i}^{-2s}|r|^2 - \frac{3}{4}s^2\lambda^2\rho_{\Gamma_i}^{-2s}\frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{2k}(T-t)^{2k}}\;|\Gamma_i|^2\;|r'|^2 - \frac{3}{4}\rho_{\Gamma_i}^{-2s}\left|\int_{\Gamma_i} pn\right|^2,$$

using the elementary inequality $|a+b|^2 \ge \frac{1}{2}|a|^2 - |b|^2$. It follows then, for s, λ large, that

$$\frac{T_3}{8} \ge \frac{1}{2}\int_0^T \rho_{\Gamma_i}^{-2s}(|r''|^2+|r|^2) - \frac{1}{4}T_4 - C\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 - C\int_0^T \rho_{\Gamma_i}^{-2s}\left|\int_{\Gamma_i} pn\right|^2.$$

As a consequence, we have

$$\frac{3}{4}T_3 + \frac{1}{2}T_4 \ge \frac{5}{8}T_3 + \frac{1}{4}T_4 + \frac{1}{2}\int_0^T \rho_{\Gamma_i}^{-2s}(|r''|^2+|r|^2) - C\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 - C\int_0^T \rho_{\Gamma_i}^{-2s}\left|\int_{\Gamma_i} pn\right|^2.$$

Thus the final estimate of the boundary terms is as follows:

$$\begin{aligned} T_1+T_2+J_2+J_4 \ge{}& 5\int_0^T \left|\int_{\Gamma_i} D\psi\, n\right|^2 + \frac{1}{2}s^3\lambda^3\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|r'|^2 \\ &+\frac{1}{2}\int_0^T \rho_{\Gamma_i}^{-2s}(|r''|^2+|r|^2) + s\int_{\Sigma_i} \frac{\lambda\, e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\,|\partial_n\psi|^2 \\ &-C\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 - C\int_0^T \rho_{\Gamma_i}^{-2s}\left|\int_{\Gamma_i} pn\right|^2 + T_7. \end{aligned}$$

Estimate of T_7. We have

$$\begin{aligned} \psi'\cdot n|_{\Gamma_i} &= \rho_{\Gamma_i}^{-s}r''\cdot n - s\rho_{\Gamma_i}^{-s}\beta'\, r'\cdot n \\ &= -2\left(\int_{\Gamma_i} D\psi\, n\right)\cdot n - s\rho_{\Gamma_i}^{-s}\left(\int_{\Gamma_i}\left(r'\otimes\nabla\beta+\nabla\beta\otimes r'\right) n\right)\cdot n \\ &\quad +\rho_{\Gamma_i}^{-s}\left(\int_{\Gamma_i} pn\right)\cdot n - \rho_{\Gamma_i}^{-s}\, r\cdot n - s\rho_{\Gamma_i}^{-s}\beta'\, r'\cdot n, \end{aligned}$$

and

$$
\begin{aligned}
T_7 &= -2s\int_{\Sigma_i}(\nabla\beta\cdot\psi)(\psi'\cdot n)\\
&= 4s\int_{\Sigma_i}(\nabla\beta\cdot\rho_{\Gamma_i}^{-s}\,r')\left(\int_{\Gamma_i}D\psi\,n\right)\cdot n + 2s^2\int_{\Sigma_i}(\nabla\beta\cdot\rho_{\Gamma_i}^{-s}\,r')\rho_{\Gamma_i}^{-s}\left(\int_{\Gamma_i}\left(r'\otimes\nabla\beta+\nabla\beta\otimes r'\right)n\right)\cdot n\\
&\quad -2s\int_{\Sigma_i}(\nabla\beta\cdot\rho_{\Gamma_i}^{-s}\,r')\rho_{\Gamma_i}^{-s}\left(\int_{\Gamma_i}pn\right)\cdot n + 2s\int_{\Sigma_i}(\nabla\beta\cdot\rho_{\Gamma_i}^{-s}\,r')\rho_{\Gamma_i}^{-s}\,r\cdot n + 2s^2\int_{\Sigma_i}(\nabla\beta\cdot\rho_{\Gamma_i}^{-s}\,r')\rho_{\Gamma_i}^{-s}\beta'\,r'\cdot n.
\end{aligned}
$$

We set

$$
\begin{aligned}
T_7^a &= 4s\int_{\Sigma_i}(\nabla\beta\cdot\rho_{\Gamma_i}^{-s}\,r')\left(\int_{\Gamma_i}D\psi\,n\right)\cdot n,\\
T_7^b &= 2s^2\int_{\Sigma_i}(\nabla\beta\cdot\rho_{\Gamma_i}^{-s}\,r')\rho_{\Gamma_i}^{-s}\left(\int_{\Gamma_i}\left(r'\otimes\nabla\beta+\nabla\beta\otimes r'\right)n\right)\cdot n,\\
T_7^c &= -2s\int_{\Sigma_i}(\nabla\beta\cdot\rho_{\Gamma_i}^{-s}\,r')\rho_{\Gamma_i}^{-s}\left(\int_{\Gamma_i}pn\right)\cdot n,\\
T_7^d &= 2s\int_{\Sigma_i}(\nabla\beta\cdot\rho_{\Gamma_i}^{-s}\,r')\rho_{\Gamma_i}^{-s}\,r\cdot n,\\
T_7^e &= 2s^2\int_{\Sigma_i}(\nabla\beta\cdot\rho_{\Gamma_i}^{-s}\,r')\rho_{\Gamma_i}^{-s}\beta'\,r'\cdot n.
\end{aligned}
$$

We have

$$
\begin{aligned}
&|T_7^a| \le 4s\int_{\Sigma_i}\frac{\lambda e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\rho_{\Gamma_i}^{-s}\,|r'|\left|\int_{\Gamma_i}D\psi\,n\right|\\
&\le 8s^2\lambda^2|\Gamma_i|\int_0^T\frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{2k}(T-t)^{2k}}\rho_{\Gamma_i}^{-2s}\,|r'|^2+\frac{1}{2}\int_0^T\left|\int_{\Gamma_i}D\psi\,n\right|^2,\\
&\left|T_7^b\right| \le 4s^2\int_{\Sigma_i}\frac{\lambda e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\rho_{\Gamma_i}^{-2s}\,|r'|\int_{\Gamma_i}|r'|\frac{\lambda e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\\
&\le 4s^2\lambda^2|\Gamma_i|\int_0^T\frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{2k}(T-t)^{2k}}\rho_{\Gamma_i}^{-2s}\,|r'|^2,\\
&|T_7^c| \le 2s\int_{\Sigma_i}\frac{\lambda e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\,\rho_{\Gamma_i}^{-2s}\,|r'|\left|\int_{\Gamma_i}pn\right|\\
&\le 2s^2\lambda^2\int_{\Sigma_i}\frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}}{t^{2k}(T-t)^{2k}}\,\rho_{\Gamma_i}^{-2s}\,|r'|^2+\frac{|\Gamma_i|}{2}\int_0^T\rho_{\Gamma_i}^{-2s}\,\left|\int_{\Gamma_i}pn\right|^2,
\end{aligned}
$$

$$
\left|T_7^d\right| \le 2s\int_{\Sigma_i}\frac{\lambda e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\,\rho_{\Gamma_i}^{-2s}\,|r'|\,|r| \le 2s^2\lambda^2\int_{\Sigma_i}\frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}}{t^{2k}(T-t)^{2k}}\rho_{\Gamma_i}^{-2s}|r'|^2+\frac{|\Gamma_i|}{2}\int_0^T\rho_{\Gamma_i}^{-2s}\,|r|^2,
$$

and

$$
\begin{aligned}
|T_7^e| &\le 2s^2\lambda\int_{\Sigma_i}\frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\,\rho_{\Gamma_i}^{-2s}\,|r'|^2\,\frac{Ce^{\lambda mK_1}}{t^{k+1}(T-t)^{k+1}}\\
&\le 2Cs^2\lambda e^{\lambda mK_1}\int_0^T\frac{e^{\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{2k+1}(T-t)^{2k+1}}\,\rho_{\Gamma_i}^{-2s}\,|r'|^2.
\end{aligned}
$$

Grouping together various estimates obtained, we can summarize the main inequality of section 3

$$\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + s^3\lambda^4 \int_Q \frac{e^{3\lambda\eta}}{t^{3k}(T-t)^{3k}}|\psi|^2 + \int_0^T \left|\int_{\Gamma_i} D\psi\, n\right|^2$$
$$+s\lambda \int_{\Sigma_i} \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\,|\partial_n\psi|^2 + \int_0^T \rho_{\Gamma_i}^{-2s}\left(|r''|^2+|r|^2\right) + s^3\lambda^3 \int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}\big|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|r'|^2$$
$$\leq C\left\{\|f_s\|^2_{L^2(Q)} + s^3\lambda^4 \int_{\omega_0\times(0,T)} \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2 + \int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 + \int_0^T \rho_{\Gamma_i}^{-2s}\left|\int_{\Gamma_i} pn\right|^2\right\}.$$

With calculations as in [19], we can also estimate $\nabla\psi$, ψ' and $\Delta\psi$ and we obtain the following theorem.

Theorem 3.1. (Carleman inequality I) *For λ sufficiently large, there is $s_0(\lambda) > 0$ such that for $s \geq s_0(\lambda)$ and for all solution (ψ, r) of system (2.4), we have*

$$\begin{aligned}
&s^{-1}\int_Q \xi^{-1}(|\psi'|^2+|\Delta\psi|^2) + \int_Q |M_1\psi|^2 + \int_Q |M_2\psi|^2 + s\lambda^2 \int_Q \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}|\nabla\psi|^2\\
&+\int_0^T \left|\int_{\Gamma_i} D\psi\, n\right|^2 + s^3\lambda^4 \int_Q \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2 + s\lambda \int_{\Sigma_i} \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\,|\partial_n\psi|^2\\
&+\int_0^T \rho_{\Gamma_i}^{-2s}(|r''|^2+|r|^2) + s^3\lambda^3 \int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|r'|^2 + \int_Q \rho^{-2s}|\nabla p|^2\\
&\leq C\left\{\int_Q \rho^{-2s}|\nabla p|^2 + s^3\lambda^4 \int_{\omega_1\times(0,T)} \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2\right.\\
&\qquad\left.+\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 + \int_0^T \rho_{\Gamma_i}^{-2s}\left|\int_{\Gamma_i} pn\right|^2\right\},
\end{aligned} \tag{3.4}$$

where $\omega_0 \subset\subset \omega_1 \subset\subset \omega$.

At this stage the pressure p is determined up to an additive constant. From now on, we choose the pressure p in the space of functions satisfying the condition

$$\int_\omega e^{-2s\widehat{\beta}(t)}\,|\xi(x,t)|^2\,\zeta(x)\,p(x,t)\,dx = 0 \quad \text{for a.e. } t\in(0,T), \tag{3.5}$$

where

$$\widehat{\beta}(t) = \min_{x\in\overline{\Omega}}\beta(x,t) = \frac{e^{\lambda m K_1} - e^{\lambda(1+m)\|\eta\|_\infty}}{t^k(T-t)^k},$$

$\zeta \in C_0^\infty(\Omega)$ is a nonnegative function, supp$\zeta \subset \omega_3$, $\omega_1 \subset\subset \omega_2 \subset\subset \omega_3 \subset\subset \omega$, and $\zeta|_{\omega_2} = 1$. (The open set ω_2 is introduced in the next section.) Condition (3.5) will be used in an essential way in the proof of Theorem 10.1 .

Let us notice that

$$e^{-2s\widehat{\beta}(t)} \geq e^{-2s\beta(x,t)} = \rho^{-2s}(x,t) \quad \text{for all } (x,t)\in\overline{Q}.$$

Moreover we can notice that condition (3.5) is equivalent to

$$\int_\omega e^{2\lambda(\eta(x)+m\|\eta\|_\infty)}\,\zeta(x)\,p(x,t)\,dx = 0 \quad \text{for a.e. } t\in(0,T),$$

and the weight function $e^{2\lambda(\eta(x)+m\|\eta\|_\infty)}\,\zeta(x)$ does not depend on t. We are going to use this property by observing that the mapping

$$p \longmapsto \left(\int_\omega e^{2\lambda(\eta(x)+m\|\eta\|_\infty)}\,\zeta(x)\,p(x)\,dx + \int_\Omega |\nabla p|^2 dx\right)^{1/2}$$

is a norm on $H^1(\Omega)$ equivalent to the usual one.

