Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

STOKES AND NAVIER-STOKES EQUATIONS WITH A NONHOMOGENEOUS DIVERGENCE CONDITION

Abstract. In this paper, we study the existence and regularity of solutions to the Stokes and Oseen equations with a nonhomogeneous divergence condition. We also prove the existence of global weak solutions to the 3D Navier-Stokes equations when the divergence is not equal to zero. These equations intervene in control problems for the Navier-Stokes equations and in fluid-structure interaction problems.

Jean-Pierre Raymond

Université de Toulouse & CNRS,
Institut de Mathématiques de Toulouse, UMR 5219
31062 Toulouse Cedex 9 - France

1. Introduction.

Let Ω be a bounded and connected domain in $\mathbb{R}^N$, with $N = 2$ or $N = 3$, with a regular boundary Γ, and let T be positive. Set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. We are interested in the following boundary value problem for the Oseen equations:

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{z}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{z} + \nabla p = 0, \quad \text{div } \mathbf{u} = h \quad \text{ in } Q, \\ &\mathbf{u} = \mathbf{g} \ \text{ on } \Sigma, \quad \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega, \end{aligned} \tag{1.1}$$

where h and $\mathbf{g}$ are nonhomogeneous terms in the divergence and boundary conditions, and $\mathbf{u}_0$ is the initial condition. The viscosity coefficient ν is positive and the function $\mathbf{z}$ satisfies $\operatorname{div}\mathbf{z} = 0$ and belongs either to $H^1(\Omega;\mathbb{R}^N)$ or to $L^\infty(0,\infty;(H^s(\Omega))^N)$ with $s > 1/2$. We are also interested in similar problems for the Navier-Stokes equations. From the divergence theorem it follows that h and $\mathbf{g}$ must satisfy the compatibility condition

$$\int_\Omega h(\cdot, t) = \int_\Gamma \mathbf{g}(\cdot, t)\cdot\mathbf{n}(\cdot) \qquad \text{for a.e. } t \in (0, T).$$

A classical way for studying equation (1.1) is to consider the solution $(\mathbf{w}(t), \pi(t))$ to equation

$$\begin{aligned} &\lambda_0\mathbf{w}(t) - \nu\Delta\mathbf{w}(t) + (\mathbf{z}\cdot\nabla)\mathbf{w}(t) + (\mathbf{w}(t)\cdot\nabla)\mathbf{z} + \nabla\pi(t) = 0, \quad \text{div } \mathbf{w} = h(t) \quad \text{ in } \Omega, \\ &\mathbf{w}(t) = \mathbf{g}(t) \ \text{ on } \Gamma, \end{aligned}$$

for some $\lambda_0 > 0$ large enough, and next to look for $(\mathbf{u}, p)$ in the form $(\mathbf{u}, p) = (\mathbf{w}, \pi) + (\mathbf{y}, q)$. This method, that we refer as the lifting method in this paper, is helpful if h and $\mathbf{g}$ are regular enough, for example if $h \in H^1(0,T;L^2(\Omega))$ and $\mathbf{g} \in H^1(0,T;H^1(\Gamma;\mathbb{R}^N))$. Motivated by control problems and related questions, we would like to study equation (1.1) when $h \in L^2(0,T;L^2(\Omega))$ and $\mathbf{g} \in$

Key words and phrases. Navier-Stokes equations, nonhomogeneous divergence condition, nonhomogeneous Dirichlet boundary condition.

The author is supported by the ANR-Project CISIFS, 09-BLAN-0213-03.

$L^2(0,T;L^2(\Gamma;\mathbb{R}^N))$, or even when $h \in L^2(0,T;(H^1(\Omega))')$ and $\mathbf{g} \in L^2(0,T;H^{-1/2}(\Gamma;\mathbb{R}^N))$. This type of equations intervenes in linearized fluid-structure models (see [21], [22]) and in some control problems (see [12]).

For the Stokes and Navier-Stokes equations, these problems have been recently studied by Farwig, Galdi, and Sohr in [7], in the case when $h \in L^s(0,T;L^r(\Omega))$, $\mathbf{g} \in L^s(0,T;W^{-1/q,q}(\Gamma;\mathbb{R}^N))$, with $1 < r \le q < \infty$, $1 < s < \infty$, $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$ for the Stokes equations, and with $1 < r < q$, $3 < q < \infty$, $2 < s < \infty$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, $\frac{2}{s} + \frac{3}{q} = 1$ for the Navier-Stokes equations. The existence of a unique very weak local solution to the Navier-Stokes equations is established [7, Theorem 1] (see also [8]). Here we only consider the case of data in Hilbert spaces for the Oseen and Navier-Stokes equations, and we prove the existence of global solutions for the Navier-Stokes equations.

In [7], very weak solutions are defined by a transposition method (also called duality method). In [19], we have already studied equation (1.1) in the case when $h \equiv 0$, and we have introduced a new definition of weak solutions which is based on the decoupling of equation (1.1) into two equations, one satisfied by $P\mathbf{u}$ and the other one satisfied by $(I-P)\mathbf{u}$, where P is the so-called Helmholtz or Leray projector. We have shown that some new regularity results may be obtained by using this decomposition. In the present paper we would like to follow the same appraoch for equation (1.1) and for the Navier-Stokes equations.

Many results in this paper are extensions to the case when $h \neq 0$ of results obtained in [19] and the proofs are very similar, except that we have to find the right spaces for h. However, Theorem 7.1 is new even in the case when $h = 0$. Indeed in [19, Theorem 5.1], the existence of a weak solution to the 3D Navier-Stokes equation is proved when $h = 0$ and $\mathbf{g} \in \mathbf{H}^{3/4,3/4}(\Sigma)$ (see section 2 for the precise definition of the different function spaces). Here we prove the same result with $\mathbf{g} \in \mathbf{H}^{s,s}(\Sigma)$ with $s > 1/2$.

In section 2, we introduce spaces of functions $(\mathbf{g}, h)$ satisfying compatibility conditions.

In section 3, we study the regularity of solutions to the Stokes equations by using the lifting method in the case of regular data, and by the transposition method in the case of data with low regularity. We introduce a new definition of weak solutions for the Stokes equations in section 4, and we derive new regularity results in section 5. We study the Oseen equations in section 6, where we adapt to Oseen equations the results obtained in section 4 for the Stokes equations. In section 7, we study the Navier-Stokes equations, and we prove the existence of weak solutions, global in time, when h and $\mathbf{g}$ are non zero. In appendices 1 and 2, we have collected some technical results for stationary problems.

It seems difficult to compare our results with the ones in [7]. Indeed, in [7, Theorem 1], the existence of a unique local weak solution $\mathbf{u}$ in $L^s(0,T';L^q(\Omega))$ is proved when $h \in L^s(0,T;L^r(\Omega))$, $\mathbf{g} \in L^s(0,T;W^{-1/q,q}(\Gamma;\mathbb{R}^N))$, with $1 < r < q$, $3 < q < \infty$, $2 < s < \infty$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, $\frac{2}{s} + \frac{3}{q} = 1$, and when the initial condition belongs to a space intermediate between $(D(A_q))'$ and $(D((-A_q)^{1-\varepsilon}))'$ with $0 < \varepsilon < 1$ (where A_q is the Stoles operator in $\{\mathbf{u} \in L^q(\Omega;\mathbb{R}^3) \mid \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}\cdot\mathbf{n}|_\Gamma = 0\}$). Here, we prove the existence of a global in time solution (not necessarily unique), when $\mathbf{g} \in \mathbf{H}^{s,s}(\Sigma)$, $h \in L^2(0,T;H^{s-1/2}(\Omega)) \cap H^s(0,T;H^{-1/2}(\Omega))$ with $s > 1/2$, $\mathbf{g}$ and h satisfy some compatibility conditions, and $\mathbf{u}_0 \in \mathbf{V}^0_n(\Omega)$ (the space of solenoidal vectors in $L^2(\Omega;\mathbb{R}^3)$ with a normal trace equal to 0). The only comparison that we can made concerns the existence of regular solutions for the Stokes equation. If we compare the results in [7, Corollary 5] for data in Hilbert spaces, it is shown that the solution to the Stokes equation belongs to $L^2(0,T;H^2(\Omega;\mathbb{R}^3)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ when h belongs to $H^1(\Omega\times(0,T))$, $\mathbf{g} \in L^2(0,T;H^{3/2}(\Gamma;\mathbb{R}^3)) \cap H^1(0,T;H^{-1/2}(\Gamma;\mathbb{R}^3))$, $\mathbf{u}_0 \in \mathbf{V}^0_n(\Omega) \cap H^2(\Omega;\mathbb{R}^3)$, and $\mathbf{g}$, h and $\mathbf{u}_0$ satisfy some compatibility conditions. Here, in Theorem 4.5, we prove that $P\mathbf{u}$ belongs to $L^2(0,T;H^2(\Omega;\mathbb{R}^3)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ when h belongs to $L^2(0,T;H^1(\Omega)) \cap H^{3/4}(0,T;(H^{1/2}(\Omega))')$, $\mathbf{g} \in L^2(0,T;H^{3/2}(\Gamma;\mathbb{R}^3)) \cap H^{3/4}(0,T;L^2(\Gamma;\mathbb{R}^3))$, $\mathbf{u}_0 \in \mathbf{V}^0_n(\Omega)\cap H^1(\Omega;\mathbb{R}^3)$, and $\mathbf{g}$, h and $\mathbf{u}_0$ satisfy some compatibility conditions. Thus, our assumptions are weaker, but the regularity is obtained for $P\mathbf{u}$ and not for $\mathbf{u}$. The regularity result that we obtain for $(I-P)\mathbf{u}$ in Theorem 4.5 is optimal for our assumptions, but weaker than the one in [7, Corollary 5].

2. **Functional setting.** Throughout the paper we assume that Ω is at least of class C^2. Let us introduce the following function spaces: $H^s(\Omega;\mathbb{R}^N) = \mathbf{H}^s(\Omega)$, $L^2(\Omega;\mathbb{R}^N) = \mathbf{L}^2(\Omega)$, the same notation conventions are used for the spaces $H^s_0(\Omega;\mathbb{R}^N)$, and the trace spaces $H^s(\Gamma;\mathbb{R}^N)$. Throughout what follows, for all $\mathbf{u}\in\mathbf{L}^2(\Omega)$ such that $\operatorname{div}\mathbf{u}\in L^2(\Omega)$, we denote by $\mathbf{u}\cdot\mathbf{n}$ the normal trace of $\mathbf{u}$ in $H^{-1/2}(\Gamma)$ [24]. Following [10], we use the letter $\mathbf{V}$ to define different spaces of divergence free vector fields:

$$\mathbf{V}^s(\Omega) = \Big\{\mathbf{u}\in\mathbf{H}^s(\Omega) \mid \operatorname{div}\mathbf{u} = 0 \ \text{ in } \Omega,\ \big\langle \mathbf{u}\cdot\mathbf{n}, 1\big\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0\Big\} \quad s\ge 0,$$

$$\mathbf{V}^s_n(\Omega) = \Big\{\mathbf{u}\in\mathbf{H}^s(\Omega) \mid \operatorname{div}\mathbf{u} = 0 \ \text{ in } \Omega,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\Big\} \qquad \text{for } s\ge 0,$$

$$\mathbf{V}^s_0(\Omega) = \Big\{\mathbf{u}\in\mathbf{H}^s(\Omega) \mid \operatorname{div}\mathbf{u} = 0 \ \text{ in } \Omega,\ \mathbf{u} = 0 \text{ on } \Gamma\Big\} \qquad \text{for } s > 1/2.$$

For spaces of time dependent functions we set

$$\mathbf{V}^{s,\sigma}(Q) = H^\sigma(0,T;\mathbf{V}^0(\Omega)) \cap L^2(0,T;\mathbf{V}^s(\Omega))$$

and

$$\mathbf{H}^{s,\sigma}(\Sigma) = H^\sigma(0,T;\mathbf{L}^2(\Gamma)) \cap L^2(0,T;\mathbf{H}^s(\Gamma)).$$

Observe that

$$\mathbf{V}^{s,\sigma}(Q) = \mathbf{H}^{s,\sigma}(Q) \cap L^2(0,T;\mathbf{V}^0(\Omega)) \qquad \text{for all } s\ge 0 \text{ and } \sigma\ge 0,$$

where $\mathbf{H}^{s,\sigma}(Q) = (H^{s,\sigma}(Q))^N$, and $H^{s,\sigma}(Q)$ corresponds to the notation in [17]. We introduce spaces of functions of zero mean value:

$$\mathcal{H}^s(\Omega) = \Big\{h\in H^s(\Omega) \mid \int_\Omega h = 0\Big\} \qquad \text{for } s\ge 0,$$

and for $s<0$, $\mathcal{H}^s(\Omega)$ is the dual space of $\mathcal{H}^{-s}(\Omega)$, with $\mathcal{H}^0(\Omega)$ as pivot space.

We introduce spaces of velocity fields defined on Γ and of functions defined in Ω, and satisfying a compatibility condition. For $-1/2\le s\le 2$ and $\sigma\ge 0$, we set

$$\mathbf{H}^{s,\sigma}_{\Gamma,\Omega} = \Big\{(\mathbf{g},h)\in\mathbf{H}^s(\Gamma)\times H^\sigma(\Omega) \mid \big\langle \mathbf{g}\cdot\mathbf{n}, 1\big\rangle_{H^s(\Gamma),H^{-s}(\Gamma)} = \int_\Omega h\Big\}.$$

If $-1/2\le s\le 2$ and $-1\le\sigma\le 0$, we set:

$$\mathbf{H}^{s,\sigma}_{\Gamma,\Omega} = \Big\{(\mathbf{g},h)\in\mathbf{H}^s(\Gamma)\times (H^{-\sigma}(\Omega))' \mid \big\langle \mathbf{g}\cdot\mathbf{n}, 1\big\rangle_{H^s(\Gamma),H^{-s}(\Gamma)} = \big\langle h, 1\big\rangle_{(H^{-\sigma}(\Omega))',H^{-\sigma}(\Omega)}\Big\}.$$

Observe that, for $-1/2 \le s \le 0$ and $-1 \le \sigma \le 0$, the space $\mathbf{H}^{s,\sigma}_{\Gamma,\Omega}$ is the dual space of $\mathbf{H}^{-s,-\sigma}_{\Gamma,\Omega}$, with respect to the pivot space $\mathbf{H}^{0,0}_{\Gamma,\Omega}$.

We denote by $\gamma_\tau \in \mathcal{L}(\mathbf{L}^2(\Gamma))$ and $\gamma_n \in \mathcal{L}(\mathbf{L}^2(\Gamma))$ the operators defined by

$$\gamma_n \mathbf{u} = (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \quad \text{and} \quad \gamma_\tau \mathbf{u} = \mathbf{u} - \gamma_n \mathbf{u} \quad \text{for all } \mathbf{u} \in \mathbf{L}^2(\Gamma).$$

As usual, for $s > 1/2$, $\gamma_0 \in \mathcal{L}(\mathbf{H}^s(\Omega), \mathbf{H}^{s-1/2}(\Gamma))$ denotes the trace operator.

Throughout the paper, for all $\mathbf{\Phi} \in \mathbf{H}^{3/2+\varepsilon'}(\Omega)$ and all $\psi \in H^{1/2+\varepsilon}(\Omega)$, with $\varepsilon > 0$, we denote by $k(\mathbf{\Phi}, \psi)$ the constant defined by

$$k(\mathbf{\Phi}, \psi) = \frac{1}{|\Gamma| + |\Omega|} \left(\int_\Gamma \left(\nu \frac{\partial \mathbf{\Phi}}{\partial \mathbf{n}} \cdot \mathbf{n} - \psi \right) - \int_\Omega \psi \right), \tag{2.1}$$

where $|\Omega|$ is the N-dimensional Lebesgue measure of Ω and $|\Gamma|$ is the $(N-1)$-dimensional Lebesgue measure of Γ.

We also introduce the space

$$W(0,T;\mathbf{V}^1(\Omega), \mathbf{V}^{-1}(\Omega)) = \Big\{ \mathbf{u} \in L^2(0,T;\mathbf{V}^1(\Omega)) \mid \frac{d\mathbf{u}}{dt} \in L^2(0,T;\mathbf{V}^{-1}(\Omega)) \Big\},$$

where $\mathbf{V}^{-1}(\Omega)$ denotes the dual space of $\mathbf{V}^1_0(\Omega)$ with $\mathbf{V}^0_n(\Omega)$ as pivot space.

Let us denote by P the orthogonal projection operator in $\mathbf{L}^2(\Omega)$ on $\mathbf{V}^0_n(\Omega)$. Recall that the Stokes operator $A = \nu P \Delta$, with domain $D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V}^1_0(\Omega)$ in $\mathbf{V}^0_n(\Omega)$, is the infinitesimal generator of a strongly continuous analytic semigroup $(e^{tA})_{t\ge0}$ on $\mathbf{V}^0_n(\Omega)$.

We also introduce the operators $L \in \mathcal{L}(\mathbf{H}^{0,0}_{\Gamma,\Omega}, \mathbf{L}^2(\Omega))$ and $L_p \in \mathcal{L}(\mathbf{H}^{0,0}_{\Gamma,\Omega}, \mathcal{H}^0(\Omega))$ defined by

$$L(\mathbf{g}, h) = \mathbf{w} \qquad \text{and} \qquad L_p(\mathbf{g}, h) = \pi,$$

where $(\mathbf{w}, \pi)$ is the solution to

$$-\nu\Delta\mathbf{w} + \nabla\pi = 0 \quad \text{and} \quad \text{div } \mathbf{w} = h \text{ in } \Omega, \quad \mathbf{w} = \mathbf{g} \quad \text{on } \Gamma.$$

Notice that L can be extended to a bounded operator from $\mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}$ into $\mathcal{H}^0(\Omega)$ (see Corollary 8.4).

3. **First regularity results for the Stokes equation.** In this section we study the Stokes equations with a nonhomogeneous divergence condition:

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla p = 0, \ \text{div}\, \mathbf{u} = h \quad \text{in } Q, \\ &\mathbf{u} = \mathbf{g} \ \text{ on } \Sigma, \quad \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega. \end{aligned} \tag{3.1}$$

There are two classical possibilities to define weak solutions to equation (3.1): the lifting method, and the transposition method. The lifting method consists in looking for a solution $(\mathbf{u}, p)$ to equation (3.1) in the form $(\mathbf{u}, p) = (\mathbf{y}, q) + (\mathbf{w}, \pi)$, where, for all $t \in [0, T]$, $(\mathbf{w}(t), \pi(t))$ is the solution to the equation:

$$-\nu\Delta\mathbf{w}(t) + \nabla\pi(t) = 0 \quad \text{and} \quad \text{div}\, \mathbf{w}(t) = h(t) \text{ in } \Omega, \quad \mathbf{w}(t) = \mathbf{g}(t) \quad \text{on } \Gamma. \tag{3.2}$$

In the case when $(\mathbf{u}, p)$, $(\mathbf{y}, q)$, and $(\mathbf{w}, \pi)$ are regular functions, $(\mathbf{y}, q)$ is the solution to the equation

$$\begin{aligned} &\frac{\partial \mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + \nabla q = -\frac{\partial \mathbf{w}}{\partial t}, \ \text{div } \mathbf{y} = 0 \quad \text{in } Q, \\ &\mathbf{y} = 0 \ \text{ on } \Sigma, \quad \mathbf{y}(0) = P(\mathbf{u}_0 - \mathbf{w}(0)) \ \text{ in } \Omega. \end{aligned} \tag{3.3}$$

This leads to the following definition.

Definition 3.1. A function $(\mathbf{u}, p) \in L^2(0,T;\mathbf{L}^2(\Omega)) \times L^2(0,T;\mathcal{H}^{-1}(\Omega))$ is a weak solution to equation (3.1), if $(\mathbf{u}, p) = (\mathbf{y}, q) + (\mathbf{w}, \pi)$, where, for all $t \in [0,T]$, $(\mathbf{w}(t), \pi(t))$ is the solution to equation (3.2), and $(\mathbf{y}, q)$ is the solution to equation (3.3).

Theorem 3.2. *For all* $(\mathbf{g}, h) \in H^1\big(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\big)$*, and* $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$*, equation* (3.1) *admits a unique weak solution in the sense of Definition 3.1. It satisfies*

$$\begin{aligned}&\|\mathbf{u}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|p\|_{L^2(0,T;\mathcal{H}^{-1}(\Omega))}\\ &\le C\left(\|P\mathbf{u}_0\|_{\mathbf{V}^0_n(\Omega)} + \|(\mathbf{g},h)\|_{H^1\left(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\right)}\right).\end{aligned}\tag{3.4}$$

Proof. Let $(\mathbf{g}, h)$ be in $H^1\big(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\big)$. Due to Corollary 8.4, the solution $(\mathbf{w}(t), \pi(t))$ to equation (3.2) is unique and it satisfies

$$\|\mathbf{w}\|_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|\pi\|_{H^1(0,T;\mathcal{H}^{-1}(\Omega))} \le C\|(\mathbf{g},h)\|_{H^1\left(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\right)}.$$

Thus, in equation (3.3), $\frac{\partial \mathbf{w}}{\partial t}$ belongs to $L^2(0,T;\mathbf{L}^2(\Omega))$, and $P(\mathbf{u}_0 - \mathbf{w}(0))$ belongs to $\mathbf{V}^0_n(\Omega)$. In that case it is well known that equation (3.3) admits a unique weak solution which satisfies

$$\begin{aligned}&\|\mathbf{y}\|_{W(0,T;\mathbf{V}^1_0(\Omega),\mathbf{V}^{-1}(\Omega))} + \|q\|_{L^2(0,T;\mathcal{H}^0(\Omega))}\\ &\le C(\|P(\mathbf{u}_0 - \mathbf{w}(0))\|_{\mathbf{V}^0_n(\Omega)} + \|\mathbf{w}'\|_{L^2(0,T;\mathbf{L}^2(\Omega))}).\end{aligned}$$

The estimate for $(\mathbf{u}, p)$ can be deduced from the estimates obtained for $(\mathbf{y}, q)$ and $(\mathbf{w}, \pi)$. □

Now, let us define solutions by transposition.