4 Carleman inequality for the pressure

In this section we recall some results obtained by O. Yu. Imanuvilov, J.-P. Puel in [15]. Using the fact that the pressure $p(t) \in H^1(\Omega)$ is the solution of the following elliptic problem

$$-\Delta p(t) = 0 \qquad \text{in} \quad \Omega \quad \text{and} \quad p(t) = p(t) \qquad \text{on} \quad \Gamma\,,$$

it follows that

$$\begin{aligned}&\int_\Omega e^{2\tau e^{\lambda\eta}}\,|\nabla p(t)|^2 + \tau^2\,\lambda^2 \int_\Omega e^{2\tau e^{\lambda\eta}}\,e^{2\lambda\eta}\,|p(t)|^2 \\ &\leq C\int_{\omega_1} e^{2\tau e^{\lambda\eta}}\,|\nabla p(t)|^2 + C\tau^2\,\lambda^2 \int_{\omega_1} e^{2\tau e^{\lambda\eta}}\,e^{2\lambda\eta}\,|p(t)|^2 + C\tau^{1/2}\,e^{2\tau}\,\|p(t)\|^2_{H^{1/2}(\Gamma)}\,.\end{aligned} \tag{4.1}$$

Using a localisation argument as in [15], [9], we can eliminate the term $\nabla p(t)$ in the right hand side, and we obtain

$$\begin{aligned}&\int_\Omega e^{2\tau e^{\lambda\eta}}\,|\nabla p(t)|^2 + \tau^2\,\lambda^2 \int_\Omega e^{2\tau e^{\lambda\eta}}\,e^{2\lambda\eta}\,|p(t)|^2 \\ &\leq C\tau^2\,\lambda^2 \int_{\omega_2} e^{2\tau e^{\lambda\eta}}\,e^{2\lambda\eta}\,|p(t)|^2 + C\tau^{1/2}\,e^{2\tau}\,\|p(t)\|^2_{H^{1/2}(\Gamma)},\end{aligned} \tag{4.2}$$

with $\omega_1 \subset\subset \omega_2 \subset\subset \omega$.

To deduce space-time integral estimates from (4.2), we have to choose τ as a function of time in an appropriate way, and we have to choose a multiplier for the estimate (4.2) and integrate with respect to time. We choose

$$\tau = \frac{s}{t^k(T-t)^k} e^{\lambda m\|\eta\|_\infty}.$$

Taking

$$e^{\frac{-2s}{t^k(T-t)^k} e^{\lambda m K_1}}\,,$$

as multiplier, we can check that:

$$e^{2\tau e^{\lambda\eta}}\,e^{\frac{-2s}{t^k(T-t)^k} e^{\lambda m K_1}} = e^{\frac{-2s}{t^k(T-t)^k}(e^{\lambda m K_1} - e^{\lambda(\eta+m\|\eta\|_\infty)})} = e^{-2s\beta}\,.$$

Thus, from estimate (4.2), we deduce

$$\begin{aligned}&\int_Q e^{-2s\beta}\,|\nabla p(t)|^2 dt + \lambda^2\,s^2 \int_Q e^{-2s\beta}\,|p(t)|^2 dt \\ &\leq C\lambda^2 \int_{\omega_2\times(0,T)} \tau^2\,e^{-2s\beta}\,e^{2\lambda\eta}\,|p(t)|^2 dt + C\int_0^T \tau^{1/2}\,e^{\frac{-2s}{t^k(T-t)^k} e^{\lambda m K_1}}\,e^{2\tau}\,\|p(t)\|^2_{H^{1/2}(\Gamma)} dt\,.\end{aligned} \tag{4.3}$$

We introduce the constant

$$\eta_* = \min_{x\in\overline{\Omega}} \eta(x),$$

and the functions

$$\beta^*(t) = \max_{x\in\overline{\Omega}} \beta(x,t) = \frac{e^{\lambda m K_1} - e^{\lambda(\eta_*+m\|\eta\|_\infty)}}{t^k(T-t)^k},$$

$$\xi^*(t) = \min_{x\in\overline{\Omega}} \xi(x,t) = \frac{e^{\lambda(\eta_*+m\|\eta\|_\infty)}}{t^k(T-t)^k},$$

$$\widehat{\xi}(t) = \max_{x\in\overline{\Omega}} \xi(x,t) = \frac{e^{\lambda(m+1)\|\eta\|_\infty}}{t^k(T-t)^k}.$$

Let us verify that

$$\tau^{1/2}\, e^{\frac{-2s}{t^k(T-t)^k} e^{\lambda m K_1}}\, e^{2\tau} \le s^{1/2}\, e^{-2s\beta^*}\, (\xi^*)^{1/2}\,. \tag{4.4}$$

We have:

$$\tau^{1/2}\, e^{\frac{-2s}{t^k(T-t)^k} e^{\lambda m K_1}}\, e^{2\tau} = \frac{s^{1/2}}{t^{k/2}(T-t)^{k/2}}\, e^{\lambda m\|\eta\|_\infty/2}\, e^{\frac{-2s}{t^k(T-t)^k}\, e^{\lambda m K_1}}\, e^{\frac{2s}{t^k(T-t)^k} e^{\lambda m\|\eta\|_\infty}}\,,$$

and

$$s^{1/2}\, e^{-2s\beta^*}\, (\xi^*)^{1/2} = s^{1/2}\, e^{\frac{-2s}{t^k(T-t)^k}(e^{\lambda m K_1} - e^{\lambda(\eta_*+m\|\eta\|_\infty)})}\, \frac{e^{\lambda/2(\eta_*+m\|\eta\|_\infty)}}{t^{k/2}(T-t)^{k/2}}\,.$$

Thus inequality (4.4) is satisfied and we have

$$\begin{aligned} &\int_Q e^{-2s\beta}\, |\nabla p(t)|^2 dt \\ &\le C\lambda^2\, s^2 \int_{\omega_2\times(0,T)} \xi^2\, e^{-2s\beta}\, |p(t)|^2 dt + C\, s^{1/2} \int_0^T (\xi^*)^{1/2}\, e^{-2s\beta^*}\, \|p(t)\|^2_{H^{1/2}(\Gamma)} dt\,. \end{aligned} \tag{4.5}$$

Let us notice that $\eta|_{\Gamma_i} \le \eta$, and therefore $\beta|_{\Gamma_i} \ge \beta$ and $e^{-2s\beta}|_{\Gamma_i} \le e^{-2s\beta}$. Thus from (4.3) it follows that

$$\begin{aligned} &\int_0^T e^{-2s\beta}|_{\Gamma_i}\, \|p(t)\|^2_{L^2(\Gamma_i)} dt \\ &\le C\lambda^2\, s^2 \int_{\omega_2\times(0,T)} \xi^2\, e^{-2s\beta}\, |p(t)|^2 dt + C\, s^{1/2} \int_0^T (\xi^*)^{1/2}\, e^{-2s\beta^*}\, \|p(t)\|^2_{H^{1/2}(\Gamma)} dt\,. \end{aligned} \tag{4.6}$$

5 Trace estimate of the pressure

The objective of this section is to estimate the term

$$\int_0^T (\xi^*)^{1/2}\, e^{-2s\beta^*}\, \|p(t)\|^2_{H^{1/2}(\Gamma)} dt$$

appearing in (4.5) and (4.6) and to prove Theorem 5.1 stated at the end of this section. For that, we introduce the functions

$$\mu(t) = e^{-s\beta^*(t)}(\xi^*(t))^{1/4}, \quad \phi^* = \mu\, \phi \quad \text{and} \quad p^* = \mu\, p.$$

By an easy calculation we can check that (ϕ^*, p^*, r) is the solution to the system

$$\begin{aligned}
&\phi^{*\prime} - \Delta\phi^* + \nabla p^* = \mu'\,\phi \quad \text{and} \quad \operatorname{div}\phi^* = 0 && \text{in} \quad Q,\\
&\phi^* = 0 && \text{on} \quad \Sigma_e,\\
&\phi^* = \mu\, r' && \text{on} \quad \Sigma_i,\\
&\phi^*(0) = 0 && \text{in} \quad \Omega,\\
&r'' + r = -\int_{\Gamma_i} \sigma(\phi, p)\, n && \text{in} \quad (0,T),\\
&r(0) = r^0 \qquad \text{and} \qquad r'(0) = r^1\,.
\end{aligned} \tag{5.1}$$

Step 1. Rewriting system (5.1) *in terms of* $P\phi^*$ *and* $(I-P)\phi^*$. We set

$$m(\phi, p) = -\int_{\Gamma_i} \sigma(\phi, p)\, n, \quad m(\phi^*, p^*) = -\mu \int_{\Gamma_i} \sigma(\phi, p)\, n = -\int_{\Gamma_i} \sigma(\phi^*, p^*)\, n,$$

$$\phi_e = P\phi, \quad \phi_s = (I-P)\phi, \quad p = p_e + p_s,$$

and

$$\phi_e^* = \mu\,\phi_e, \quad \phi_s^* = \mu\,\phi_s, \quad p_e^* = \mu\, p_e, \quad p_s^* = \mu\, p_s,$$

where p_e is the pressure appearing in the equation satisfied by ϕ_e and p_s is the pressure associated with ϕ_s (see [21]). More precisely, we denote by $q(t) = N(r' \cdot n) \in H^1(\Omega)$, the solution to the Neumann boundary value problem

$$\int_\omega e^{2\lambda(\eta + m\|\eta\|_\infty)}\, \zeta\, q(t)\, dx = 0, \quad \Delta q(t) = 0 \quad \text{in } \Omega, \quad \frac{\partial q}{\partial n} = 0 \quad \text{on } \Gamma_e, \quad \frac{\partial q}{\partial n} = r' \cdot n \quad \text{on } \Gamma_i.$$

From [21], it follows that $p_s = -q_t$. Therefore, $\int_\omega e^{2\lambda(\eta + m\|\eta\|_\infty)}\, \zeta\, p_s(t)\, dx = 0$. The pressure $p_e^* = \mu\, p_e$ is determined by

$$\phi_e^{*\prime} - \Delta\phi_e^* + \nabla {p_e}^* = \mu'\,\phi \quad \text{in} \quad Q \quad \text{and} \quad \int_\Omega e^{2\lambda(\eta + m\|\eta\|_\infty)}\, \zeta\, p_e^*\, dx = 0 \quad \text{for a.e. } t \in (0,T).$$

Denoting by γ_i the trace operator on Γ_i, we have

$$p_s|_{\Gamma_i} = -\gamma_i N(r'' \cdot n).$$

Now we introduce the operator $K \in \mathcal{L}(\mathbb{R}^2)$ defined by

$$Kr = \int_{\Gamma_i} \gamma_i N(r \cdot n) n.$$

We can easily verify that $K = K^* \geq 0$ and that $I + K$ is an automorphism in $\mathbb{R}^2$.

Let us denote by $A = P\Delta$ the Stokes operator (as an unbounded operator in $V_n^0(\Omega)$). To rewrite system (1.4) in terms of ϕ_e and ϕ_s, we introduce the operator $L \in \mathcal{L}(V^0(\Gamma), V^0(\Omega))$ defined by $Lg = w$, where

$$-\Delta w + \nabla\pi = 0 \quad \text{and} \quad \operatorname{div} w = 0 \quad \text{in } \Omega, \quad w = g \quad \text{on } \Gamma.$$

Following [21], we rewrite system (1.4) (with $f = 0$) as follows

$$\begin{aligned} &\phi_e' - A\phi_e = (-A)PL(r'\chi_{\Gamma_i}), \quad \phi_e(0) = P\phi^0, \\ &\phi_s = (I-P)L(r'\chi_{\Gamma_i}), \\ &r'' + r = -\int_{\Gamma_i} \sigma(\phi_e, p_e)\, n - \int_{\Gamma_i} D\phi_s\, n - \int_{\Gamma_i} \gamma_i N(r''\cdot n)n, \qquad \text{in} \quad (0,T), \\ &r(0) = r^0 \qquad \text{and} \qquad r'(0) = r^1\,. \end{aligned} \tag{5.2}$$

The equation satisfied by r can be rewritten in the form

$$(I+K)r'' + r = -\int_{\Gamma_i} \sigma(\phi_e, p_e)\, n - \int_{\Gamma_i} D\phi_s\, n.$$

The equation for ϕ_e^* is

$$\phi_e^{*\prime} - A\phi_e^* = (-A)PL(\mu\, r'\chi_{\Gamma_i}) + \mu'\,\phi_e, \quad \phi_e^*(0) = 0,$$

and ϕ_s^* obeys

$$\phi_s^* = (I-P)L(\mu\, r'\chi_{\Gamma_i}).$$

From [21, Proposition 2.2], it follows that

$$\|\phi_e^*\|_{H^{2,1}(Q)} + \|p_e^*\|_{L^2(0,T;H^1(\Omega))} + \|\phi_s^*\|_{L^2(0,T;H^2(\Omega))} \le C\left(\|\mu'\,\phi_e\|_{L^2(Q)} + \|\mu\, r'\|_{H^{3/4}(0,T)}\right).$$

Since $\phi_e = P\phi$, we have

$$\|\mu'\,\phi_e\|_{L^2(Q)} \le \|\mu'\,\phi\|_{L^2(Q)}.$$

Step 2. Estimate of $\|p_s^*\|_{L^2(0,T;H^1(\Omega))}$. Since $p_s(t) = -N(r''\cdot n)$, we have

$$\|p_s^*(t)\|_{H^1(\Omega)} \le C\mu(t)|r''(t)| \le C(|\mu(t)r(t)| + \|\phi_e^*(t)\|_{H^2(\Omega)} + \|\nabla p_e^*(t)\|_{L^2(\Omega)} + \|\phi_s^*(t)\|_{H^2(\Omega)}).$$

Thus, we obtain

$$\begin{aligned} &\|p_s^*\|_{L^2(0,T;H^1(\Omega))} \le C(\|\mu\, r\|_{L^2(0,T)} + \|\phi_e^*\|_{H^{2,1}(Q)} + \|\nabla p_e^*\|_{L^2(0,T;L^2(\Omega))} + \|\phi_s^*\|_{L^2(0,T;H^2(\Omega))}) \\ &\le C\left(\|\mu'\,\phi_e\|_{L^2(Q)} + \|\mu\, r'\|_{H^{3/4}(0,T)} + \|\mu\, r\|_{L^2(0,T)}\right). \end{aligned}$$