Definition 3.3. Assume that $(\mathbf{g}, h) \in L^2\big(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\big)$, and $\mathbf{u}_0 \in \mathbf{H}^{-1}(\Omega)$. A function $\mathbf{u} \in \mathbf{L}^2(Q)$ is a solution to equation (3.1), in the sense of transposition, if

$$\begin{aligned}&\int_Q \mathbf{u}\cdot\mathbf{f} + \int_0^T \Big\langle \psi(t), h(t)\Big\rangle_{H^1(\Omega),(H^1(\Omega))'} dt\\ &+ \int_0^T \Big\langle \nu\frac{\partial \mathbf{\Phi}}{\partial \mathbf{n}}(t) - \psi(t)\mathbf{n}, \mathbf{g}(t)\Big\rangle_{\mathbf{H}^{1/2}(\Gamma),\mathbf{H}^{-1/2}(\Gamma)} dt\\ &= \big\langle \mathbf{u}_0, \mathbf{\Phi}(0)\big\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)},\end{aligned}\tag{3.5}$$

for all $\mathbf{f} \in \mathbf{L}^2(Q)$, where $(\mathbf{\Phi}, \psi)$ is the solution to equation

$$\begin{aligned}&-\frac{\partial \mathbf{\Phi}}{\partial t} - \nu\Delta\mathbf{\Phi} + \nabla\psi = \mathbf{f},\ \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in } Q,\\ &\mathbf{\Phi} = 0 \text{ on } \Sigma, \quad \mathbf{\Phi}(T) = 0 \text{ in } \Omega.\end{aligned}\tag{3.6}$$

Remark 3.4. Notice that the pair $(\mathbf{\Phi}, \psi)$, solution to equation (3.6) is chosen so that ψ belongs to $L^2(0,T;\mathcal{H}^1(\Omega))$, in particular $\int_\Omega \psi(t) = 0$. Since $(\mathbf{g}, h) \in L^2\big(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\big)$, in formula (3.5) we can replace ψ by $\psi + C$ for any $C \in \mathbb{R}$, because

$$\int_0^T \Big\langle C, h(t)\Big\rangle_{H^1(\Omega),(H^1(\Omega))'} dt - \int_0^T \Big\langle C\mathbf{n}, \mathbf{g}(t)\Big\rangle_{\mathbf{H}^{1/2}(\Gamma),\mathbf{H}^{-1/2}(\Gamma)} dt = 0.$$

Thus formula (3.5) is satisfied for all solutions $(\mathbf{\Phi}, \psi)$ to equation (3.6) ($\mathbf{\Phi}$ is unique but not ψ. The adjoint pressure ψ is unique up to an additive constant). In particular we can replace $\psi \in L^2(0,T;\mathcal{H}^1(\Omega))$ by $\psi + k(\mathbf{\Phi},\psi)$, where $k(\mathbf{\Phi},\psi)$ is the constant defined in (2.1). Observe that

$$\left(\nu\frac{\partial \mathbf{\Phi}}{\partial \mathbf{n}} - \psi\mathbf{n} - k(\mathbf{\Phi},\psi)\mathbf{n}, \psi + k(\mathbf{\Phi},\psi)\right) \quad \text{belongs to} \quad L^2\big(0,T;\mathbf{H}^{1/2,1}_{\Gamma,\,\Omega}\big).$$

Theorem 3.5. *For all* $(\mathbf{g},h) \in L^2\big(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\big)$*, and* $\mathbf{u}_0 \in \mathbf{H}^{-1}(\Omega)$*, equation* (3.1) *admits a unique solution in the sense of Definition 3.3. It satisfies*

$$\|\mathbf{u}\|_{\mathbf{L}^2(Q)} \le C\left(\|\mathbf{u}_0\|_{\mathbf{H}^{-1}(\Omega)} + \|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\right)}\right). \tag{3.7}$$

Moreover, there exists a distribution $p \in \mathcal{D}'(Q)$*, such that*

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla p = 0 \quad \text{in } \mathcal{D}'(Q), \tag{3.8}$$

and $\mathbf{u}$ *obeys*

$$\operatorname{div}\mathbf{u} = h \quad \text{in } L^2(0,T;H^{-1}(\Omega)).$$

Proof. To prove the uniqueness, we assume that $\mathbf{u}_0 = 0$, $\mathbf{g} = 0$ and $h = 0$. In that case, if $\mathbf{u}$ is a solution to equation (3.1) in the sense of Definition 3.3, by setting $\mathbf{f} = \mathbf{u}$ in (3.5), we deduce that $\mathbf{u} = 0$. Thus the uniqueness is established.

Let us denote by Λ the mapping

$$\Lambda \;:\; \mathbf{f} \longmapsto \left(\mathbf{\Phi}(0), -\nu\frac{\partial \mathbf{\Phi}}{\partial \mathbf{n}} + \psi\mathbf{n} + k(\mathbf{\Phi},\psi)\mathbf{n}, -\psi - k(\mathbf{\Phi},\psi)\right),$$

where $(\mathbf{\Phi},\psi)$ is the solution to equation (3.6). We can easily see that Λ is a bounded operator from $\mathbf{L}^2(Q)$ into $\mathbf{H}^1_0(\Omega)\times L^2\big(0,T;\mathbf{H}^{1/2,1}_{\Gamma,\,\Omega}\big)$. Thus Λ^* is a bounded operator from $\mathbf{H}^{-1}(\Omega) \times L^2\big(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\big)$ into $\mathbf{L}^2(Q)$. If we set $\mathbf{u} = \Lambda^*(\mathbf{g},h)$, we can verify that $\mathbf{u} \in \mathbf{L}^2(Q)$ obeys (3.5). Thus we have proved the existence of a function $\mathbf{u} \in \mathbf{L}^2(Q)$ which satisfies (3.5).

Let ψ be in $L^2(0,T;H^1(\Omega))$. In that case $(\mathbf{\Phi},\psi)$, with $\mathbf{\Phi} = 0$, is the solution to equation (3.6) corresponding to $\mathbf{f} = \nabla\psi$. By choosing $\mathbf{f} = \nabla\psi$ in (3.5), we prove that

$$\begin{aligned}&\int_Q \mathbf{u}\cdot\nabla\psi + \Big\langle \psi, h\Big\rangle_{L^2(0,T;H^1(\Omega)),L^2(0,T;(H^1(\Omega))')}\\ &= \Big\langle \gamma_0\psi\,\mathbf{n}, \mathbf{g}\Big\rangle_{L^2(0,T;\mathbf{H}^{1/2}(\Gamma)),L^2(0,T;\mathbf{H}^{-1/2}(\Gamma))},\end{aligned}$$

for all $\psi \in L^2(0,T;H^1(\Omega))$.

Choosing ψ in $L^2(0,T;H^1_0(\Omega))$, we obtain $\operatorname{div}\mathbf{u} = h$ in $L^2(0,T;H^{-1}(\Omega))$.

Let $\mathbf{\Phi}$ be in $(\mathcal{D}(Q))^N$ such that $\operatorname{div}\mathbf{\Phi} = 0$. The pair $(\mathbf{\Phi},\psi)$, with $\psi = 0$, is the solution to equation (3.6) corresponding to $\mathbf{f} = -\partial_t\mathbf{\Phi} - \nu\Delta\mathbf{\Phi}$. With this choice for $\mathbf{f}$ in (3.5), we prove that

$$\Big\langle \partial_t\mathbf{u} - \nu\Delta\mathbf{u}, \mathbf{\Phi}\Big\rangle_{(\mathcal{D}'(Q))^N,(\mathcal{D}(Q))^N} = \int_Q \mathbf{u}(-\partial_t\mathbf{\Phi} - \nu\Delta\mathbf{\Phi}) = 0,$$

for all $\mathbf{\Phi} \in (\mathcal{D}(Q))^N$ such that $\operatorname{div}\mathbf{\Phi} = 0$. From de Rham Theorem, it follows that there exists a distribution p such that (3.8) is satisfied. □

Theorem 3.6. *For all* $(\mathbf{g},h) \in H^1\big(0,T;\mathbf{H}^{-1/2,0}_{\Gamma,\,\Omega}\big)$*, and* $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$*,* $\mathbf{u}$ *is a solution to equation* (3.1) *in the sense of Definition 3.1 if, and only if, it is a solution in the sense of Definition 3.3.*

Proof. We first establish the theorem in the case when $(\mathbf{g},h)\in C^1\big([0,T];\mathbf{H}^{3/2,1}_{\Gamma,\,\Omega}\big)$ and $P(\mathbf{u}_0-L(\mathbf{g}(0),h(0)))\in \mathbf{V}^1_0(\Omega)$. In that case, the solution $(\mathbf{u},p)=(\mathbf{w},\pi)+(\mathbf{y},q)$ to equation (3.1) in the sense of Definition 3.1 is such that $\mathbf{w}\in C^1([0,T];\mathbf{H}^2(\Omega))$, $\pi\in C^1([0,T];\mathcal{H}^1(\Omega))$, $\mathbf{y}\in \mathbf{V}^{2,1}(Q)$, and $q\in L^2(0,T;\mathcal{H}^1(\Omega))$. Therefore, we can easily verify that $\mathbf{u}$ is a solution to equation (3.1) in the sense of Definition 3.3. Since the solution to equation (3.1) in the sense of Definition 3.3 is unique, the theorem is proved in that case.