Step 3. Estimate of $\|\mu\, r'\|_{H^{3/4}(0,T)}$. Now, we want to eliminate the term $\|\mu\, r'\|_{H^{3/4}(0,T)}$ from the previous estimates. For that, we are going to use the interpolation inequality

$$\|\mu\, r'\|_{H^{3/4}(0,T)} \le C\|\mu\, r'\|_{L^2(0,T)}^{1/4}\|\mu\, r'\|_{H^1(0,T)}^{3/4}.$$

Let us now calculate $\|\mu\, r'\|_{H^1(0,T)}$. We have

$$(\mu\, r')' = \mu'\, r' + \mu r''.$$

For the term $\mu r''$ we use the equation satisfied by r:

$$\mu r'' = -\mu(I+K)^{-1}r + (I+K)^{-1}m(\phi_e^*, p_e^*) + (I+K)^{-1}\int_{\Gamma_i} D\phi_s^*\, n.$$

With classical majorations we have:

$$\|\mu\, r'\|_{H^{3/4}(0,T)} \le C\|\mu\, r'\|^{1/4}_{L^2(0,T)}\|\mu\, r'\|^{3/4}_{H^1(0,T)}$$

$$\le C\|\mu\, r'\|^{1/4}_{L^2(0,T)}\Big(\|\mu\, r'\|^{3/4}_{L^2(0,T)} + \|\mu'\, r'\|^{3/4}_{L^2(0,T)} + \|\mu\, r\|^{3/4}_{L^2(0,T)}$$

$$+\|m(\phi_e^*, p_e^*)\|^{3/4}_{L^2(0,T)} + \Big\|\int_{\Gamma_i} D\phi_s^*\, n\Big\|^{3/4}_{L^2(0,T)}\Big)$$

$$\le C\|\mu\, r'\|_{L^2(0,T)} + C\|\mu\, r\|_{L^2(0,T)} + C\|\mu'\, r'\|_{L^2(0,T)} + \frac{C}{\varepsilon^3}\|\mu\, r'\|_{L^2(0,T)} + C\varepsilon\|m(\phi_e^*, p_e^*)\|_{L^2(0,T)}$$

$$+C\varepsilon\Big\|\int_{\Gamma_i} D\phi_s^*\, n\Big\|_{L^2(0,T)}.$$

Thus

$$\|\phi_e^*\|_{H^{2,1}(Q)} + \|p_e^*\|_{L^2(0,T;H^1(\Omega))} + \|\phi_s^*\|_{L^2(0,T;H^2(\Omega))} + \|p_s^*\|_{L^2(0,T;H^1(\Omega))}$$

$$\le C\left(\|\mu'\,\phi\|_{L^2(Q)} + \|\mu\, r'\|_{L^2(0,T)} + \|\mu\, r\|_{L^2(0,T)} + \|\mu'\, r'\|_{L^2(0,T)}\right.$$

$$\left.+\frac{1}{\varepsilon^3}\|\mu\, r'\|_{L^2(0,T)} + \varepsilon\|m(\phi_e^*, p_e^*)\|_{L^2(0,T)} + C\varepsilon\Big\|\int_{\Gamma_i} D\phi_s^*\, n\Big\|_{L^2(0,T)}\right).$$

We can choose $\varepsilon > 0$ to have

$$\|\phi_e^*\|_{H^{2,1}(Q)} + \|p_e^*\|_{L^2(0,T;H^1(\Omega))} + \|\phi_s^*\|_{L^2(0,T;H^2(\Omega))} + \|p_s^*\|_{L^2(0,T;H^1(\Omega))}$$

$$\le C\left(\|\mu'\,\phi\|_{L^2(Q)} + \|\mu\, r'\|_{L^2(0,T)} + \|\mu\, r\|_{L^2(0,T)} + \|\mu'\, r'\|_{L^2(0,T)}\right).$$

Step 4. Estimate of $\|p^*\|_{L^2(0,T;H^{1/2}(\Gamma))}$. We have

$$\mu'(t) = -se^{-s\beta^*(t)}(\xi^*(t))^{1/4}\,(\beta^*)'(t) + \frac{1}{4}e^{-s\beta^*(t)}(\xi^*(t))^{-3/4}(\xi^*)'(t)$$

$$= -se^{-s\beta^*(t)}(\xi^*(t))^{1/4}\,k(2t-T)\frac{e^{\lambda m K_1} - e^{\lambda(\eta_* + m\|\eta\|_\infty)}}{t^{k+1}(T-t)^{k+1}}$$

$$+\frac{1}{4}e^{-s\beta^*(t)}(\xi^*(t))^{-3/4}k(2t-T)\frac{e^{\lambda(\eta_* + m\|\eta\|_\infty)}}{t^{k+1}(T-t)^{k+1}}.$$

$$|\mu'(t)|^2 \le Cs^2e^{-2s\beta^*(t)}|\xi^*(t)|^{1/2}\frac{(e^{\lambda m K_1} - e^{\lambda(\eta_* + m\|\eta\|_\infty)})^2}{t^{2k+2}(T-t)^{2k+2}} + Ce^{-2s\beta^*(t)}|\xi^*(t)|^{-3/2}\frac{e^{2\lambda(\eta_* + m\|\eta\|_\infty)}}{t^{2k+2}(T-t)^{2k+2}}$$

$$\le Cs^2e^{-2s\beta^*(t)}\frac{e^{\lambda(\eta_* + m\|\eta\|_\infty)/2 + 2\lambda m K_1}}{t^{\frac{5k}{2}+2}(T-t)^{\frac{5k}{2}+2}}$$

$$+Ce^{-2s\beta^*(t)}\frac{e^{\lambda(\eta_* + m\|\eta\|_\infty)/2}}{t^{2k+\frac{1}{2}}(T-t)^{2k+\frac{1}{2}}}$$

$$\le Cs^2e^{-2s\beta^*(t)}\frac{e^{3\lambda(\eta_* + m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}.$$

Thus

$$\|\mu'\,\phi\|^2_{L^2(Q)} \le C\int_Q |\mu'|^2\,|\phi|^2$$

$$\le Cs^2\int_Q e^{-2s\beta^*(t)}\frac{e^{3\lambda(\eta_* + m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\,|\phi|^2 \le Cs^2\int_Q e^{-2s\beta(t)}\frac{e^{3\lambda(\eta + m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\,|\phi|^2.$$

We finally obtain

$$
\begin{aligned}
&s^{1/2}\int_0^T (\xi^*)^{1/2}\, e^{-2s\beta^*}\, \|p\|^2_{H^{1/2}(\Gamma)} = s^{1/2}\|p^*\|^2_{L^2(0,T;H^{1/2}(\Gamma))} \\
&\le C\, s^{5/2}\int_Q e^{-2s\beta}\frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\,|\phi|^2 + Cs^{5/2}\int_0^T e^{-2s\beta^*(t)}\frac{e^{3\lambda(\eta_*+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\,|r'|^2 \qquad (5.3)\\
&\qquad\qquad +Cs^{1/2}\|\mu\, r\|^2_{L^2(0,T)}.
\end{aligned}
$$

Substituting in estimate (4.5), it yields:

$$
\begin{aligned}
&\int_Q e^{-2s\beta}\,|\nabla p|^2 dt \\
&\le C\lambda^2\, s^2\int_{\omega_2\times(0,T)}\xi^2\, e^{-2s\beta}\,|p|^2 + C\, s^{5/2}\int_Q e^{-2s\beta}\frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\,|\phi|^2 \qquad (5.4)\\
&\quad +Cs^{5/2}\int_0^T e^{-2s\beta^*(t)}\frac{e^{3\lambda(\eta_*+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\,|r'|^2 + Cs^{1/2}\|\mu\, r\|^2_{L^2(0,T)}.
\end{aligned}
$$

Step 5. Last estimates. Combining this inequality with the one obtained in (3.4), we notice that the term

$$
C\, s^{5/2}\int_Q e^{-2s\beta}\frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\,|\phi|^2
$$

can be absorbed by the term

$$
s^3\,\lambda^4\int_Q \rho^{-2s}\frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\phi|^2 = s^3\,\lambda^4\int_Q \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2
$$

in the left hand side of (3.4), and the term

$$
Cs^{5/2}\int_0^T e^{-2s\beta^*(t)}\frac{e^{3\lambda(\eta_*+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\,|r'|^2
$$

can be absorbed by the term

$$
s^3\lambda^3\int_0^T \rho_{\Gamma_i}^{-2s}\frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\,|r'|^2.
$$

We finally arrive at

$$
\begin{aligned}
&s^{-1}\int_Q \xi^{-1}(|\psi'|^2+|\Delta\psi|^2) + \int_Q |M_1\psi|^2 + \int_Q |M_2\psi|^2 + s\lambda^2\int_Q \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}|\nabla\psi|^2 \\
&+\int_0^T\left|\int_{\Gamma_i} D\psi\, n\right|^2 + s^3\lambda^4\int_Q \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2 + s\lambda\int_{\Sigma_i}\frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\,|\partial_n\psi|^2 \\
&+\int_0^T \rho_{\Gamma_i}^{-2s}(|r''|^2+|r|^2) + s^3\lambda^3\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|r'|^2 + \int_Q \rho^{-2s}|\nabla p|^2 \qquad (5.5)\\
&\le C\Bigg\{ s^3\lambda^4\int_{\omega\times(0,T)}\frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2 \\
&\qquad +s^2\lambda^2\int_{\omega_2\times(0,T)}\rho^{-2s}\,\xi^2|p|^2 + s^{1/2}\|\mu\, r\|^2_{L^2(0,T)} + \int_0^T \rho_{\Gamma_i}^{-2s}|r|^2\Bigg\}.
\end{aligned}
$$

Since

$$\mu(t) \le |\xi^*|^{1/4} \rho_{\Gamma_i}^{-s}(t),$$

the two last terms can be estimated by $2s^{1/2} \int_0^T |\xi^*|^{1/2} \rho_{\Gamma_i}^{-2s} |r|^2$. But we are going to face a new difficulty in section 9. We shall have to estimate the term

$$\int_0^T |\widehat{\xi}|^3 \, e^{-2s\beta^*} |r|^2.$$

A priori this term cannot be easily estimated by the terms in the LHS of (5.5). However, we are going to see that such an estimate is possible via a compactness argument. For that we first add the term $s^{1/2} \int_0^T |\widehat{\xi}|^3 \, \rho_{\Gamma_i}^{-2s} |r|^2$ in both sides of (5.5), and we obtain the following theorem.

Theorem 5.1. *Consider the coupled system* (1.4). *Then there exist positive constants* λ_0, $s_0(\lambda)$ *such that the following inequality holds for all* $\lambda \ge \lambda_0, s \ge s_0(\lambda)$ *and for all solutions* (ϕ, r) *of the system* (1.4)*:*

$$\begin{aligned}
& s^{-1} \int_Q \xi^{-1} (|\psi'|^2 + |\Delta\psi|^2) + \int_Q |M_1 \psi|^2 + \int_Q |M_2 \psi|^2 + s\lambda^2 \int_Q \frac{e^{\lambda(\eta + m\|\eta\|_\infty)}}{t^k (T-t)^k} |\nabla \psi|^2 \\
& + \int_0^T \left| \int_{\Gamma_i} D\psi \, n \right|^2 + s^3 \lambda^4 \int_Q \frac{e^{3\lambda(\eta + m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}} |\psi|^2 + s\lambda \int_{\Sigma_i} \frac{e^{\lambda(\eta + m\|\eta\|_\infty)}}{t^k (T-t)^k} |\partial_n \psi|^2 \\
& + \int_0^T \rho_{\Gamma_i}^{-2s} |r''|^2 + s^3 \lambda^3 \int_0^T \frac{e^{3\lambda(\eta + m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}} \rho_{\Gamma_i}^{-2s} |r'|^2 + \int_Q \rho^{-2s} |\nabla p|^2 \\
& + s^{1/2} \int_0^T |\widehat{\xi}|^3 \, \rho_{\Gamma_i}^{-2s} |r|^2 \\
& \le C \left\{ s^3 \lambda^4 \int_{\omega \times (0,T)} \frac{e^{3\lambda(\eta + m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}} |\psi|^2 + s^2 \lambda^2 \int_{\omega_2 \times (0,T)} \rho^{-2s} \, \xi^2 |p|^2 + s^{1/2} \int_0^T |\widehat{\xi}|^3 \, \rho_{\Gamma_i}^{-2s} |r|^2 \right\}.
\end{aligned} \tag{5.6}$$

If we compare the above estimate with the one of Theorem 3.1, we can observe that the gradient and the trace of the pressure have been removed from the RHS of the inequality, only a local term of the pressure is still remaining. But for that it has been necessary to modify the weight in the term involving r.

6 Estimate of r

Our goal in the next two sections is to strengthen the above inequality (5.6) by removing the term $\int_0^T \theta^2 |r|^2$ from the RHS, where $\theta = |\widehat{\xi}|^{3/2} \, \rho_{\Gamma_i}^{-s}$. This signifies that the observability of the whole system is possible without making any observation on the solid. A priori this is not obvious.

In [19], we have used the analogue of equation (2.4) to obtain an estimate of the term $\int_0^T \rho_{\Gamma_i}^{-2s} |r|^2$ in the right hand side of (5.5). Due to the presence of the pressure term, such an idea does not seem to work, as shown below. To overcome this difficulty, we present arguments based on a combination of monotonicity and compacteness in this section and the next one.

6.1 First estimate of r

If we multiply the equation satisfied by r in (2.4) by $\rho_{\Gamma_i}^{-2s} r$ we obtain

$$\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 = -\int_0^T r''\,\rho_{\Gamma_i}^{-2s} r - 2\int_0^T r\rho_{\Gamma_i}^{-s}\int_{\Gamma_i} D\psi\, n$$
$$-s\int_0^T r\rho_{\Gamma_i}^{-2s}\int_{\Gamma_i}\left(r'\otimes\nabla\beta+\nabla\beta\otimes r'\right)n + \int_0^T r\rho_{\Gamma_i}^{-2s}\int_{\Gamma_i} pn.$$

With an integration by parts and Cauchy-Schwarz and Young inequalities we have

$$\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2$$
$$\leq \int_0^T \rho_{\Gamma_i}^{-2s}|r'|^2 - 2s\int_0^T \rho_{\Gamma_i}^{-2s}\beta'|_{\Gamma_i} r'\cdot r + \varepsilon\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 + \frac{1}{\varepsilon}\int_0^T \left|\int_{\Gamma_i} D\psi\, n\right|^2$$
$$+\varepsilon\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 + \frac{2s^2\lambda^2}{\varepsilon}\int_0^T \rho_{\Gamma_i}^{-2s}\frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{2k}(T-t)^{2k}}|\Gamma_i||r'|^2 + \frac{\varepsilon}{2}\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 + \frac{1}{2\varepsilon}\int_0^T\int_0^T \rho_{\Gamma_i}^{-2s}\left|\int_{\Gamma_i} pn\right|^2$$
$$\leq \int_0^T \rho_{\Gamma_i}^{-2s}|r'|^2 + \frac{7\varepsilon}{2}\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 + \frac{s^2}{2\varepsilon}\int_0^T \rho_{\Gamma_i}^{-2s}|r'|^2|\beta'|_{\Gamma_i}|^2 + \frac{1}{\varepsilon}\int_0^T \left|\int_{\Gamma_i} D\psi\, n\right|^2$$
$$+\frac{2s^2\lambda^2}{2\varepsilon}\int_0^T \rho_{\Gamma_i}^{-2s}\frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{2k}(T-t)^{2k}}|\Gamma_i||r'|^2 + \frac{1}{2\varepsilon}\int_0^T\int_0^T \rho_{\Gamma_i}^{-2s}\left|\int_{\Gamma_i} pn\right|^2.$$

Since $|\beta'|_{\Gamma_i}| \leq \frac{e^{\lambda m K_1}}{t^k(T-t)^k}$, by choosing $\varepsilon = 1/7$, it follows that

$$\int_0^T \rho_{\Gamma_i}^{-2s}|r|^2 \leq 2\int_0^T \rho_{\Gamma_i}^{-2s}|r'|^2 + Cs\, e^{2\lambda m K_1}\int_0^T \rho_{\Gamma_i}^{-2s}\frac{1}{t^{2k}(T-t)^{2k}}|r'|^2$$
$$+Cs^2\lambda^2\int_0^T \rho_{\Gamma_i}^{-2s}\frac{e^{2\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{2k}(T-t)^{2k}}|r'|^2 + C\int_0^T \left|\int_{\Gamma_i} D\psi\, n\right|^2 + C\int_0^T \rho_{\Gamma_i}^{-2s}\left|\int_{\Gamma_i} pn\right|^2.$$

The first three terms of the RHS are dominated by $s^3\lambda^3\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|r'|^2$, the fourth term is dominated by $s\lambda\int_{\Sigma_i}\frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}|\partial_n\psi|^2$. But the term $C\int_0^T \rho_{\Gamma_i}^{-2s}\left|\int_{\Gamma_i} pn\right|^2$ cannot be estimated by $\int_Q \rho^{-2s}|\nabla p|^2$ because there is no parameter s or λ multiplying it. Therefore the above estimate cannot be helpful because we find again a term involving the trace of the pressure.

6.2 Second estimate of r

Now, we exploit the fact that the state space of the 'solid part' of the model is of finite dimension. Our goal will be achieved in two steps. As a first step, we prove in this section an intermediate inequality (6.1) written down below. The final inequality will be established in the next section (see (7.1)).

Let us recall that we have set $\theta(t) = |\widehat{\xi}|^{3/2}e^{-s\beta_i}$, where $\beta_i = \beta|_{\Gamma_i}$. A direct calculation leads to

$$\begin{aligned}\theta'(t) &= e^{-s\beta_i}\,|\widehat{\xi}|^{3/2}\left(-s\beta_i' + \frac{3}{2}|\widehat{\xi}|^{-1}\widehat{\xi}'\right)\\ &= e^{-s\beta_i}\,|\widehat{\xi}|^{3/2}\left(-s(e^{\lambda mK_1} - e^{\lambda(\eta_i+m\|\eta\|_\infty)})(\frac{-k}{t^{k+1}(T-t)^k}\frac{k}{t^k(T-t)^{k+1}}) - \frac{3}{2}\frac{k}{t} + \frac{3}{2}\frac{k}{T-t}\right)\\ &= e^{-s\beta_i}\,|\widehat{\xi}|^{3/2}\frac{k}{t^{k+1}(T-t)^{k+1}}R(t),\end{aligned}$$

with $\eta_i = \eta|_{\Gamma_i}$ and

$$R(t) = -s(e^{\lambda mK_1} - e^{\lambda(\eta_i+m\|\eta\|_\infty)})(-kT+2kt) - \frac{3}{2}kt^k(T-t)^{k+1} + \frac{3}{2}kt^{k+1}(T-t)^k.$$

The roots of θ' are the roots of the polynomial R. Let us denote by

$$T_1 < T_2 < \cdots < T_\ell$$

the roots of R lying in the interval $(0,T)$. Necessarily, θ is monotone in the sub-intervals (T_j, T_{j+1}) for $0 \le j \le \ell$, with $T_0 = 0$ and $T_{\ell+1} = T$. Let E be the vector space of solutions to system (2.4) obtained by varying (r^0, r^1). We introduce the following subspace of E :

$$E_i = \Big\{(\psi,p,r) \in E \mid r(T_j) = 0 \quad \text{for all } 1 \le j \le \ell\Big\}.$$

We see that E_i is of infinite dimension and is of codimension $\le 2\ell$. In the following arguments, we will suppose that E_i is of codimension $= 2\ell$ (other cases can be treated in a similar manner). In such a case, there exist $(\widehat{\psi}_j^1, \widehat{p}_j^1, \widehat{r}_j^1) \in E$ and $(\widehat{\psi}_j^2, \widehat{p}_j^2, \widehat{r}_j^2) \in E$ such that

$$\widehat{r}_j^1(T_j) = (1,0) \quad \text{and} \quad \widehat{r}_j^2(T_j) = (0,1).$$

Let E_0 be the space spanned by $\{\widehat{r}_j^1, \widehat{r}_j^2 \mid j = 1, \cdots, \ell\}$, and E_f be the subspace spanned by $(\widehat{\psi}_j^1, \widehat{p}_j^1, \widehat{r}_j^1)_{j=1,\cdots,\ell}$ and $(\widehat{\psi}_j^2, \widehat{p}_j^2, \widehat{r}_j^2)_{j=1,\cdots,\ell}$ so that we have

$$E = E_i \oplus E_f.$$

Let us denote by $\pi_f : E \to E_f$ the mapping defined by

$$\pi_f(\psi,p,r) = \sum_{j=1}^{\ell}\Big(\big(r(T_j)\cdot(1,0)\big)(\widehat{\psi}_j^1, \widehat{p}_j^1, \widehat{r}_j^1) + \big(r(T_j)\cdot(0,1)\big)(\widehat{\psi}_j^2, \widehat{p}_j^2, \widehat{r}_j^2)\Big).$$

Observe that $(\psi,p,r) - \pi_f(\psi,p,r) \in E_i$ for all $(\psi,p,r) \in E$. Further we set $\pi_0(\psi,p,r) = r$ for all $(\psi,p,r) \in E$, and we define $\pi : E \to E_0$ by $\pi = \pi_0 \circ \pi_f$. We have then

$$\pi(\psi,p,r) = \sum_{j=1}^{\ell}\Big(\big(r(T_j)\cdot(1,0)\big)\widehat{r}_j^1 + \big(r(T_j)\cdot(0,1)\big)\widehat{r}_j^2\Big).$$

Lemma 6.1. *If $(\psi,p,r) \in E_i$, then*

$$\int_0^T \theta^2\,|r|^2 \le C\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\,\rho_{\Gamma_i}^{-2s}\,|r'|^2.$$

Proof. Indeed, we establish similar inequalities over the intervals (T_j, T_{j+1}), with $j = 0, \cdots, \ell$, on which θ is monotone.

If θ is nondecreasing over (T_j, T_{j+1}), we write

$$r(t) = -\int_t^{T_{j+1}} r'(\tau)d\tau,$$

and we have

$$\theta(t)|r(t)| \leq \theta(t)\int_t^{T_{j+1}} |r'(\tau)|d\tau \leq \int_t^{T_{j+1}} \theta(\tau)|r'(\tau)|d\tau \leq (T_{j+1}-t)^{1/2}\left(\int_t^{T_{j+1}} |\theta(\tau)|^2|r'(\tau)|^2d\tau\right)^{1/2},$$

for all $T_j \leq t \leq T_{j+1}$. Therefore we have

$$\int_{T_j}^{T_{j+1}} |\theta(t)|^2|r(t)|^2dt \leq (T_{j+1}-T_j)^2\int_{T_j}^{T_{j+1}} |\theta(\tau)|^2|r'(\tau)|^2d\tau.$$

By summing them up, we obtain the required estimate.

If θ is nonincreasing over (T_j, T_{j+1}), we write

$$r(t) = \int_{T_j}^t r'(\tau)d\tau,$$

and we have

$$\theta(t)|r(t)| \leq \theta(t)\int_{T_j}^t |r'(\tau)|d\tau \leq \int_{T_j}^t \theta(\tau)|r'(\tau)|d\tau \leq (t-T_j)^{1/2}\left(\int_{T_j}^t |\theta(\tau)|^2|r'(\tau)|^2d\tau\right)^{1/2},$$

for all $T_j \leq t \leq T_{j+1}$. Therefore we have

$$\int_{T_j}^{T_{j+1}} |\theta(t)|^2|r(t)|^2dt \leq (T_{j+1}-T_j)^2\int_{T_j}^{T_{j+1}} |\theta(\tau)|^2|r'(\tau)|^2d\tau.$$

Taking into account all these inequalities, it yields

$$\int_0^T |\theta(t)|^2|r(t)|^2dt = \sum_{j=0}^{\ell}\int_{T_j}^{T_{j+1}} |\theta(t)|^2|r(t)|^2dt \leq \sum_{j=0}^{\ell}(T_{j+1}-T_j)^2\int_{T_j}^{T_{j+1}} |\theta(\tau)|^2|r'(\tau)|^2d\tau$$
$$\leq T^2\int_0^T |\theta(\tau)|^2|r'(\tau)|^2d\tau \leq T^2\int_0^T |\xi|_{\Gamma_i}|^3\rho_{\Gamma_i}^{-2s}|r'(\tau)|^2d\tau.$$

∎

With these preparations, we can now consider the inequality (5.6) and estimate the last term of the right hand side of the inequality as follows. Writing $r = r - \pi(\psi, p, r) + \pi(\psi, p, r)$ and noting that $r - \pi(\psi, p, r) \in E_i$, we have by Lemma 6.1

$$s^{1/2}\int_0^T \theta^2|r|^2 \leq Cs^{1/2}\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|r'|^2 + CJ(\psi, p, r),$$

where

$$J(\psi,p,r) = s^{1/2}\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|\pi(\psi,p,r)'|^2 + s^{1/2}\int_0^T \theta^2|\pi(\psi,p,r)|^2.$$

Note that the first term can be absorbed in the left hand side of (5.6) by choosing λ large. More precisely, we have

$$C\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|r'|^2 \le \frac{1}{2}s^3\lambda^3\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|r'|^2,$$

for λ large. Thus the estimate (5.6) gives

$$I(\psi,p,r) \le C\,(K(\psi,p,r) + J(\psi,p,r)), \tag{6.1}$$

with

$$\begin{aligned} I(\psi,p,r) &= s^{-1}\int_Q \xi^{-1}(|\psi'|^2+|\Delta\psi|^2) + \int_Q |M_1\psi|^2 + \int_Q |M_2\psi|^2 + s\lambda^2\int_Q \frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}|\nabla\psi|^2 \\ &+ \int_0^T \left|\int_{\Gamma_i} D\psi\, n\right|^2 + s^3\lambda^4\int_Q \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2 + s\lambda\int_{\Sigma_i}\frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t^k(T-t)^k}\,|\partial_n\psi|^2 \\ &+ \int_0^T (\rho_{\Gamma_i}^{-2s}|r''|^2 + s^{1/2}\theta^2|r|^2) + s^3\lambda^3\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|r'|^2 + \int_Q \rho^{-2s}|\nabla p|^2 \end{aligned}$$

and

$$K(\psi,p,r) = s^3\lambda^4\int_{\omega\times(0,T)} \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}|\psi|^2 + s^2\lambda^2\int_{\omega_2\times(0,T)}\rho^{-2s}\,\xi^2|p|^2.$$