Now assume that $(\mathbf{g},h)\in H^1\big(0,T;\mathbf{H}^{-1/2,0}_{\Gamma,\,\Omega}\big)$, and $\mathbf{u}_0\in\mathbf{L}^2(\Omega)$. Consider a sequence $\{(\mathbf{g}_k,h_k)\}\subset C^1\big([0,T];\mathbf{H}^{3/2,1}_{\Gamma,\,\Omega}\big)$ converging to $(\mathbf{g},h)$ in $H^1\big(0,T;\mathbf{H}^{-1/2,0}_{\Gamma,\,\Omega}\big)$, and a sequence $\{\mathbf{u}_{0,k}\}\subset\mathbf{H}^1(\Omega)$ converging to $\mathbf{u}_0$ in $\mathbf{L}^2(\Omega)$ and such that $\{P\mathbf{u}_{0,k}-L(\mathbf{g}_k(0),h_k(0))\}\subset\mathbf{V}^1_0(\Omega)$ converges to $P(\mathbf{u}_0-L(\mathbf{g}(0),h(0)))$ in $\mathbf{V}^0_n(\Omega)$. Let $\mathbf{u}_k$ be the solution to equation (3.1) corresponding to $(\mathbf{g}_k,h_k)$ and $\mathbf{u}_{0,k}$. According to step 1, the solutions in the sense of Definitions 3.1 and 3.3 coincide. From (3.4), it follows that $\{\mathbf{u}_k\}$ converges in $\mathbf{L}^2(Q)$ to the solution $\mathbf{u}$ to equation (3.1) in the sense of Definition 3.1, and from (3.7), it follows that $\{\mathbf{u}_k\}$ converges in $\mathbf{L}^2(Q)$ to the solution of equation (3.1) in the sense of Definition 3.3. The proof is complete. □

Definition 3.7. Assume that $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{-1/2,0}_{\Gamma,\,\Omega}\big)$ and $\mathbf{u}_0\in\mathbf{H}^{-1}(\Omega)$. A function $\mathbf{u}\in\mathbf{L}^2(Q)$ is a very weak solution to equation (3.1), if

$$\begin{aligned}&\int_Q \mathbf{u}\cdot(-\partial_t\mathbf{\Phi}-\nu\Delta\mathbf{\Phi})\\&=\big\langle\mathbf{u}_0,\mathbf{\Phi}(0)\big\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)}-\int_0^T\Big\langle\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}}(t),\mathbf{g}(t)\Big\rangle_{\mathbf{H}^{1/2}(\Gamma),\mathbf{H}^{-1/2}(\Gamma)}dt\end{aligned}\tag{3.9}$$

for all $\mathbf{\Phi}\in L^2(0,T;\mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega))\cap H^1(0,T;\mathbf{V}^0_n(\Omega))$ such that $\mathbf{\Phi}(T)=0$, and if in addition

$$\operatorname{div}\mathbf{u}(t)=h(t),\quad \gamma_n\mathbf{u}(t)=\gamma_n\mathbf{g}(t)\quad\text{for a.e. } t\in(0,T).$$

Remark 3.8. In [7], weak solutions to equation (3.1) are defined as in Definition 3.7 with test functions $\mathbf{\Phi}\in C^1_0([0,T);C^2_{0,\sigma}(\overline{\Omega}))$, $C^2_{0,\sigma}(\overline{\Omega})$ denotes the space of divergence free functions, belonging to $C^2(\overline{\Omega})$, and whose trace on Γ is equal to zero. Thus Definition 3.7 is equivalent to the definition in [7] when Ω is regular enough.

Theorem 3.9. *For all $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{-1/2,0}_{\Gamma,\,\Omega}\big)$, and $\mathbf{u}_0\in\mathbf{H}^{-1}(\Omega)$, $\mathbf{u}$ is a solution to equation (3.1) in the sense of Definition 3.7 if, and only if, it is a solution in the sense of Definition 3.3. Moreover, it satisfies*

$$\|\mathbf{u}\|_{\mathbf{L}^2(Q)}\le C\left(\|\mathbf{u}_0\|_{\mathbf{H}^{-1}(\Omega)}+\|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{-1/2,0}_{\Gamma,\,\Omega}\right)}\right).$$

Proof. Let us assume that $\mathbf{u}\in\mathbf{L}^2(Q)$ is a very weak solution to equation (3.1) in the sense of Definition 3.7. Let $\mathbf{f}$ belong to $(\mathcal{D}(Q))^N$, and let $(\mathbf{\Phi},\psi)$ be the solution to equation (3.6) corresponding to $\mathbf{f}$. From Definition 3.7, it follows that

$$\begin{aligned}&\big\langle\mathbf{u}_0,\mathbf{\Phi}(0)\big\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)}-\int_0^T\Big\langle\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}}(t),\mathbf{g}(t)\Big\rangle_{\mathbf{H}^{1/2}(\Gamma),\mathbf{H}^{-1/2}(\Gamma)}dt\\&=\int_Q\mathbf{u}\cdot(-\partial_t\mathbf{\Phi}-\nu\Delta\mathbf{\Phi})=\int_Q\mathbf{u}\cdot(\mathbf{f}-\nabla\psi)\\&=\int_Q\mathbf{u}\cdot\mathbf{f}+\int_0^T\big\langle\psi(t),h(t)\big\rangle_{H^1(\Omega),(H^1(\Omega))'}dt-\int_0^T\big\langle\psi(t)\mathbf{n},\mathbf{g}(t)\big\rangle_{\mathbf{H}^{1/2}(\Gamma),\mathbf{H}^{-1/2}(\Gamma)}dt.\end{aligned}$$

Thus equation (3.5) is satisfied for all $\mathbf{f} \in (\mathcal{D}(Q))^N$. By a density argument it follows that $\mathbf{u}$ is a solution to equation (3.1) in the sense of Definition 3.3.

Conversely, assume that $\mathbf{u}$ is a solution to equation (3.1) in the sense of Definition 3.3. With Theorem 3.5, we already know that $\operatorname{div}\mathbf{u} = h$ in $L^2(0,T;H^{-1}(\Omega))$. Since $h \in L^2(Q)$, $\gamma_n\mathbf{u}$ is well defined in $L^2(0,T;\mathbf{H}^{-1/2}(\Gamma))$, and we have

$$\int_Q \mathbf{u}\cdot\nabla\psi = -\int_Q h\psi + \Big\langle \gamma_n\mathbf{u}, \gamma_0\psi\,\mathbf{n}\Big\rangle_{L^2(0,T;\mathbf{H}^{-1/2}(\Gamma)),L^2(0,T;\mathbf{H}^{1/2}(\Gamma))},$$

for all $\psi \in L^2(0,T;H^1(\Omega))$. Since we have

$$\begin{aligned}&\int_Q \mathbf{u}\cdot\nabla\psi + \Big\langle \psi, h\Big\rangle_{L^2(0,T;H^1(\Omega)),L^2(0,T;(H^1(\Omega))')}\\ &= \Big\langle \gamma_0\psi\,\mathbf{n}, \mathbf{g}\Big\rangle_{L^2(0,T;\mathbf{H}^{1/2}(\Gamma)),L^2(0,T;\mathbf{H}^{-1/2}(\Gamma))},\end{aligned}$$

for all $\psi \in L^2(0,T;\mathcal{H}^1(\Omega))$, we deduce that $\gamma_n\mathbf{u} = \gamma_n\mathbf{g}$.

Let $\mathbf{\Phi}$ belong to $C_0^1([0,T);C^2_{0,\sigma}(\overline{\Omega}))$, and set $\mathbf{f} = -\frac{\partial\mathbf{\Phi}}{\partial t} - \nu\Delta\mathbf{\Phi}$ and $\psi = 0$. From (3.5), it follows that

$$\begin{aligned}&\int_Q \mathbf{u}\cdot(-\frac{\partial\mathbf{\Phi}}{\partial t} - \nu\Delta\mathbf{\Phi}) + \int_Q \psi\, h + \int_0^T \Big\langle \nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}}(t) - \psi(t)\mathbf{n}, \mathbf{g}(t)\Big\rangle_{\mathbf{H}^{1/2}(\Gamma),\mathbf{H}^{-1/2}(\Gamma)} dt\\ &= \big\langle \mathbf{u}_0, \mathbf{\Phi}(0)\big\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)}.\end{aligned}$$

Thus $\mathbf{u} \in \mathbf{L}^2(Q)$ is a very weak solution to equation (3.1) in the sense of Definition 3.7. □

Remark 3.10. We notice that Definitions 3.3 and 3.7 are equivalent in the case when $(\mathbf{g},h) \in L^2\big(0,T;\mathbf{H}^{-1/2,0}_{\Gamma,\,\Omega}\big)$, but that Definition 3.7 cannot be used if $(\mathbf{g},h) \in L^2\big(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\big)$. Indeed in that case $\gamma_n\mathbf{u}$ cannot be defined.

4. **A new definition of weak solution.** In this section, we are going to give a new definition of weak solution to equation (3.1). Thanks to this new definition, we are able to obtain new regularity results for $P\mathbf{u}$ and $(I-P)\mathbf{u}$, where $\mathbf{u}$ is the solution to equation (3.1).

We first consider the case of regular data. Assume that $(\mathbf{g},h) \in C^1\big([0,T];\mathbf{H}^{3/2,1}_{\Gamma,\,\Omega}\big)$ and $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$. Let us set $\mathbf{w}(t) = L(\mathbf{g}(t),h(t))$ and $\pi(t)) = L_p(\mathbf{g}(t),h(t))$. It is clear that $(\mathbf{w},\pi) \in C^1([0,T];\mathbf{H}^2(\Omega)\times\mathcal{H}^1(\Omega))$. Let $(\mathbf{y},q)$ be the weak solution in $W(0,T;\mathbf{V}^1_0(\Omega),\mathbf{V}^{-1}(\Omega))\times L^2(0,T;\mathcal{H}^0(\Omega))$ to the equation (3.3). We set $\mathbf{u} = \mathbf{w}+\mathbf{y}$. We have already seen that $\mathbf{u} = \mathbf{w}+\mathbf{y}$ is a solution to equation (3.1) in the sense of Definition 3.1.