7 Compactness argument and Carleman inequality II

From now on, we do not vary the parameters (s,λ) and fix them so that inequality (6.1) holds. The aim in this section is to show that we can strengthen the inequality (6.1) by proving that there exists a constant $C(\lambda,s) > 0$ such that

$$I(\psi,p,r) \le C(\lambda,s)\,K(\psi,p,r)\,. \tag{7.1}$$

This is the Carleman inequality II that we have for system (2.4). We will translate it to the original system (1.4) in the next section. To prove inequality (7.1), we argue by contradiction. We suppose that there exists a sequence $(\psi_j,p_j,r_j)_j$ associated with the data (r_j^0,r_j^1) such that

$$I(\psi_j,p_j,r_j) = 1 \qquad \text{and} \qquad \lim_{j\to\infty}K(\psi_j,p_j,r_j) = 0\,.$$

We can assume that there exists $(\psi,p,r) \in L^2_{\text{loc}}(Q) \times L^2_{\text{loc}}(0,T)$ and that – after extraction of a subsequence – the sequence $(\psi_j,p_j,r_j)_j$ enjoys the following convergence properties in the indicated weighted spaces:

$$\begin{aligned}
&\psi_j' \rightharpoonup \psi' \quad \text{for the weak topology of } L^2(\xi^{-1};Q)\,,\\
&\Delta\psi_j \rightharpoonup \Delta\psi \quad \text{for the weak topology of } L^2(\xi^{-1};Q)\,,\\
&\nabla\psi_j \rightharpoonup \nabla\psi \quad \text{for the weak topology of } L^2(e^{\lambda(\eta+m\|\eta\|_\infty)}t^{-k}(T-t)^{-k};Q)\,,\\
&\psi_j \rightharpoonup \psi \quad \text{for the weak topology of } L^2(e^{3\lambda(\eta+m\|\eta\|_\infty)}t^{-3k}(T-t)^{-3k};Q)\,,\\
&r_j'' \rightharpoonup r'' \quad \text{for the weak topology of } L^2(\rho_{\Gamma_i}^{-2s};(0,T))\,,\\
&r_j \rightharpoonup r \quad \text{for the weak topology of } L^2(|\widehat{\xi}|^3\rho_{\Gamma_i}^{-2s};(0,T))\,,\\
&r_j' \rightharpoonup r' \quad \text{for the weak topology of } L^2(|\xi|_{\Gamma_i}|^3\rho_{\Gamma_i}^{-2s}t^{-3k}(T-t)^{-3k};(0,T))\,.\\
&\nabla p_j \rightharpoonup \nabla p \quad \text{for the weak topology of } L^2(\rho^{-2s};Q)\\
&\int_{\Gamma_i} D\psi_j\, n \rightharpoonup \int_{\Gamma_i} D\psi\, n \quad \text{for the weak topology of } L^2(0,T).
\end{aligned}$$

Notice that these weights act only with respect to the time variable and not in space variables. In the next two subsections, we will deduce that $\psi \equiv 0$, $r \equiv 0$, and that

$$\int_0^T \theta^2|\pi(\psi_j,p_j,r_j)|^2 + s^3\lambda^3\int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|\pi(\psi_j,p_j,r_j)'|^2 \to 0. \tag{7.2}$$

From (6.1), we conclude that $I(\psi_j,p_j,r_j) \to 0$. This is in contradiction with $I(\psi_j,p_j,r_j) = 1$, which proves (7.1).

7.1 Passage to the limit in problem (2.4)

To prove that $\psi \equiv 0$, $p \equiv 0$ and $r \equiv 0$, we first show that we can pass to the limit in problem (2.4). To pass to the limit in the equation

$$M_1\psi_j + M_2\psi_j = -\rho^{-s}\,\nabla p_j + s(\Delta)\psi_j = f_s^j + s(\Delta)\psi_j,$$

we use the L^2-estimate on $\{M_1\psi_j\}$ and $\{M_2\psi_j\}$. Hence the subsequences $\{M_1\psi_j\}$ and $\{M_2\psi_j\}$ weakly converge in $L^2(Q)$. To identify their limits, it is enough to take test functions in $\mathcal{D}(Q)$ and to pass to the limit. Thanks to the above convergence we get

$$\begin{aligned}
&M_1\psi_j \overset{L^2(Q)}{\rightharpoonup} M_1\psi, \qquad M_2\psi_j \overset{L^2(Q)}{\rightharpoonup} M_2\psi,\\
&s(\Delta\beta)\psi_j \rightharpoonup s(\Delta\beta)\psi \quad \text{weakly in} \quad L^2(e^{3\lambda(\eta+m\|\eta\|_\infty)}t^{-3k}(T-t)^{-3k};Q),\\
&\nabla p_j \rightharpoonup \nabla p \quad \text{weakly in } L^2(\rho^{-2s};Q).
\end{aligned}$$

Next we use $K(\psi_j,p_j,r_j) \to 0$. This shows that $f_s^j \to 0$ in $L^2(Q)$ and $\psi = 0$ in $\omega\times(0,T)$. With this information, we see that

$$M_1\psi + M_2\psi = s(\Delta\beta)\psi \quad \text{in } Q, \qquad \text{and} \quad \psi = 0 \quad \text{in } \omega\times(0,T)\,.$$

To pass to the limit in the equation satisfied by r_j, we notice that

$$\int_{\Gamma_i} D\psi_j\, n \rightharpoonup \int_{\Gamma_i} D\psi_j\, n \quad \text{for the weak topology of } L^2(0,T),$$

and

$$p_j|_{\Gamma_i} \rightharpoonup p|_{\Gamma_i} \quad \text{for the weak topology of } L^2(\rho_{\Gamma_i}^{-2s}; 0, T; L^2(\Gamma_i)).$$

We can also pass to the limit in the boundary conditions on Σ, in particular we can prove that

$$\psi = \rho_{\Gamma_i}^{-s}\, r' \quad \text{on } \Sigma_i.$$

This proves that (ψ, p, r) satisfies the system

$$\begin{aligned}
&M_1\psi + M_2\psi = -\rho^{-s}\,\nabla\pi + s(\Delta\beta)\psi, \quad \operatorname{div}\psi = -s\nabla\beta\cdot\psi && \text{in} \quad Q,\\
&\psi = 0 && \text{on} \quad \Sigma_e,\\
&\psi = \rho_{\Gamma_i}^{-s}\, r' && \text{on} \quad \Sigma_i,\\
&r'' + r = -2\rho_{\Gamma_i}^{s}\int_{\Gamma_i} D\psi\, n - s\int_{\Gamma_i}\big(r'\otimes\nabla\beta + \nabla\beta\otimes r'\big)\, n + \int_{\Gamma_i}\pi n && \text{in} \quad (0,T).
\end{aligned}$$

To deduce that $\psi \equiv 0$, $p \equiv 0$ and $r \equiv 0$, we pass from ψ to $\phi = \rho^s\psi$. We see that (ϕ, p, r) satisfies the system (1.4) with $f = 0$. In addition, we have $\phi \equiv 0$ in $\omega \times (0, T)$. Applying the unique continuation principle for the Stokes equation [6, 7], we obtain $\phi = 0$ and $\nabla p = 0$ in Q, and hence $\psi = 0$ and $p = 0$ in Q ($p = 0$ because of (3.5)). Going back to the system satisfied by (ψ, p, r), we deduce successively that $r' = 0$, $r'' = 0$ and $r = 0$. In particular, we have

$$\begin{aligned}
&r_j \rightharpoonup 0 \quad \text{for the weak topology of } L^2(|\widehat{\xi}|^3\rho_{\Gamma_i}^{-2s}; (0,T))\\
&r_j' \rightharpoonup 0 \quad \text{for the weak topology of } L^2(\rho_{\Gamma_i}^{-2s}; (0,T)).
\end{aligned} \tag{7.3}$$

7.2 Proof of (7.2)

We equip the space

$$H = \Big\{ r \in H^1_{\text{loc}}(0, T; \mathbb{R}^2) \mid \|r'\|_{L^2(\rho_{\Gamma_i}^{-2s} t^{-3k}(T-t)^{-3k}; (0,T))} + \|r\|_{L^2(|\widehat{\xi}|^3\rho_{\Gamma_i}^{-2s}; (0,T))} < \infty \Big\}$$

with the norm

$$\|r\|_H = \|r'\|_{L^2(\rho_{\Gamma_i}^{-2s} t^{-3k}(T-t)^{-3k}; (0,T))} + \|r\|_{L^2(|\widehat{\xi}|^3\rho_{\Gamma_i}^{-2s}; (0,T))}.$$

The mapping

$$r \longmapsto (r(T_n))_{1\le n\le \ell}$$

is continuous from H into $\mathbb{R}^\ell$ since

$$|r\,(T_n)\,| \le C\Big\{ \|r'\|_{L^2(\rho_{\Gamma_i}^{-2s} t^{-3k}(T-t)^{-3k}; (0,T))} + \|r\|_{L^2(|\widehat{\xi}|^3\rho_{\Gamma_i}^{-2s}; (0,T))} \Big\}.$$

Therefore it is also compact. Due to (7.3), $|r_j\,(T_n)\,| \longrightarrow 0$ for all $n = 1, \cdots, \ell$. The proof of (7.2) is complete.

8 Carleman inequality III

The purpose here is to merely translate the Carleman inequality (7.1) from the transformed system (2.4) to original system (1.4). Recalling that $\phi = e^{s\beta}\,\psi$, we have

$$\phi' = e^{s\beta}\,\left(s\,\beta'\,\psi + \psi'\right)\,, \quad \nabla\phi = e^{s\beta}\,\left(\nabla\psi + s\psi\,\nabla\beta\right)\,,$$
$$\Delta\phi = e^{s\beta}\,\left(\Delta\psi + s(\Delta\beta)\psi + 2s\nabla\beta\cdot\nabla\psi + s^2|\nabla\beta|^2\psi\right)\,.$$

As in [19, Section 11], we can prove the following theorem.

Theorem 8.1. *Consider the coupled system* (1.4) *with* $f = 0$*. Then there exist positive constants* λ_0*,* $s_0(\lambda)$ *such that the following inequality holds for all* $\lambda \geq \lambda_0, s \geq s_0(\lambda)$ *and for all solutions* (ϕ, r) *of the system* (1.4)*:*

$$\int_Q \rho^{-2s}\,\xi^{-1}\,\left(\left|\phi'\right|^2 + |\Delta\phi|^2\right) + \int_Q \rho^{-2s}\,\frac{e^{\lambda(\eta+m\|\eta\|_\infty)}}{t(T-t)}|\nabla\phi|^2 + \int_Q \rho^{-2s}\,\frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\,|\phi|^2$$
$$+ \int_0^T \rho_{\Gamma_i}^{-2s}|r''|^2 + \int_0^T |\widehat{\xi}|^3\,\rho_{\Gamma_i}^{-2s}|r|^2 + \int_0^T \frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}|_{\Gamma_i}}{t^{3k}(T-t)^{3k}}\rho_{\Gamma_i}^{-2s}|r'|^2 + \int_Q \rho^{-2s}|\nabla p|^2$$
$$\leq C(\lambda, s)\left\{\int_{\omega\times(0,T)} \rho^{-2s}\frac{e^{3\lambda(\eta+m\|\eta\|_\infty)}}{t^{3k}(T-t)^{3k}}\,|\phi|^2 + \int_{\omega_2\times(0,T)} \rho^{-2s}\,\xi^2|p|^2\right\}.$$

9 Regularity of solutions to system (1.4)

One way to prove Theorem 1.1 is to improve the Carleman inequality of Theorem 8.1 by removing the local term of the pressure in the RHS of Carleman estimate in Theorem 8.1 as in [9]. This leads to lengthy calculations. Another way consists in using a fictitious control as in [14]. We follow this method in the following section. It consists in using an additional control in the divergence condition (see system (10.1)). Next this control is eliminated in section 10.2 by using the regularity results obtained in Theorem 9.1 below. Let us first state a regularity result for the system

$$\begin{aligned}
&\phi' - \Delta\phi + \nabla p = f \quad \text{and} \quad \operatorname{div}\phi = 0 && \text{in} \quad Q,\\
&\phi_1 = 0 && \text{on} \quad \Sigma_e,\\
&\phi = r_b && \text{on} \quad \Sigma_i,\\
&\phi(0) = 0 && \text{in} \quad \Omega,\\
&r_a' - r_b = g && \text{in} \quad (0,T),\\
&r_b' + r_a = h - \int_{\Gamma_i} \sigma(\phi, p)\,n && \text{in} \quad (0,T),\\
&r_a(0) = 0 \qquad \text{and} \qquad r_b(0) = 0.
\end{aligned} \tag{9.1}$$

Lemma 9.1. *The solution to system* (9.1) *obeys*

$$\begin{aligned}&\|\phi\|_{H^{2,1}(Q)} + \|p\|_{L^2(0,T;H^1(\Omega))} + \|r_a\|_{H^1(0,T;\mathbb{R}^2)} + \|r_b\|_{H^1(0,T;\mathbb{R}^2)}\\ &\le C(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(0,T;\mathbb{R}^2)} + \|h\|_{L^2(0,T;\mathbb{R}^2)}).\end{aligned} \tag{9.2}$$

Proof. Let us first notice that, using an energy identity as in section 1, we can verify that the solution to system (9.1) obeys

$$\begin{aligned}&\|\phi\|_{L^2(0,T;H^1(\Omega))} + \|r_a\|_{L^2(0,T;\mathbb{R}^2)} + \|r_b\|_{L^2(0,T;\mathbb{R}^2)}\\ &\le C(\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(0,T;\mathbb{R}^2)} + \|h\|_{L^2(0,T;\mathbb{R}^2)}).\end{aligned} \tag{9.3}$$

As in section 5, system (9.1) can be rewritten in terms of $P\phi = \phi_e$ and $(I-P)\phi = \phi_s$ as follows

$$\begin{aligned}&\phi_e' - A\phi_e = (-A)PL(r_b\chi_{\Gamma_i}) + Pf, \quad \phi_e(0) = 0,\\ &\phi_s = (I-P)L(r_b\chi_{\Gamma_i}),\\ &r_a' - r_b = g \qquad \text{in} \quad (0,T),\\ &(I+K)r_b' + r_a = h - \int_{\Gamma_i} \sigma(\phi_e, p_e)\, n - \int_{\Gamma_i} D\phi_s\, n + \int_{\Gamma_i} p_f\, n \qquad \text{in} \quad (0,T),\\ &r_a(0) = 0 \qquad \text{and} \qquad r_b(0) = 0,\end{aligned} \tag{9.4}$$

where p_e is the pressure appearing in the equation satisfied by ϕ_e, $p_s = -q_t$ where $q(t) = N(r_b \cdot n) \in H^1(\Omega)$ (N is the operator introduced in section 5), p_f is determined by $\nabla p_f = \nabla p_1 + \nabla p_2$, p_1 and p_2 are the solutions to

$$\begin{aligned}&p_1 \in H^1_0(\Omega), \quad \Delta p_1 = \operatorname{div} f \quad \text{in } \Omega,\\ &p_2 \in H^1(\Omega), \quad \Delta p_2 = 0 \quad \text{in } \Omega, \quad \frac{\partial p_2}{\partial n} = (f - \nabla p_1)\cdot n \quad \text{on } \Gamma.\end{aligned}$$

As in section 5, we can choose all the pressure terms obeying the condition (3.5). Estimate (9.2) can be proved with (9.3) and with calculations similar as the ones in section 5. ∎

Let (ϕ, p, r) be the solution to (1.4) corresponding to $f = 0$ and to $(\phi^0, r^0, r^1) \in H$. It will be advantageous to rewrite the structure equation as a first order evolution system. Let us introduce

$$(\phi_1, p_1, r_{a,1}, r_{b,1}) = (s\,\widehat{\xi})^{-\delta} e^{-s\beta^*}(\phi, p, r, r') \quad \text{and} \quad \rho_1 = \frac{d}{dt}\left((s\,\widehat{\xi})^{-\delta} e^{-s\beta^*}\right).$$

We can check that $(\phi_1, p_1, r_{a,1}, r_{b,1})$ is the solution to system (9.1) with $f = \rho_1\phi = f_1$, $g = \rho_1 r = g_1$ and $h = \rho_1 r' = h_1$.

Theorem 9.1. *There exist positive constants* λ_0, $s_0(\lambda)$ *such that the following inequality holds for all* $\lambda \ge \lambda_0, s \ge s_0(\lambda)$ *and for all solutions* (ϕ, p, r) *of system* (1.4) *with* $f = 0$, *the quadruplet* $(\phi_1, p_1, r_{a,1}, r_{b,1}) = (s\,\widehat{\xi})^{-\delta} e^{-s\beta^*}(\phi, p, r, r')$ *satisfies the estimate:*

$$\begin{aligned}&\|\phi_1\|^2_{H^{4,2}(Q)} + \|\nabla p_1\|^2_{H^1(0,T;L^2(\Omega))} + \|r_{a,1}\|^2_{H^2(0,T;\mathbb{R}^2)} + \|r_{b,1}\|^2_{H^2(0,T;\mathbb{R}^2)}\\ &\le C(\lambda, s)\left\{\int_{\omega\times(0,T)} \rho^{-2s}\,\xi^3\,|\phi|^2 + \int_{\omega_2\times(0,T)} \rho^{-2s}\,\xi^2 |p|^2\right\}.\end{aligned} \tag{9.5}$$

Proof. From Lemma 9.1, it follows that

$$\begin{aligned}&\|\phi_1\|_{H^{2,1}(Q)} + \|\nabla p_1\|_{L^2(0,T;L^2(\Omega))} + \|r_{a,1}\|_{H^1(0,T;\mathbb{R}^2)} + \|r_{b,1}\|_{H^1(0,T;\mathbb{R}^2)}\\ &\le C(\|\rho_1\phi\|_{L^2(Q)} + \|\rho_1 r_a\|_{L^2(0,T;\mathbb{R}^2)} + \|\rho_1 r_b\|_{L^2(0,T;\mathbb{R}^2)}).\end{aligned} \tag{9.6}$$

To improve the regularity of the solution to system (9.1), we write the equation satisfied by

$$\phi_2 = \phi_1', \quad p_2 = p_1', \quad r_{a,2} = r_{a,1}', \quad r_{b,2} = r_{b,1}'.$$

We observe that

$$\begin{aligned}
&\phi_2' - \Delta\phi_2 + \nabla p_2 = f_1' = (\rho_1\phi)' \quad \text{and} \quad \operatorname{div}\phi_2 = 0 && \text{in} \quad Q,\\
&\phi_2 = 0 && \text{on} \quad \Sigma_e,\\
&\phi_2 = r_{b,2} && \text{on} \quad \Sigma_i,\\
&\phi_2(0) = 0 && \text{in} \quad \Omega,\\
&r_{a,2}' - r_{b,2} = g_1' = (\rho_1 r_{a,1})' && \text{in} \quad (0,T),\\
&r_{b,2}' + r_{a,2} = h_1' - \int_{\Gamma_i} \sigma(\phi_2, p_2)\, n = (\rho_1 r_{b,1})' - \int_{\Gamma_i} \sigma(\phi_2, p_2)\, n && \text{in} \quad (0,T),\\
&r_{a,2}(0) = 0 \qquad \text{and} \qquad r_{b,2}(0) = 0.
\end{aligned}$$

From Lemma 9.1, it follows that

$$\begin{aligned}&\|\phi_2\|_{H^{2,1}(Q)} + \|\nabla p_2\|_{L^2(0,T;L^2(\Omega))} + \|r_{a,2}\|_{H^1(0,T;\mathbb{R}^2)} + \|r_{b,2}\|_{H^1(0,T;\mathbb{R}^2)}\\ &\le C(\|f_1'\|_{L^2(Q)} + \|g_1'\|_{L^2(0,T;\mathbb{R}^2)} + \|h_1'\|_{L^2(0,T;\mathbb{R}^2)}).\end{aligned} \tag{9.7}$$

Let us estimate f_1 in $H^{2,1}(Q)$. We have

$$\begin{aligned}&\Delta f_1 = \rho_1\,\Delta\phi, \quad f_1' = \rho_1'\,\phi + \rho_1\,\phi', \quad |\rho_1| \le C s^{-\delta+1}\,(\widehat{\xi})^{-\delta+\frac{k+1}{k}}\,e^{-s\beta^*},\\ &|\rho_1'| \le C s^{-\delta+2}\,(\widehat{\xi})^{-\delta+\frac{2(k+1)}{k}}\,e^{-s\beta^*}.\end{aligned}$$

Therefore, with Theorem 8.1, we have

$$\begin{aligned}
&\|\Delta f_1\|^2_{L^2(Q)} + \|f_1'\|^2_{L^2(Q)}\\
&\le C(s,\lambda)\left(\int_Q |\widehat{\xi}|^{-2\delta+\frac{4(k+1)}{k}}\,e^{-2s\beta^*}\,|\phi|^2 + \int_Q |\widehat{\xi}|^{-2\delta+\frac{2(k+1)}{k}}\,e^{-2s\beta^*}\,(|\phi'|^2 + |\Delta\phi|^2)\right)\\
&\le C(s,\lambda)\left(\int_Q \xi^3\,e^{-2s\beta^*}\,|\phi|^2 + \int_Q \widehat{\xi}^{-1} e^{-2s\beta^*}\,(|\phi'|^2 + |\Delta\phi|^2)\right)\\
&\le C(s,\lambda)\left\{\int_{\omega\times(0,T)} \rho^{-2s}\,\xi^3\,|\phi|^2 + \int_{\omega_2\times(0,T)} \rho^{-2s}\,\xi^2|p|^2\right\},
\end{aligned}$$

provided that

$$-2\delta + \frac{2(k+1)}{k} \le -1,$$

which is satisfied if

$$k = 4 \quad \text{and} \quad 7 \leq 4\delta.$$

Now, let us estimate g_1' and h_1' in $L^2(0,T;\mathbb{R}^2)$. We have

$$g_1' = \rho_1' \, r + \rho_1 \, r', \quad h_1' = \rho_1' \, r' + \rho_1 \, r''.$$

Therefore, still with Theorem 8.1, we obtain

$$\begin{aligned}
&\|g_1'\|^2_{L^2(0,T;\mathbb{R}^2)} \\
&\leq C(s,\lambda) \left(\int_0^T |\widehat{\xi}|^{-2\delta + \frac{4(k+1)}{k}} e^{-2s\beta^*} |r|^2 + \int_0^T |\widehat{\xi}|^{-2\delta + \frac{2(k+1)}{k}} e^{-2s\beta^*} |r'|^2 \right) \\
&\leq C(s,\lambda) \left(\int_0^T |\xi|_{\Gamma_i}|^3 \, e^{-2s\beta^*} |r|^2 + \int_0^T \widehat{\xi}^{-1} e^{-2s\beta^*} |r'|^2 \right) \\
&\leq C(s,\lambda) \left\{ \int_{\omega\times(0,T)} \rho^{-2s} \, \xi^3 \, |\phi|^2 + \int_{\omega_2\times(0,T)} \rho^{-2s} \, \xi^2 |p|^2 \right\},
\end{aligned}$$

and

$$\begin{aligned}
&\|h_1'\|_{L^2(0,T;\mathbb{R}^2)} \\
&\leq C(s,\lambda) \left(\int_0^T |\widehat{\xi}|^{-2\delta + \frac{4(k+1)}{k}} e^{-2s\beta^*} |r'|^2 + \int_0^T |\widehat{\xi}|^{-2\delta + \frac{2(k+1)}{k}} e^{-2s\beta^*} |r''|^2 \right) \\
&\leq C(s,\lambda) \left(\int_0^T |\xi|_{\Gamma_i}|^3 \, e^{-2s\beta^*} |r'|^2 + \int_0^T \widehat{\xi}^{-1} e^{-2s\beta^*} |r''|^2 \right) \\
&\leq C(s,\lambda) \left\{ \int_{\omega\times(0,T)} \rho^{-2s} \, \xi^3 \, |\phi|^2 + \int_{\omega_2\times(0,T)} \rho^{-2s} \, \xi^2 |p|^2 \right\}.
\end{aligned}$$

In these estimates we have used that $|\widehat{\xi}| \leq C(\lambda)|\xi|_{\Gamma_i}|$. Thus, from (9.7) it follows that

$$\begin{aligned}
&\|\phi_2\|_{H^{2,1}(Q)} + \|\nabla p_2\|_{L^2(0,T;L^2(\Omega))} + \|r_{a,2}\|_{H^1(0,T;\mathbb{R}^2)} + \|r_{b,2}\|_{H^1(0,T;\mathbb{R}^2)} \\
&C(s,\lambda) \left\{ \int_{\omega\times(0,T)} \rho^{-2s} \, \xi^3 \, |\phi|^2 + \int_{\omega_2\times(0,T)} \rho^{-2s} \, \xi^2 |p|^2 \right\},
\end{aligned} \tag{9.8}$$

from which we deduce

$$\begin{aligned}
&\|\phi_1\|_{H^1(0,T;H^2(\Omega))} + \|\phi_1\|_{H^2(0,T;L^2(\Omega))} + \|\nabla p_1\|_{H^1(0,T;L^2(\Omega))} + \|r_{a,1}\|_{H^2(0,T;\mathbb{R}^2)} + \|r_{b,1}\|_{H^2(0,T;\mathbb{R}^2)} \\
&\leq C(s,\lambda) \left\{ \int_{\omega\times(0,T)} \rho^{-2s} \, \xi^3 \, |\phi|^2 + \int_{\omega_2\times(0,T)} \rho^{-2s} \, \xi^2 |p|^2 \right\}.
\end{aligned}$$

Next, using equation (9.1), we can write that $\phi_1(t)$ obeys the stationary Stokes equation

$$\begin{aligned}
&-\Delta\phi_1(t) + \nabla p_1(t) = f_1 - \phi_1' \quad \text{and} \quad \operatorname{div}\phi_1(t) = 0 && \text{in} \quad \Omega, \\
&\phi_1(t) = 0 && \text{on} \quad \Gamma_e, \\
&\phi_1(t) = r_{b,1}(t) && \text{on} \quad \Gamma_i.
\end{aligned} \tag{9.9}$$

Thus from elliptic regularity results it follows that

$$
\begin{aligned}
&\|\phi_1\|_{L^2(0,T;H^4(\Omega))} \\
&\leq C\left\{\|f_1\|_{L^2(0,T;H^2(\Omega))} + \|\phi_1'\|_{L^2(0,T;H^2(\Omega))} + \|r_{b,1}\|_{L^2(0,T;\mathbb{R}^2)}\right\} \\
&\leq C(s,\lambda)\left\{\int_{\omega\times(0,T)} \rho^{-2s}\,\xi^3\,|\phi|^2 + \int_{\omega_2\times(0,T)} \rho^{-2s}\,\xi^2|p|^2\right\}.
\end{aligned}
$$

This completes the proof of (9.5). ∎

10 Null controllability result

In this section, we establish null controllability of our original system (1.4) as a consequence of the Carleman inequality stated in Theorem 8.1 and of the regularity results in Theorem 9.1.