Equation (3.3) can be rewritten in the form

$$\mathbf{y}' = A\mathbf{y} - P\mathbf{w}', \qquad \mathbf{y}(0) = P(\mathbf{u}_0 - \mathbf{w}(0)),$$

and $\mathbf{y}$ is defined by

$$\mathbf{y}(t) = e^{tA}P(\mathbf{u}_0 - \mathbf{w}(0)) - \int_0^t e^{(t-s)A}P\mathbf{w}'(s)\,ds,$$

where $A = \nu P\Delta$, with domain $\mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega)$, is the Stokes operator in $\mathbf{V}^0_n(\Omega)$. Integrating by parts we obtain

$$\mathbf{y}(t) = e^{tA}P(\mathbf{u}_0 - \mathbf{w}(0)) + \int_0^t (-A)e^{(t-s)A}P\mathbf{w}(s)\,ds - P\mathbf{w}(t) + e^{tA}P\mathbf{w}(0).$$

Thus we have

$$P\mathbf{u}(t) = \mathbf{y}(t) + P\mathbf{w}(t) = e^{tA}P\mathbf{u}_0 + \int_0^t (-A)e^{(t-s)A}PL(\mathbf{g}(s), h(s))\,ds.$$

With the extrapolation method, we can extend the operator A to an unbounded operator $\tilde{A}$ of domain $D(\tilde{A}) = \mathbf{V}^0_n(\Omega)$ in $(D(A^*))' = (D(A))'$, in order that $(\tilde{A}, D(\tilde{A}))$ is the infinitesimal generator of a strongly continuous semigroup $(e^{t\tilde{A}})_{t\geq 0}$ on $(D(A^*))'$, satisfying $e^{tA}\mathbf{u}_0 = e^{t\tilde{A}}\mathbf{u}_0$ for all $\mathbf{u}_0 \in \mathbf{V}^0_n(\Omega)$. This means that $P\mathbf{u}$ is solution to the equation

$$P\mathbf{u}' = \tilde{A}P\mathbf{u} + (-\tilde{A})PL(\mathbf{g}, h), \qquad P\mathbf{u}(0) = P\mathbf{u}_0.$$

The equation satisfied by $(I - P)\mathbf{u}$ is nothing else than

$$(I - P)\mathbf{u}(t) = (I - P)\mathbf{w}(t) = (I - P)L(\mathbf{g}(t), h(t)).$$

The operator PL is continuous and linear from $\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}$ to $\mathbf{V}^0_n(\Omega)$. Thus $(-\tilde{A})PL$ is continuous and linear from $\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}$ to $(D(-A^*))'$. Consequently $(-\tilde{A})PL(\mathbf{g}, h)$ belongs to $L^2(0, T; (D(-A^*)')$ if $(\mathbf{g}, h)$ belongs to $L^2\big(0, T; \mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\big)$.

We can now state a new definition of weak solution.

Definition 4.1. A function $\mathbf{u} \in L^2(0, T; \mathbf{L}^2(\Omega))$ is a weak solution to equation (3.1) if

$P\mathbf{u}$ is a weak solution of the following evolution equation

$$P\mathbf{u}' = \tilde{A}P\mathbf{u} + (-\tilde{A})PL(\mathbf{g}, h), \qquad P\mathbf{u}(0) = P\mathbf{u}_0, \tag{4.1}$$

and if $(I - P)\mathbf{u}$ is defined by

$$(I - P)\mathbf{u}(\cdot) = (I - P)L(\mathbf{g}, h)(\cdot) \quad \text{in } L^2(0, T; \mathbf{L}^2(\Omega)). \tag{4.2}$$

By definition (see [4]), a function $P\mathbf{u} \in L^2(0, T; \mathbf{V}^0_n(\Omega))$ is weak solution to equation (4.1) if and only if, for all $\mathbf{\Phi} \in D(A^*)$, the mapping $t \mapsto \int_\Omega P\mathbf{u}(t)\mathbf{\Phi}$ belongs to $H^1(0, T)$ and satisfies

$$\frac{d}{dt}\int_\Omega P\mathbf{u}(t)\mathbf{\Phi} = \int_\Omega P\mathbf{u}(t)\, A^*\mathbf{\Phi} + \big\langle(-\tilde{A})PL(\mathbf{g}(t), h(t)), \mathbf{\Phi}\big\rangle_{(D(A^*))', D(A^*)}$$

$$\text{and} \quad \int_\Omega P\mathbf{u}(t)\mathbf{\Phi}\Big|_{t=0} = \int_\Omega P\mathbf{u}_0\mathbf{\Phi}.$$

Observe that $A^* = A$ and that

$$\big\langle(-\tilde{A})PL(\mathbf{g}(t), h(t)), \mathbf{\Phi}\big\rangle_{(D(A^*))', D(A^*)} = \big\langle(\mathbf{g}(t), h(t)), L^*(-A^*)\mathbf{\Phi}\big\rangle_{\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}, \mathbf{H}^{1/2,1}_{\Gamma,\,\Omega}}.$$

Due to Lemma 8.5, we have

$$\begin{aligned}
&\big\langle(\mathbf{g}(t), h(t)), L^*(-A^*)\mathbf{\Phi}\big\rangle_{\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}, \mathbf{H}^{1/2,1}_{\Gamma,\,\Omega}} \\
&= \Big\langle(\mathbf{g}, h), \Big(-\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}} + \psi\mathbf{n} + k(\Phi, \psi)\mathbf{n}, -\psi(t) - k(\Phi, \psi)\Big)\Big\rangle_{\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}, \mathbf{H}^{1/2,1}_{\Gamma,\,\Omega}} \\
&= \big\langle\mathbf{g}(t), -\nu\frac{\partial\mathbf{\Phi}(t)}{\partial\mathbf{n}} + \psi(t)\mathbf{n}\big\rangle_{\mathbf{H}^{-1/2}(\Gamma), \mathbf{H}^{1/2}(\Gamma)} - \big\langle h(t), \psi(t)\big\rangle_{(H^1(\Omega))', H^1(\Omega)},
\end{aligned}$$

where $\psi \in \mathcal{H}^1(\Omega)$ is determined by

$$\nabla\psi = \nu(I - P)\Delta\mathbf{\Phi}\,. \tag{4.3}$$

Thus the variational equation satisfied by $P\mathbf{u}$ is nothing else than

$$\begin{aligned}\frac{d}{dt}\int_\Omega P\mathbf{u}(t)\mathbf{\Phi} = \int_\Omega P\mathbf{u}(t)\,A\mathbf{\Phi} + \big\langle \mathbf{g}(t), -\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}} + \psi(t)\mathbf{n}\big\rangle_{\mathbf{H}^{-1/2}(\Gamma),\mathbf{H}^{1/2}(\Gamma)} \\ -\big\langle h(t),\psi\big\rangle_{(H^1(\Omega))',H^1(\Omega)},\end{aligned} \tag{4.4}$$

for all $\mathbf{\Phi}\in D(A^*)$, with ψ defined by (4.3).

Remark 4.2. From the calculations of the beginning of the section, it follows that, if $(\mathbf{g},h)\in C^1\big([0,T];\mathbf{H}^{3/2,1}_{\Gamma,\,\Omega}\big)$ and $\mathbf{u}_0\in\mathbf{L}^2(\Omega)$, then the solution $\mathbf{u}\in\mathbf{L}^2(Q)$ to equation (3.1) in the sense of Definition 3.1 is also a solution in the sense of Definition 4.1.

Remark 4.3. Notice that in Definition 4.1, we do not require that $\mathbf{u}(0)=\mathbf{u}_0$, we only impose the initial condition $P\mathbf{u}(0)=P\mathbf{u}_0$. Indeed if $(\mathbf{g},h)\in L^2(0,T;\mathbf{H}^{1/2,0}_{\Gamma,\,\Omega})$, then $(I-P)\mathbf{u}=(I-P)L(\mathbf{g},h)$ belongs to $L^2(0,T;\mathbf{H}^1(\Omega))$, $(I-P)\mathbf{u}(0)$ is not defined, and therefore the initial condition of $(I-P)\mathbf{u}$ cannot be defined. On the other hand if $(\mathbf{g},h)\in H^s(0,T;L^2(0,T;\mathbf{H}^{1/2,0}_{\Gamma,\,\Omega})$ with $s>1/2$, then $(I-P)\mathbf{u}=(I-P)L(\mathbf{g},h)$ belongs to $H^s(0,T;\mathbf{H}^1(\Omega))$, and $(I-P)\mathbf{u}(0)$ is well defined in $\mathbf{H}^1(\Omega)$. If $(I-P)L(\mathbf{g}(0),h(0))=(I-P)\mathbf{u}_0$, then the solution defined in Definition 4.1 satisfies $\mathbf{u}(0)=\mathbf{u}_0$. Otherwise we only have $P\mathbf{u}(0)=P\mathbf{u}_0$.

Theorem 4.4. *For all $P\mathbf{u}_0\in\mathbf{V}^{-1}(\Omega)$, all $(\mathbf{g},h)\in L^2(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega})$, equation (3.1), admits a unique weak solution in $L^2(0,T;\mathbf{L}^2(\Omega))$ in the sense of Definition 4.1. This solution obeys*

$$\begin{aligned}&\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}^0_n(\Omega))}+\|P\mathbf{u}\|_{H^1(0,T;(\mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega))')}+\|(I-P)\mathbf{u}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\\ &\le C(\|P\mathbf{u}_0\|_{\mathbf{V}^{-1}(\Omega)}+\|h\|_{L^2(0,T;(H^1(\Omega))')}+\|\mathbf{g}\|_{L^2(0,T;\mathbf{H}^{-1/2}(\Omega))}).\end{aligned}$$

Proof. Step 1. The system

$$P\mathbf{u}'=\tilde{A}P\mathbf{u},\quad P\mathbf{u}(0)=0,\quad\text{and}\quad (I-P)\mathbf{u}=0,$$

admits $\mathbf{u}=0$ as unique solution. Thus uniqueness of solution to equation (3.1) is obvious. Let us prove the existence. Let us first asume that $\mathbf{g}\in C^1([0,T];\mathbf{H}^{3/2}(\Omega))$, $h\in C^1([0,T];\mathcal{H}^1(\Omega))$ and $\mathbf{u}_0\in\mathbf{L}^2(\Omega)$. We have already seen that, if $(\mathbf{w}(t),\pi(t))$ is the solution to (3.2), and if $(\mathbf{y},q)$ is the solution to (3.3), then the solution $\mathbf{u}=\mathbf{w}+\mathbf{y}$ to equation (3.1) in the sense of Definition 3.1 is also a solution to equation (3.1) in the sense of Definition 4.1.

Now suppose that $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\big)$ and $\mathbf{u}_0\in\mathbf{H}^{-1}(\Omega)$. Let $(\mathbf{g}_k,h_k)_k$ be a sequence in $C^1\big([0,T];\mathbf{H}^{3/2,1}_{\Gamma,\,\Omega}\big)$ converging to $(\mathbf{g},h)$ in $L^2\big(0,T;\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}\big)$, and let $(\mathbf{u}_{0,k})_k$ be a sequence in $\mathbf{L}^2(\Omega)$ converging to $\mathbf{u}_0$ in $\mathbf{H}^{-1}(\Omega)$. Let $(\mathbf{w}_k(t),\pi_k(t))$ be the solution to equation (3.2) corresponding to $(\mathbf{g}_k(t),h_k(t))$, let $(\mathbf{y}_k,q_k)$ be the weak solution to equation (3.3) corresponding to $\mathbf{w}_k$ and $\mathbf{u}_{0,k}$, and set $\mathbf{u}_k=\mathbf{w}_k+\mathbf{y}_k$. We have already seen that $(\mathbf{u}_k)_k$ converges in $L^2(0,T;\mathbf{L}^2(\Omega))$ to the solution $\mathbf{u}$ to equation (3.1). Moreover, passing to the limit when k tends to infinity in the equality $(I-P)\mathbf{u}_k=(I-P)L(\mathbf{g}_k,h_k)$, we obtain $(I-P)\mathbf{u}=(I-P)L(\mathbf{g},h)$. Knowing that $(P\mathbf{u}_k)_k$ converges to $P\mathbf{u}$ in $L^2(0,T;\mathbf{V}^0_n(\Omega))$, and passing to the limit in the variational formulation

$$\frac{d}{dt}\int_\Omega P\mathbf{u}_k(t)\mathbf{\Phi}=\int_\Omega P\mathbf{u}_k\,A\mathbf{\Phi}-\int_\Gamma\Big(\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}}-\psi\mathbf{n}\Big)\cdot\mathbf{g}_k(t)-\int_\Omega h_k\psi\,,$$

we can show that $P\mathbf{u}$ is the solution of $P\mathbf{u}' = \tilde{A}P\mathbf{u} + (-\tilde{A})PL(\mathbf{g}, h)$, $P\mathbf{u} = P\mathbf{u}_0$. Thus $\mathbf{u}$ is the solution of equation (3.1) in the sense of Definition 4.1.