10.1 Null controllability with two controls

We first consider the system with two controls (u, v)

$$
\begin{aligned}
& y' - \operatorname{div}\sigma(y,\pi) = u\,\chi_\omega \quad \text{and} \quad \operatorname{div} y = v\,\zeta && \text{in} \quad Q, \\
& y = 0 && \text{on} \quad \Sigma_e, \\
& y = q' && \text{on} \quad \Sigma_i, \\
& y(0) = y^0 && \text{in} \quad \Omega, \\
& q'' + q = -\int_{\Gamma_i} \sigma(y,\pi)\,n && \text{in} \quad (0,T), \\
& q(0) = q^0 \qquad \text{and} \qquad q'(0) = q^1\,.
\end{aligned}
\tag{10.1}
$$

We have to define solutions to system (10.1) in the case when v belongs to $L^2(0,T;L^2(\Omega))$. For that, we use the transposition method. Let us consider the adjoint system, in which the structure equation is rewritten as a first order system

$$
\begin{aligned}
& -\phi' - \operatorname{div}\sigma(\phi,p) = f \quad \text{and} \quad \operatorname{div}\phi = 0 && \text{in} \quad Q, \\
& \phi = 0 && \text{on} \quad \Sigma_e, \\
& \phi = -r_b && \text{on} \quad \Sigma_i, \\
& \phi(T) = 0 && \text{in} \quad \Omega, \\
& r_a' = r_b + g, \\
& r_b' + r_a = h - \int_{\Gamma_i} \sigma(\phi,p)\,n && \text{in} \quad (0,T), \\
& r_a(T) = 0 \qquad \text{and} \qquad r_b(T) = 0.
\end{aligned}
\tag{10.2}
$$

We shall say that $(y, \pi, q) \in L^2(0,T;L^2(\Omega)) \times C^1([0,T];\mathbb{R}^2)$ is a solution to system (10.1), in the sense of transposition, when

$$\int_0^T \int_\Omega y\, f + \int_0^T q\, g + \int_0^T q'\, h = \int_0^T \int_\omega u\, \phi + \int_0^T \int_\Omega v\, \zeta\, p + \int_\Omega y^0\, \phi(0) + q^0 \cdot r_a(0) + q^1 \cdot r_b(0),$$

for all $(f, g, h) \in L^2(0,T;L^2(\Omega)) \times L^2(0,T;\mathbb{R}^2) \times L^2(0,T;\mathbb{R}^2)$, where (ϕ, p, r_a, r_b) is the solution to system (10.2). By this way, we can show that system (10.1) admits a unique solution, in the sense of transposition, and this solution obeys the estimate

$$\|y\|_{L^2(0,T;L^2(\Omega))} + \|q\|_{C^1([0,T];\mathbb{R}^2)} \leq C(\|y^0\|_{L^2(\Omega)} + |q^0| + |q^1| + \|\zeta\, v\|_{L^2(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;L^2(\Omega))}).$$

Next using the equation

$$\begin{aligned}
&y' - \operatorname{div} \sigma(y, \pi) = u\, \chi_\omega \quad \text{and} \quad \operatorname{div} y = v\, \zeta && \text{in} \quad Q,\\
&y = 0 && \text{on} \quad \Sigma_e,\\
&y = q' && \text{on} \quad \Sigma_i,\\
&y(0) = y^0 && \text{in} \quad \Omega,
\end{aligned} \tag{10.3}$$

and regularity result from [22] we get

$$\|Py\|_{C([0,T];V^{-1}(\Omega))} \leq C(\|y^0\|_{L^2(\Omega)} + \|q\|_{L^2(0,T;\mathbb{R}^2)} + \|\zeta\, v\|_{L^2(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;L^2(\Omega))}).$$

Here $V^{-1}(\Omega)$ denotes the dual of $V_0^1(\Omega)$ with $V_n^0(\Omega)$ as pivot space. Let us notice that this estimate is more precise than the one stated in [8, Theorem 2.14] where it is shown that Py belongs to $C([0,T];V^{-2}(\Omega))$ for less regular data ($V^{-2}(\Omega)$ is the dual of $H^2(\Omega) \cap V_0^1(\Omega)$). Finally with the previous estimate we have

$$\begin{aligned}
&\|y\|_{L^2(0,T;L^2(\Omega))} + \|q\|_{C^1([0,T];\mathbb{R}^2)} + \|Py\|_{C([0,T];V^{-1}(\Omega))}\\
&\leq C(\|y^0\|_{L^2(\Omega)} + |q^0| + |q^1| + \|\zeta\, v\|_{L^2(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;L^2(\Omega))}).
\end{aligned} \tag{10.4}$$

Theorem 10.1. *For all $y^0 \in V^0(\Omega)$, $q^0 \in \mathbb{R}^2$, and $q^1 \in \mathbb{R}^2$ satisfying the conditions $y^0 \cdot n = q^1 \cdot n$ on Γ_i and $y^0 \cdot n = 0$ on Γ_e, there exist a function $u \in L^2(Q)$ and a function $\zeta\, v \in H^1(0,T;H^1(\Omega))$, satisfying $\int_\Omega v(t)\, \zeta = 0$ for all $t \in [0,T]$, $\zeta\, v(0) = 0$ and $\zeta\, v(T) = 0$, such that the solution of* (10.1) *obeys*

$$y(T) = 0, \quad q(T) = 0 \quad \text{and} \quad q'(T) = 0.$$

Proof. *Step 1. Penalized problem.* We are going to prove the null controllability result by using a penalized optimal control problem. Let us introduce the problem

$$(\mathcal{P}_\varepsilon) \qquad \inf \Big\{ J_\varepsilon(y, u, v) \mid (y, p, u, v) \text{ obeys } (10.1) \Big\}$$

where

$$J_\varepsilon(y, u, v) = \frac{1}{2\varepsilon} \|Py(T)\|^2_{V^{-1}(\Omega)} + \frac{1}{2\varepsilon} |q(T)| + \frac{1}{2\varepsilon} |q'(T)|^2 + \frac{1}{2} \int_{\omega \times (0,T)} \left(\xi^{-3}\, e^{2s\beta} |u|^2 + \zeta\, \xi^{-2}\, e^{2s\widehat{\beta}} |v|^2 \right).$$

In J_ε only $Py(T)$ is penalized and not $y(T)$ for two convergent reasons. Firstly, we know that $t \longmapsto Py(t)$ is continuous from $[0,T]$ into $V^{-1}(\Omega)$ while there is no hope to have continuity results for $t \longmapsto y(t)$ (see [8, 21, 22]). Secondly, if $Py(T) = 0$, $q(T) = 0$, $q'(T) = 0$, and $u(t) = 0$ and $v(t) = 0$ for $t > T$, then the solution to (10.1) obeys $y(t) = 0$, $q(t) = 0$ and $q'(t) = 0$ for $t > T$ (see [21]).

Problem $(\mathcal{P}_\varepsilon)$ admits a unique solution $(y_\varepsilon, \pi_\varepsilon, u_\varepsilon, v_\varepsilon)$ which is characterized by the optimality system

$$\begin{aligned}
&y_\varepsilon' - \operatorname{div}\sigma(y_\varepsilon,\pi_\varepsilon) = u_\varepsilon\,\chi_\omega \quad \text{and} \quad \operatorname{div} y_\varepsilon = v_\varepsilon\,\zeta && \text{in} \quad Q,\\
&y_\varepsilon = 0 && \text{on} \quad \Sigma_e,\\
&y_\varepsilon = q_\varepsilon' && \text{on} \quad \Sigma_i,\\
&y_\varepsilon(0) = y^0 && \text{in} \quad \Omega,\\
&q_\varepsilon'' + q_\varepsilon = -\int_{\Gamma_i} \sigma(y_\varepsilon,\pi_\varepsilon)\,n && \text{in} \quad (0,T),\\
&q_\varepsilon(0) = q^0 \qquad \text{and} \qquad q_\varepsilon'(0) = q^1,
\end{aligned} \tag{10.5}$$

$$\begin{aligned}
&-\phi_\varepsilon' - \operatorname{div}\sigma(\phi_\varepsilon,p_\varepsilon) = 0 \quad \text{and} \quad \operatorname{div}\phi_\varepsilon = 0 && \text{in} \quad Q,\\
&\phi_\varepsilon = 0 && \text{on} \quad \Sigma_e,\\
&\phi_\varepsilon = -r_\varepsilon' && \text{on} \quad \Sigma_i,\\
&\phi_\varepsilon(T) = -\tfrac{1}{\varepsilon}(-P\Delta)^{-1}Py_\varepsilon(T) && \text{in} \quad \Omega,\\
&r_\varepsilon'' + r_\varepsilon = -\int_{\Gamma_i} \sigma(\phi_\varepsilon,p_\varepsilon)\,n && \text{in} \quad (0,T),\\
&r_\varepsilon(T) = \tfrac{1}{\varepsilon}q_\varepsilon(T) \qquad \text{and} \qquad r_\varepsilon'(T) = \tfrac{1}{\varepsilon}q_\varepsilon'(T),
\end{aligned} \tag{10.6}$$

$$u_\varepsilon = \xi^3\,e^{-2s\beta}\,\phi_\varepsilon\,\chi_\omega \quad \text{and} \quad v_\varepsilon = -\xi^2\,e^{-2s\widehat{\beta}}\,p_\varepsilon + \frac{1}{\int_\omega \zeta}\int_\omega \xi^2\,e^{-2s\widehat{\beta}}\,p_\varepsilon\,\zeta\,dx. \tag{10.7}$$

We choose the pressure p_ε (see (3.5)) such that

$$\int_\omega \xi(x,t)^2\,e^{-2s\widehat{\beta}(t)}\,p_\varepsilon(x,t)\,\zeta(x)\,dx = 0 \quad \text{for all } t \in (0,T).$$

Thus

$$v_\varepsilon = -\xi^2\,e^{-2s\widehat{\beta}}\,p_\varepsilon.$$

With equations (10.5)–(10.7), we obtain

$$\int_{\omega\times(0,T)} \xi^3\,e^{-2s\beta}|\phi_\varepsilon|^2 + \int_{\omega\times(0,T)} \xi^2\,e^{-2s\widehat{\beta}}\,\zeta\,|p_\varepsilon|^2 + \frac{1}{\varepsilon}\|Py_\varepsilon(T)\|^2_{V^{-1}(\Omega)} + \frac{1}{\varepsilon}|q_\varepsilon(T)|^2 + \frac{1}{\varepsilon}|q_\varepsilon'(T)|^2$$
$$= -\int_\Omega y^0\cdot\phi_\varepsilon(0) - q^0\cdot r_\varepsilon(0) - q^1\cdot r_\varepsilon'(0).$$

With Young's inequality we have

$$\begin{aligned}&\int_{\omega\times(0,T)}\Big(\xi^3\,e^{-2s\beta}|\phi_\varepsilon|^2+\xi^2\,e^{-2s\widehat{\beta}}\,\zeta\,|p_\varepsilon|^2\Big)+\frac{1}{\varepsilon}\|Py_\varepsilon(T)\|^2_{V^{-1}(\Omega)}+\frac{1}{\varepsilon}|q_\varepsilon(T)|^2+\frac{1}{\varepsilon}|q'_\varepsilon(T)|^2\\&\le\frac{\eta}{2}\Big(\|\phi_\varepsilon(0)\|^2_{L^2(\Omega)}+|r_\varepsilon(0)|^2+|r'_\varepsilon(0)|^2\Big)+\frac{1}{2\eta}\Big(\|y^0\|^2_{L^2(\Omega)}+|q^0|^2+|q^1|^2\Big).\end{aligned}\tag{10.8}$$

Step 2. Uniform estimates. As in [19, Lemma 12.2], applying the Carleman inequality of Theorem 8.1 to the solution $(\phi_\varepsilon, p_\varepsilon, r_\varepsilon)$ of the adjoint system (10.6) and using that $\zeta|_{\omega_2}=1$, we obtain

$$\|\phi_\varepsilon(0)\|^2_{L^2(\Omega)}+|r_\varepsilon(0)|^2+|q'_\varepsilon(0)|^2\le C\int_{\omega\times(0,T)}\xi^3\,e^{-2s\beta}\,|\phi_\varepsilon|^2+C\int_{\omega\times(0,T)}\xi^2\,e^{-2s\beta}\,\zeta\,|p_\varepsilon|^2,\tag{10.9}$$

where C is independent of ε. Since $e^{-2s\widehat{\beta}}\ge e^{-2s\beta}$, with (10.9) estimate (10.8) is transformed as follows

$$\begin{aligned}&\int_{\omega\times(0,T)}\Big(\xi^3\,e^{-2s\beta}|\phi_\varepsilon|^2+\xi^2\,e^{-2s\widehat{\beta}}\,\zeta\,|p_\varepsilon|^2\Big)+\frac{1}{\varepsilon}\|Py_\varepsilon(T)\|^2_{V^{-1}(\Omega)}+\frac{1}{\varepsilon}|q_\varepsilon(T)|^2+\frac{1}{\varepsilon}|q'_\varepsilon(T)|^2\\&\le C\Big(\|y^0\|^2_{L^2(\Omega)}+|q^0|^2+|q^1|^2\Big).\end{aligned}\tag{10.10}$$