Step 2. Observe that if $\mathbf{\Phi}$ belongs to $\mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega)$, then the associated pressure ψ, determined by $\nabla\psi = \nu(I - P)\Delta\mathbf{\Phi}$, belongs to $\mathcal{H}^1(\Omega)$, and the mapping $\mathbf{\Phi} \mapsto \psi$ is continuous and linear from $\mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega)$ into $\mathcal{H}^1(\Omega)$. Therefore, in equation (4.4), the mapping

$$\mathbf{\Phi} \longmapsto -\int_\Gamma \Big(\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}} - \psi\mathbf{n}\Big)\cdot\mathbf{g}(t) - \int_\Omega h(t)\psi,$$

is continuous and linear from $(D(A^*), \|\cdot\|_{\mathbf{V}^2(\Omega)})$ into $L^2(0,T)$. Thus it can be identified with an element in $L^2(0,T;(\mathbf{V}^2(\Omega)\cap\mathbf{V}_0^1(\Omega))')$. If we denote this element by $\mathbf{f}$, we have

$$\|\mathbf{f}\|_{L^2(0,T;(\mathbf{V}^2(\Omega)\cap\mathbf{V}_0^1(\Omega))')} \le C(\|\mathbf{g}\|_{L^2(0,T;\mathbf{H}^{-1/2}(\Gamma))} + \|h\|_{L^2(0,T;(H^1(\Omega))')}.$$

In other words, $P\mathbf{u}$ is the solution to equation

$$P\mathbf{u}' = AP\mathbf{u} + \mathbf{f}, \qquad P\mathbf{u}(0) = \mathbf{u}_0,$$

and we have

$$\begin{aligned}&\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}_n^0(\Omega))} + \|P\mathbf{u}\|_{H^1(0,T;(\mathbf{V}^2(\Omega)\cap\mathbf{V}_0^1(\Omega))')}\\ &\le C(\|P\mathbf{u}_0\|_{\mathbf{V}_n^0(\Omega)} + \|\mathbf{f}\|_{L^2(0,T;(\mathbf{V}^2(\Omega)\cap\mathbf{V}_0^1(\Omega))')}).\end{aligned}$$

The proof is complete. □

Theorem 4.5. (i) *If* $(\mathbf{g}, h) \in L^2\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$ *and if* $P\mathbf{u}_0 \in [\mathbf{V}_n^0(\Omega), \mathbf{V}^{-1}(\Omega)]_{1/2}$, *the solution* $\mathbf{u}$ *to equation (3.1) obeys the estimate:*

$$\begin{aligned}&\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}^{1/2-\varepsilon}(\Omega))} + \|P\mathbf{u}\|_{H^{1/4-\varepsilon/2}(0,T;\mathbf{V}_n^0(\Omega))} + \|(I-P)\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{1/2}(\Omega))}\\ &\le C(\|P\mathbf{u}_0\|_{[\mathbf{V}_n^0(\Omega),\mathbf{V}^{-1}(\Omega)]_{1/2}} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{L}^2(\Gamma))} + \|h\|_{L^2(0,T;(H^{1/2}(\Omega))')}) \quad \forall\varepsilon > 0.\end{aligned}$$

(ii) *If* $(\mathbf{g}, h) \in L^2\big(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big) \cap H^{s/2}\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$ *with* $0 \le s \le 2$, *and* Ω *is of class* C^3 *when* $3/2 < s \le 2$, *then*

$$\begin{aligned}&\|(I-P)\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{s+1/2}(\Omega))} + \|(I-P)\mathbf{u}\|_{H^{s/2}(0,T;\mathbf{H}^{1/2}(\Omega))}\\ &\le C(\|\mathbf{g}\|_{\mathbf{H}^{s,s/2}(\Sigma)} + \|h\|_{L^2(0,T;H^{s-1/2}(\Omega))} + \|h\|_{H^{s/2}(0,T;(H^{1/2}(\Omega))')}).\end{aligned}$$

(iii) *If* Ω *is of class* C^3, $(\mathbf{g}, h) \in L^2\big(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big) \cap H^{s/2}\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$, $P\mathbf{u}_0 \in \mathbf{V}_n^{0\vee(s-1/2)}(\Omega)$, *with* $0\vee(s-1/2) = \max(0, s-1/2)$ *and* $s \in [0,1[\cup]1,2]$, *and if* $\mathbf{u}_0$ *and* $(\mathbf{g}(0), h(0))$ *satisfy the compatibility condition*

$$\gamma_0\big[P\big(\mathbf{u}_0 - L(\mathbf{g}(0), h(0))\big)\big] = 0, \tag{4.5}$$

when $s \in]1,2]$, *then*

$$\begin{aligned}\|P\mathbf{u}\|_{\mathbf{H}^{s+1/2-\varepsilon,s/2+1/4-\varepsilon/2}(Q)} \le C(&\|P\mathbf{u}_0\|_{\mathbf{V}_n^{0\vee(s-1/2)}(\Omega)} + \|\mathbf{g}\|_{\mathbf{H}^{s,s/2}(\Sigma)}\\ &+\|h\|_{L^2(0,T;H^{s-1/2}(\Omega))} + \|h\|_{H^{s/2}(0,T;(H^{1/2}(\Omega))')}) \quad \forall\varepsilon > 0.\end{aligned} \tag{4.6}$$

Proof. The proof is very similar to the one in [19, Theorem 2.3]. Throughout the proof, $D\mathbf{g}$ has to be replaced by $L(\mathbf{g}, h)$. The other modifications are obvious. □

5. **Other regularity results.** We would like to give an equivalent formulation to equation (4.1) which allows us to use regularity results from [15].

Proposition 5.1. *Assume that* Ω *is of class* C^3, $(\mathbf{g},h) \in L^2\big(0,T;\mathbf{H}^{2,3/2}_{\Gamma,\Omega}\big) \cap H^1\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$, $P\mathbf{u}_0 \in \mathbf{V}^1_n(\Omega)$, *and* $P(\mathbf{u}_0 - L(\mathbf{g}(0),h(0))) \in \mathbf{V}^1_0(\Omega)$. *A function* $P\mathbf{u} \in \mathbf{V}^{2,1}(Q)$ *is a weak solution to equation* (4.1) *if and only if the following conditions are satisfied:*

(i) $P\mathbf{u}(0) = P\mathbf{u}_0$. *There exists a function* $\pi \in L^2(0,T;\mathcal{H}^1(\Omega))$ *such that*

$$\frac{\partial P\mathbf{u}}{\partial t} - \nu\Delta P\mathbf{u} + \nabla\pi = 0, \tag{5.1}$$

in the sense of distributions in Q.

(ii) $P\mathbf{u}$ *satisfies the following boundary condition:*

$$P\mathbf{u}|_\Sigma = \gamma_\tau(\mathbf{g}) - \gamma_\tau(\nabla q), \tag{5.2}$$

where $q \in L^2(0,T;\mathcal{H}^2(\Omega))$ *is the solution to the boundary value problem*

$$\Delta q(t) = 0 \quad \text{in } \Omega, \quad \frac{\partial q(t)}{\partial \mathbf{n}} = \mathbf{g}\cdot\mathbf{n} - \frac{\partial \rho(t)}{\partial \mathbf{n}} \quad \text{on } \Gamma, \quad \text{for all } t \in [0,T], \tag{5.3}$$

and $\rho \in L^2(0,T;H^2(\Omega))$ *is the solution to*

$$\Delta\rho(t) = h(t) \quad \text{in } \Omega, \quad \rho(t) = 0 \quad \text{on } \Gamma, \quad \text{for all } t \in [0,T]. \tag{5.4}$$

Proof. First prove (5.1). Let $P\mathbf{u}$ be the solution of (4.1) and $(I-P)\mathbf{u}$ be defined by (4.2). We know that $\mathbf{u} = P\mathbf{u} + (I-P)\mathbf{u}$ is the solution of (3.1). Due to Theorem 4.5, we know that $P\mathbf{u} \in \mathbf{V}^{5/2-\varepsilon,5/4-\varepsilon/2}(Q)$ for all $\varepsilon > 0$, and that $(I-P)\mathbf{u} \in L^2(0,T;\mathbf{H}^{5/2}(\Omega)) \cap H^1(0,T;\mathbf{H}^{1/2}(\Omega))$. Thus the pressure p in (3.1) belongs to $L^2(0,T;H^1(\Omega))$, and we have

$$\frac{\partial P\mathbf{u}}{\partial t} - \nu\Delta P\mathbf{u} + \nabla p = \nu\Delta(I-P)\mathbf{u} - \frac{\partial (I-P)\mathbf{u}}{\partial t}.$$

From the characterization of $(I-P)$ (see [24]), it follows that $(I-P)\mathbf{u} = (I-P)L(\mathbf{g},h) = \nabla q + \nabla\rho$, where q is the solution of (5.3) and ρ is the solution of (5.4). Since $(\mathbf{g},h) \in L^2\big(0,T;\mathbf{H}^{2,3/2}_{\Gamma,\Omega}\big) \cap H^1\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$, the functions ρ and q belong to $L^2(0,T;H^3(\Omega)) \cap H^1(0,T;H^{3/2}(\Omega))$ (we only assume that Ω is of class C^3, thus we cannot hope to have ρ in $L^2(0,T;H^{7/2}(\Omega))$). Since $\Delta\nabla q + \Delta\nabla\rho = \nabla\Delta q + \nabla\Delta\rho$ in the sense of distributions in Q, equation (5.1) is established with $\pi = p - \Delta q - \Delta\rho + \frac{\partial q}{\partial t} + \frac{\partial \rho}{\partial t} = p - h + \frac{\partial q}{\partial t} + \frac{\partial \rho}{\partial t}$.

To prove (5.2), we observe that

$$P\mathbf{u}|_\Sigma = \mathbf{u}|_\Sigma - (I-P)\mathbf{u}|_\Sigma = \mathbf{g} - (I-P)L(\mathbf{g},h)|_\Sigma,$$

and that $(I-P)L(\mathbf{g},h) = \nabla q + \nabla\rho$. Therefore (5.2) is proved because $\mathbf{g} - \gamma_0(\nabla q(t) + \nabla\rho(t)) = \gamma_\tau(\mathbf{g}) - \gamma_\tau(\nabla q(t) + \nabla\rho(t)) = \gamma_\tau(\mathbf{g}) - \gamma_\tau(\nabla q(t))$. Indeed $\gamma_\tau(\nabla\rho(t)) = \nabla_\tau(\gamma_0\rho(t)) = 0$.

Now we assume that $P\mathbf{u} \in \mathbf{V}^{2,1}(Q)$ obeys the statements (i) and (ii) of the proposition. For all $\mathbf{\Phi} \in \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega)$, we have

$$\frac{d}{dt}\int_\Omega P\mathbf{u}(t)\mathbf{\Phi} = \int_\Omega \nu\Delta P\mathbf{u}(t)\mathbf{\Phi} = \int_\Omega \mathbf{u}(t)A\mathbf{\Phi} + \int_\Gamma \gamma_\tau(\nabla q(t) + \nabla\rho(t) - \mathbf{g}(t))\cdot\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}}.$$

Introducing the function $\psi \in H^1(\Omega)/\mathbb{R}$ defined by $\nabla\psi = \nu(I-P)\Delta\mathbf{\Phi}$, we obtain

$$\int_\Gamma \gamma_\tau(\nabla q(t) + \nabla\rho(t))\cdot \nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}} = \int_\Omega \nu\Delta\mathbf{\Phi}\cdot\big(\nabla q(t)+\nabla\rho(t)\big) - \int_\Omega \nu\,\mathbf{\Phi}\cdot\nabla h(t)$$
$$= \int_\Omega \big((I-P)\nu\Delta\mathbf{\Phi}\big)\cdot\big(\nabla q(t)+\nabla\rho(t)\big) = \int_\Omega \nabla\psi\cdot\big(\nabla q(t)+\nabla\rho(t)\big)$$
$$= -\int_\Omega \psi\, h(t) + \int_\Gamma \mathbf{g}\cdot\mathbf{n}\,\psi.$$

The first equality comes from the fact that $\nu\frac{\partial\mathbf{\Phi}}{\partial\mathbf{n}}\cdot\mathbf{n} = 0$ when $\mathbf{\Phi}\in\mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega)$. Thus, if $P\mathbf{u}$ obeys conditions (i) and (ii) in the proposition, then $P\mathbf{u}$ is the weak solution to equation (4.1) (see (4.4)). □

Proposition 5.2. *Assume that* $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big)\cap H^{s/2}\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$ *and* $P\mathbf{u}_0\in\mathbf{V}^{s-1/2}_n(\Omega)$ *for some* $s>1$. *Let* q *be the solution of* (5.3). *The compatibility condition*

$$\gamma_\tau(\mathbf{g}(0)) - \gamma_\tau\nabla q(0) = \gamma_0 P\mathbf{u}_0, \tag{5.5}$$

is equivalent to (4.5).

Proof. We have

$$\gamma_0 PL(\mathbf{g}(0),h(0)) = \gamma_0 L(\mathbf{g}(0),h(0)) - \gamma_0\big((I-P)L(\mathbf{g}(0),h(0))\big)$$
$$= \gamma_0(\mathbf{g}(0)) - \gamma_0\big(\nabla q(0)+\nabla\rho(0)\big) = \gamma_\tau(\mathbf{g}(0)) - \gamma_\tau\nabla q(0),$$

which proves that (4.5) and (5.5) are equivalent. □

Proposition 5.3. *Assume that* $(\mathbf{g},h)$ *belongs to* $\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}$ *with* $0\le s\le 3$ *and that*

$$\begin{aligned}&\Omega \text{ is of class } C^3 \text{ if } 0\le s\le 3/2,\\ &\Omega \text{ is of class } C^4 \text{ if } 3/2< s\le 5/2,\\ &\text{and } \Omega \text{ is of class } C^5 \text{ if } 5/2< s\le 3.\end{aligned}\tag{5.6}$$

There exists a constant $C>0$ *such that*

$$\|\gamma_\tau(\nabla q)\|_{\mathbf{V}^s(\Gamma)} \le C\|(\mathbf{g},h)\|_{\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}} \quad \text{for all } s\in]0,3], \text{ and all } (\mathbf{g},h)\in\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega},$$

where q *is the solution of equation* (5.3), *and*

$$\|\nabla_\tau(\gamma_0 q)\|_{\mathbf{V}^0(\Gamma)} \le C\|(\mathbf{g},h)\|_{\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}} \qquad \text{for all } (\mathbf{g},h)\in\mathbf{H}^{0,-1/2}_{\Gamma,\Omega},$$

where ∇_τ *denotes the tangential gradient operator.*

Proof. If $(\mathbf{g},h)\in\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}$ and $s\in]0,3]$, we know that $q\in H^{s+3/2}(\Omega)$, $\nabla q\in\mathbf{H}^{s+1/2}(\Omega)$, and $\gamma_0(\nabla q)\in\mathbf{H}^s(\Gamma)$, which provides the estimate of the proposition in the case when $s>0$. For $s=0$, we have $\gamma_0 q\in H^1(\Gamma)$, and $\nabla_\tau(\gamma_0 q)\in\mathbf{V}^0(\Gamma)$. Indeed, from [5, Lemma A.3] it follows that $\frac{\partial\rho}{\partial\mathbf{n}}$ belongs to $L^2(\Gamma)$. Thus $\gamma_0 q$ belongs to $H^1(\Gamma)$. The proof is complete. □

Remark 5.4. Since we use regularity results for the auxiliary problem (5.3), we need that Ω satisfies (5.6) (see [25, Exercise 3.11]). From Proposition 5.1 and a density argument it follows that the system (5.1)–(5.2) is equivalent to

$$P\mathbf{u}' = \tilde{A}P\mathbf{u} + (-\tilde{A})D(\gamma_\tau\mathbf{g} - \gamma_\tau(\nabla q)), \qquad P\mathbf{u}(0) = P\mathbf{u}_0,$$

if $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big)$ and $s>0$, and to

$$P\mathbf{u}'=\tilde{A}P\mathbf{u}+(-\tilde{A})D(\gamma_\tau\mathbf{g}-\nabla_\tau(\gamma_0 q)),\qquad P\mathbf{u}(0)=P\mathbf{u}_0,$$

if $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$. In theses equations, $\tilde{A}$ is the extension of A to $(D(A^*))'$ (see section 4).

Theorem 5.5. *Assume that* $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big)\cap H^{s/2}\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$, $P\mathbf{u}_0\in\mathbf{V}^{s-1/2}_n(\Omega)$, *with* $3/2\le s<3$, *and* Ω *satisfies* (5.6). *If* $\mathbf{u}_0$ *and* $(\mathbf{g}(0),h(0))$ *satisfy the compatibility condition* (5.5), *then the solution* $P\mathbf{u}$ *to equation* (4.1) *satisfies the estimate*

$$\begin{aligned}&\|P\mathbf{u}\|_{\mathbf{V}^{s+1/2,s/2+1/4}(Q)}\\&\le C\left(\|P\mathbf{u}_0\|_{\mathbf{V}^{s-1/2}_n(\Omega)}+\|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\right)\cap H^{s/2}\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)}\right).\end{aligned}\tag{5.7}$$

Proof. By a density argument, it is sufficient to prove estimate (5.7) when $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big)\cap H^{s/2}\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)\cap L^2\big(0,T;\mathbf{H}^{2,3/2}_{\Gamma,\Omega}\big)\cap H^1\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$. In this way we can use Proposition 5.1. With Proposition 5.3, we can show that

$$\|\gamma_\tau(\nabla q)\|_{\mathbf{V}^{s,s/2}(\Sigma)}\le C\|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\right)\cap H^{s/2}\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)},$$

for all $s\in]0,3]$, and all $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big)\cap H^{s/2}\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$, where q is the solution of equation (5.3). Thus, the theorem is a direct consequence of the above estimate, of Proposition 5.1, and of known regularity results for the instationary Stokes equations with nonhomogeneous boundary conditions [23]. ☐

Theorem 5.6. *Assume that* $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big)\cap H^{s/2}\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$, Ω *is of class* C^3 *and* $P\mathbf{u}_0\in\mathbf{V}^{0\vee(s-1/2)}_n(\Omega)$, *with* $s\in[0,1[$. *Then the solution* $P\mathbf{u}$ *to equation* (4.1) *satisfies the estimate*

$$\begin{aligned}&\|P\mathbf{u}\|_{\mathbf{V}^{s+1/2,s/2+1/4}(Q)}\\&\le C\left(\|P\mathbf{u}_0\|_{\mathbf{V}^{0\vee(s-1/2)}_n(\Omega)}+\|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\right)\cap H^{s/2}\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)}\right).\end{aligned}\tag{5.8}$$

Proof. By a density argument, it is sufficient to prove estimate (5.8) in the case when $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{2,3/2}_{\Gamma,\Omega}\big)\cap H^1\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$ and $(\mathbf{g}(0),h(0))$ and $P\mathbf{u}_0$ satisfy $\gamma_0(PL(\mathbf{g}(0),h(0))-P\mathbf{u}_0)=0$. With Proposition 5.3 we can show that

$$\|\nabla_\tau(\gamma_0 q)\|_{\mathbf{V}^{s,s/2}(\Sigma)}\le C\|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\right)\cap H^{s/2}\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)}\quad\text{for all } s\in[0,1[,$$

where q is the solution of equation (5.3). (For $s>0$, we have to observe that $\gamma_\tau(\nabla q)=\nabla_\tau(\gamma_0 q)$.) Thus estimate (5.8) follows from Proposition 5.1, and from [15, Theorem 2.1] in the case where $0\le s<1$. ☐

Theorem 5.7. *Assume that* $(\mathbf{g},h)\in L^2\big(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big)\cap H^{s/2}\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$, $P\mathbf{u}_0\in\mathbf{V}^{0\vee(s-1/2)}_n(\Omega)$, *with* $s\in[0,1[\cap]1,3[$, *and* Ω *satisfies* (5.6). *If* $\mathbf{u}_0$, $\mathbf{g}(0)$ *and* $h(0)$ *satisfy the compatibility condition* (4.5) *when* $1<s<3$, *then*

$$\begin{aligned}&\|P\mathbf{u}\|_{\mathbf{V}^{s+1/2,s/2+1/4}(Q)}\\&\le C\Big(\|P\mathbf{u}_0\|_{\mathbf{V}^{0\vee(s-1/2)}_n(\Omega)}+\|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\right)\cap H^{s/2}\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)}\Big).\end{aligned}\tag{5.9}$$

Proof. Estimate (5.9) is already proved for $s \in [0,1[$ and $s \in [3/2,3[$. For $s \in]1,3/2[$, it is obtained by interpolation between the regularity results stated in Theorems 5.5 and 5.6. □

Remark 5.8. In Theorems 5.5 and 5.7, for $s = 2$, we have to assume that Ω is of class C^4 (because we make use of Proposition 5.1), while in Theorem 4.5.(iii) we only assume that Ω is of class C^3. Actually, combining the results stated in Theorem 5.6 for $s = 0$ with arguments in the proof of Theorem 4.5, we can show that (5.9) is still true when Ω is of class C^3.