In particular $\{u_\varepsilon\}$ is bounded in $L^2(Q)$ since we have

$$\int_Q|u_\varepsilon|^2=\int_{\omega\times(0,T)}\xi^6\,e^{-4s\beta}|\phi_\varepsilon|^2\le C\int_{\omega\times(0,T)}\xi^3\,e^{-2s\beta}|\phi_\varepsilon|^2.$$

Step 3. Regularity of v_ε. Recall that

$$u_\varepsilon=\xi^3\,e^{-2s\beta}\,\phi_\varepsilon\,\chi_\omega\quad\text{and}\quad v_\varepsilon=-\xi^2\,e^{-2s\widehat{\beta}}\,p_\varepsilon.$$

We introduce

$$\tilde{\phi}_\varepsilon=(s\,\widehat{\xi})^{-\delta}e^{-s\beta^*}\,\phi_\varepsilon\quad\text{and}\quad\tilde{p}_\varepsilon=(s\,\widehat{\xi})^{-\delta}e^{-s\beta^*}\,p_\varepsilon.$$

We have

$$(s\,\widehat{\xi})^{\delta}e^{s\beta^*}\,\tilde{\phi}_\varepsilon=\phi_\varepsilon\quad\text{and}\quad(s\,\widehat{\xi})^{\delta}e^{s\beta^*}\,\tilde{p}_\varepsilon=p_\varepsilon.$$

Thus

$$u_\varepsilon=\xi^3\,e^{-2s\beta}\,(s\,\widehat{\xi})^{\delta}e^{s\beta^*}\,\tilde{\phi}_\varepsilon\,\chi_\omega$$

and

$$v_\varepsilon=\gamma_2\,\tilde{p}_\varepsilon\quad\text{with}\quad\gamma_2=-\xi^2\,e^{-2s\widehat{\beta}}\,(s\,\widehat{\xi})^{\delta}\,e^{s\beta^*}.$$

Let us calculate $\zeta\,v'_\varepsilon$ and $\nabla(\zeta\,v_\varepsilon)$. We have

$$\begin{aligned}&\zeta\,v'_\varepsilon=\zeta\,\gamma'_2\,\tilde{p}_\varepsilon+\zeta\,\gamma_2\,\tilde{p}'_\varepsilon,\quad\nabla(\zeta\,v_\varepsilon)=\nabla(\zeta\,\gamma_2)\,\tilde{p}_\varepsilon+\zeta\,\gamma_2\,\nabla\tilde{p}_\varepsilon,\\&\nabla(\zeta\,v'_\varepsilon)=\nabla(\zeta\,\gamma'_2)\,\tilde{p}_\varepsilon+\nabla(\zeta\,\gamma_2)\,\tilde{p}'_\varepsilon+\zeta\,\gamma_2\,\nabla\tilde{p}'_\varepsilon+\zeta\,\gamma'_2\,\nabla\tilde{p}_\varepsilon.\end{aligned}$$

Notice that the functions $\zeta\,\gamma_2$, $\zeta\,\gamma_2'$, $\nabla(\zeta\,\gamma_2)$, and $\nabla(\zeta\,\gamma_2')$ are bounded in Q. Thus, with estimate (9.5) we have

$$
\begin{aligned}
\|\zeta\, v_\varepsilon\|^2_{H^1(0,T;H^1(\Omega))} &\le C\|\tilde{p}_\varepsilon\|^2_{H^1(0,T;H^1(\Omega))} \\
&\le C\left(\int_{\omega\times(0,T)} \xi^3\, e^{-2s\beta}\,|\phi_\varepsilon|^2 + \int_{\omega\times(0,T)} e^{-2s\beta}\,\xi^2\,\zeta\,|p_\varepsilon|^2\right) \\
&\le C\left(\int_{\omega\times(0,T)} \xi^3\, e^{-2s\beta}\,|\phi_\varepsilon|^2 + \int_{\omega\times(0,T)} e^{-2s\widehat{\beta}}\,\xi^2\,\zeta\,|p_\varepsilon|^2\right).
\end{aligned}
$$

Using estimate (10.10) for $(\phi_\varepsilon, p_\varepsilon)$, we finally obtain

$$
\begin{aligned}
&\|\zeta\, v_\varepsilon\|^2_{H^1(0,T;H^1(\Omega))} \le C\left(\int_{\omega\times(0,T)} \xi^3\, e^{-2s\beta}\,|\phi_\varepsilon|^2 + \int_{\omega\times(0,T)} e^{-2s\widehat{\beta}}\,\xi^2\,\zeta\,|p_\varepsilon|^2\right) \\
&\le C\left(\|y^0\|^2_{L^2(\Omega)} + |q^0|^2 + |q^1|^2\right).
\end{aligned}
\tag{10.11}
$$

Step 4. Passage to the limit when ε tends to zero. From (10.10) and (10.11), it follows that the sequences $\{u_\varepsilon\}$ and $\{\zeta\, v_\varepsilon\}$ are bounded respectively in $L^2(0,T;L^2(\Omega))$ and in $H^1(0,T;H^1(\Omega))$. Therefore, using the estimate (10.4) and equation (10.5), we can show that $\{(y_\varepsilon, q_\varepsilon, q_\varepsilon')\}$ converges to the solution (y, q, q') of equation (10.1), weakly-star in $L^2(0,T;L^2(\Omega)) \times L^\infty(0,T;\mathbb{R}^2) \times L^\infty(0,T;\mathbb{R}^2)$ and $\{(y_\varepsilon(T), q_\varepsilon(T), q_\varepsilon(T))\}$ converges to $(y(T), q(T), q'(T))$ weakly in $V^{-1}(\Omega)\times\mathbb{R}^2\times\mathbb{R}^2$. Since $\{(y_\varepsilon(T), q_\varepsilon(T), q_\varepsilon(T))\}$ converges to $(0,0,0)$, we have shown that the pair $(u, \zeta\, v) \in L^2(0,T;L^2(\Omega)) \times H^1(0,T;H^1(\Omega))$ is the solution to the null controlability problem stated in Theorem 10.1. Finally, since the sequence $\{\zeta^{1/2}\,\xi^{-1}\,e^{s\widehat{\beta}}\,v_\varepsilon\}$ is bounded in $L^2(0,T;L^2(\Omega))$, the sequence $\{\zeta\,\xi^{-1}\,e^{s\widehat{\beta}}\,v_\varepsilon\}$ is also bounded in $L^2(0,T;L^2(\Omega))$, and therefore the function $\zeta\,\xi^{-1}\,e^{s\widehat{\beta}}\,v$ belongs to $L^2(0,T;L^2(\Omega))$. Since $\zeta\ v$ belongs to $H^1(0,T;H^1(\Omega))$, we necessarily have $\zeta v(0) = 0$ and $\zeta v(T) = 0$. ∎

10.2 Proof of Theorem 1.1.

In this final part, we eliminate the fictitious control v of Theorem 10.1 and we prove Theorem 1.1.

Let $z \in H^1(0,T;H^1_0(\omega_3))$ be the solution to the divergence equation

$$
\operatorname{div} z(t) = \zeta\, v(t) \quad \text{in } \omega_3, \qquad z(t) = 0 \quad \text{on } \partial\omega_3. \tag{10.12}
$$

Let us denote by $\tilde{z}(t) \in H^1_0(\Omega)$ the extension of $z(t)$ by 0 to Ω. It is clear that $\tilde{z} \in H^1(0,T;H^1_0(\Omega))$ is the solution to the divergence equation

$$
\operatorname{div} \tilde{z}(t) = \zeta\, v(t) \quad \text{in } \Omega, \qquad \tilde{z}(t) = 0 \quad \text{on } \Gamma. \tag{10.13}
$$

Since $\zeta\, v$ belongs to $H^1(0,T;H^1_0(\Omega))$, from [13] it follows that $\tilde{z}$ belongs to $H^1(0,T;H^2(\Omega))$. Moreover, we have $\zeta\, v|_{(\Omega\setminus\omega_3)\times[0,T]} = 0$ (indeed supp$\zeta \subset \omega_3$). Setting $Z = y - \tilde{z}$, it is easy to check that $Z(0) = y_0$, $Z(T) = 0$, and that the pair (Z, π, q) is the solution to

$$
\begin{aligned}
& Z' - \operatorname{div} \sigma(Z,\pi) = (u - \tilde{z}' + \operatorname{div}(D\tilde{z}))\,\chi_\omega \quad \text{and} \quad \operatorname{div} Z = 0 && \text{in} \quad Q, \\
& Z = 0 && \text{on} \quad \Sigma_e, \\
& Z = q' && \text{on} \quad \Sigma_i, \\
& y(0) = y^0 && \text{in} \quad \Omega, \\
& q'' + q = -\int_{\Gamma_i} \sigma(Z,\pi)\, n && \text{in} \quad (0,T), \\
& q(0) = q^0 \qquad \text{and} \qquad q'(0) = q^1\,.
\end{aligned}
$$

We notice that $u - \tilde{z}' + \operatorname{div}(D\tilde{z})$ belongs to $L^2(0,T;L^2(\Omega))$. Thus $u - \tilde{z}' + \operatorname{div}(D\tilde{z})$ is a control solution to the null controllability problem stated in Theorem 1.1, and the proof is complete. ■

Acknowledgements. The authors have been supported by CEFIPRA within the project 3701-1 'Control of systems of partial differential equations'. The first author is also supported by the ANR project CISIFS 09-BLAN-0213-03.They would like to thank Prof. J.-P. Puel for providing them a copy of Ref. [14] before its publication and for helpful discussions on results in [14] and [15].

References

[1] M. Boulakia and A. Osses, Local null controllability of a two-dimensional fluid-structure interaction problem, ESAIM COCV, 14 (2008), 142.

[2] C. Conca, J. Planchard, B. Thomas, M. Vanninathan, Problèmes mathématiques en couplage fluide-structure, Eyrolles, Paris, 1994.

[3] C. Conca, J. Planchard, M. Vanninathan, Fluids and periodic structures, Masson and J. Wiley, Paris, 1995.

[4] R. Dautray, J.-L. Lions, Analyse Mathématique et Calcul Scientifique, Masson, Paris, 1987.

[5] O. Yu. Èmanuvilov, Controllability of parabolic equations, Mat. Sbornik, 186 (6) (1995), 109-132.

[6] C. Fabre, G. Lebeau, Prolongement unique des solutions de l'équation de Stokes, Comm. Partial Differential Equations 21 (1996), 573–596.

[7] C. Fabre, G. Lebeau, Régularité et unicité pour le problème de Stokes, Comm. Partial Differential Equations, 27 (2002), 437–475.

[8] R. Farwig, H. Kozono, H. Sohr, Very weak and strong solutions to the instationary Navier-Stokes system, in Topics on partial differential equations, P. Kaplický and Š. Nečasová ed., Matfyzpress, Praha, 2007.

[9] E. Fernandez-Cara, S. Guerrero, O. Yu. Imanuvilov, J.-P. Puel, Local exact controllability of the Navier-Stokes system. J. Math. Pures Appl. 83 (2004), 1501–1542.

[10] E. Fernandez-Cara, S. Guerrero, Global Carleman estimates for parabolic systems and applications to controllability, SIAM Journal on Control and Optimization 45 (2006), 1399–1446.

[11] A. V. Fursikov, Optimal Control of distributed systems, Theory and Applications, Translations of Mathematical Monographs # 187, AMS Providence RI, 2000.

[12] A. V. Fursikov, and O. Yu. Imanuvilov, Controllabiliy of evolution equations, Lecture Notes series 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Centre, Seoul 1996.

[13] G. P. Galdi, An Introduction to the Mathematical Theory of the Navier-Stokes Equations, vol. 1, Springer-Verlag, 1994.

[14] M. Gonzalez-Burgos, S. Guerrero and J.-P. Puel, Local exact controllability to the trajectories of the Boussinesq system via a fictitious control on the divergence equation, Communications on Pure and Applied Analysis (2009), no. 1, 311–333.

[15] O. Yu. Imanuvilov, J.-P. Puel, Global Carleman estimate for weak elliptic non homogeneous Dirichlet problem, Int. Math. Research Notices, Vol. 16 (2003), 883–913.

[16] O. Imanuvilov, T. Takahashi, Exact controllability of a fluid-rigid body system, J. Math. Pures Appl., 87 (2007), 408–437.

[17] J.-L. Lions, Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués, Masson, Paris, 1988.

[18] J.-P. Raymond and M. Vanninathan, Exact controllability in fluid-solid structure: the Helmoholtz model, ESAIM Control Optim. Calc. Var 11 (2005) no. 2, 180–203 (electronic).

[19] J.-P. Raymond and M. Vanninathan, Null controllability in a heat - solid structure model, Appl. Math. Optim., 59 (2009), 247–273 (electronic).

[20] J.-P. Raymond and M. Vanninathan, Null controllability for a coupled heat - finite dimensional beam system, in Control of coupled partial differential equations, Internat. Ser. Numer. Math., Vol. 158, Birkhäuser, Basel, 2009, 221–238.

[21] J.-P. Raymond, Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions, Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (2007), 921–951.

[22] J.-P. Raymond, Stokes and Navier-Stokes equations with a nonhomogeneous divergence condition, submitted to DCDS-B.

[23] J.C. Saut and B. Scheurer, Unique Continuation for some evolution equations, J. Diff. Equations, 66 (1987), 118–139.