Corollary 5.9. *Assume that Ω is of class C^3. If $(\mathbf{g}, h) \in L^2\big(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\,\Omega}\big) \cap H^s\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$ and $P\mathbf{u}_0 \in \mathbf{V}^s_n(\Omega)$ with $\frac{1}{2} < s < 1$, and if $P\mathbf{u}_0 = PL(\mathbf{g}(0), h(0))$, then $P\mathbf{u} \in \mathbf{V}^{s+1/2,s/2+1/4}(Q)$, $(I-P)\mathbf{u} \in L^2(0,T;\mathbf{H}^{s+1/2}(\Omega)) \cap H^s(0,T;\mathbf{H}^{1/2}(\Omega))$, $\mathbf{u}$ belongs to $C([0,T];\mathbf{H}^s(\Omega))$ and*

$$\begin{aligned}&\|\mathbf{u}\|_{C([0,T];\mathbf{H}^s(\Omega))}\\ &\le C\Big(\|P\mathbf{u}_0\|_{\mathbf{V}^s_n(\Omega)} + \|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\,\Omega}\right)\cap H^s\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)}\Big).\end{aligned} \tag{5.10}$$

Proof. From Theorem 5.6 it follows that

$$\begin{aligned}&\|P\mathbf{u}\|_{\mathbf{V}^{s+1/2,s/2+1/4}(Q)}\\ &\le C\Big(\|P\mathbf{u}_0\|_{\mathbf{V}^{s-1/2}_n(\Omega)} + \|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\,\Omega}\right)\cap H^s\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)}\Big).\end{aligned}$$

It is clear that $(I-P)\mathbf{u}$ belongs to $L^2(0,T;\mathbf{H}^{s+1/2}(\Omega)) \cap H^s(0,T;\mathbf{H}^{1/2}(\Omega)) \hookrightarrow C([0,T];\mathbf{H}^s(\Omega))$ since $s > 1/2$. Moreover, we have

$$\begin{aligned}&\|(I-P)\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{s+1/2}(\Omega))\cap H^s(0,T;\mathbf{H}^{1/2}(\Omega))}\\ &\le C\|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\,\Omega}\right)\cap H^s\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)},\end{aligned}$$

and

$$\|(I-P\mathbf{u})\|_{C([0,T];\mathbf{H}^s(\Omega))} \le C\|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\,\Omega}\right)\cap H^s\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)}.$$

Let us show that $P\mathbf{u} \in C([0,T];\mathbf{H}^s(\Omega))$. If $(\mathbf{g},h) \in L^2\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\,\Omega}\big)$, then $P\mathbf{u} \in \mathbf{V}^{1/2,1/4}(Q)$, and

$$\|P\mathbf{u}\|_{\mathbf{V}^{1/2,1/4}(Q)} \le C\Big(\|P\mathbf{u}_0\|_{\mathbf{V}^0_n(\Omega)} + \|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\,\Omega}\right)}\Big). \tag{5.11}$$

If $(\mathbf{g},h) \in L^2\big(0,T;\mathbf{H}^{1,1/2}_{\Gamma,\,\Omega}\big) \cap H^1\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$ and $P\mathbf{u}_0 = PL(\mathbf{g}(0),h(0))$, then $P\mathbf{u}(t) = PL(\mathbf{g}(t),h(t)) - \int_0^t e^{(t-s)A}PL(\mathbf{g}'(s),h'(s))ds$, $\int_0^t e^{(t-s)A}PL(\mathbf{g}'(s),h'(s))ds \in L^2(0,T;\mathbf{V}^2(\Omega)) \cap H^1(0,T;\mathbf{V}^{1/2-\varepsilon}(\Omega))$ for all $\varepsilon > 0$, $PL(\mathbf{g},h) \in H^\theta(0,T;\mathbf{V}^{3/2-\theta}(\Omega))$ for all $\theta \in [0,1]$. Thus $P\mathbf{u} \in H^\theta(0,T;\mathbf{V}^{3/2-\theta}(\Omega))$ for all $\theta \in [0,1[$, and

$$\|P\mathbf{u}\|_{H^\theta(0,T;\mathbf{V}^{3/2-\theta}(\Omega))} \le C\|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{1,1/2}_{\Gamma,\,\Omega}\right)\cap H^1\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)}. \tag{5.12}$$

By interpolation between (5.11) and (5.12), we obtain

$$\|P\mathbf{u}\|_{C([0,T];\mathbf{H}^s(\Omega))} \le C\Big(\|P\mathbf{u}_0\|_{\mathbf{V}^s_n(\Omega)} + \|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\,\Omega}\right)\cap H^s\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)}\Big).$$

The proof is complete. Let us notice that

$$\|P\mathbf{u}_0\|_{\mathbf{V}^s_n(\Omega)} = \|PL(\mathbf{g}(0),h(0))\|_{\mathbf{V}^s_n(\Omega)} \le C\|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\,\Omega}\right)\cap H^s\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)}.$$

□

6. **Oseen equation.**

6.1. **Linearized Navier-Stokes equations around a stationary state.** In this section, we want to extend the results of sections 4 and 5 to the equation

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{z}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{z} + \nabla p = 0, \quad \text{div }\mathbf{u} = h \quad \text{ in } Q, \\ &\mathbf{u} = \mathbf{g} \quad \text{on } \Sigma, \quad \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega, \end{aligned} \tag{6.1}$$

where $\mathbf{z}$ belongs to $\mathbf{V}^1(\Omega)$, and $(\mathbf{g}, h)$ belongs to $L^2(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\ \Omega})$.

To study equation (6.1) we introduce the unbounded operators $A_\mathbf{z}$ and $A_\mathbf{z}^*$ in $\mathbf{V}_n^0(\Omega)$ defined by

$$\begin{aligned} &D(A_\mathbf{z}) = \Big\{\mathbf{u}\in\mathbf{V}_0^1(\Omega) \mid \nu P\Delta\mathbf{u} - P((\mathbf{z}\cdot\nabla)\mathbf{u}) - P((\mathbf{u}\cdot\nabla)\mathbf{z}) \in \mathbf{V}_n^0(\Omega)\Big\}, \\ &D(A_\mathbf{z}^*) = \Big\{\mathbf{u}\in\mathbf{V}_0^1(\Omega) \mid \nu P\Delta\mathbf{u} + P((\mathbf{z}\cdot\nabla)\mathbf{u}) - P((\nabla\mathbf{z})^T\mathbf{u}) \in \mathbf{V}_n^0(\Omega)\Big\}, \\ &A_\mathbf{z}\mathbf{u} = \nu P\Delta\mathbf{u} - P((\mathbf{z}\cdot\nabla)\mathbf{u}) - P((\mathbf{u}\cdot\nabla)\mathbf{z}) \\ &\text{and} \quad A_\mathbf{z}^*\mathbf{u} = \nu P\Delta\mathbf{u} + P((\mathbf{z}\cdot\nabla)\mathbf{u}) - P((\nabla\mathbf{z})^T\mathbf{u})\,. \end{aligned}$$

Throughout this section we assume that $\lambda_0 > 0$ is such that

$$\begin{aligned} &\int_\Omega\Big(\lambda_0|\mathbf{u}|^2 + \nu|\nabla\mathbf{u}|^2 + ((\mathbf{z}\cdot\nabla)\mathbf{u})\cdot\mathbf{u} + ((\mathbf{u}\cdot\nabla)\mathbf{z})\cdot\mathbf{u}\Big)\,dx \\ &\geq \frac{\nu}{2}\int_\Omega(|\mathbf{u}|^2 + |\nabla\mathbf{u}|^2)\,dx \\ &\text{and} \\ &\int_\Omega\Big(\lambda_0|\mathbf{u}|^2 + \nu|\nabla\mathbf{u}|^2 - ((\mathbf{z}\cdot\nabla)\mathbf{u})\cdot\mathbf{u} + ((\nabla\mathbf{z})^T\mathbf{u})\cdot\mathbf{u}\Big)\,dx \\ &\geq \frac{\nu}{2}\int_\Omega(|\mathbf{u}|^2 + |\nabla\mathbf{u}|^2)\,dx \end{aligned} \tag{6.2}$$

for all $\mathbf{u}\in\mathbf{V}_0^1(\Omega)$.

Lemma 6.1. *The operator* $(A_\mathbf{z} - \lambda_0 I)$ *(respectively* $(A_\mathbf{z}^* - \lambda_0 I)$*) with domain* $D(A_\mathbf{z} - \lambda_0 I) = D(A_\mathbf{z})$ *(respectively* $D(A_\mathbf{z}^* - \lambda_0 I) = D(A_\mathbf{z}^*)$*) is the infinitesimal generator of a bounded analytic semigroup on* $\mathbf{V}_n^0(\Omega)$*. Moreover, for all* $0\le\alpha\le 1$*, we have*

$$D((\lambda_0 I - A_\mathbf{z})^\alpha) = D((\lambda_0 I - A_\mathbf{z}^*)^\alpha) = D((\lambda_0 I - A)^\alpha) = D((-A)^\alpha)\,.$$

Proof. See [19, proof of Lemma 4.1]. In particular, we show that $\mathbf{u}\in L^2(0,T;\mathbf{V}^0(\Omega))$ is a weak solution to equation (6.1) if and only if

$$\begin{aligned} &P\mathbf{u} \text{ is a weak solution of the evolution equation} \\ &P\mathbf{u}' = \tilde{A}P\mathbf{u} + (-\tilde{A})PL(\mathbf{g},h) + P(\text{div}\,(\mathbf{z}\otimes\mathbf{u})) + P(\text{div}\,(\mathbf{u}\otimes\mathbf{z})), \quad (6.3) \\ &P\mathbf{u}(0) = P\mathbf{u}_0, \\ &\text{and} \quad (I-P)\mathbf{u}(\cdot) = (I-P)L(\mathbf{g}(\cdot),h(\cdot)) \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega)), \quad (6.4) \end{aligned}$$

where $\tilde{A}$ is the extension of A to $(D(A^*))'$, see section 4. □

Let us denote by $\tilde{A}_\mathbf{z}$ the extension of $A_\mathbf{z}$ to $(D(A_\mathbf{z}^*))' = (D(A^*))'$. Following what is done for the Stokes equations, we introduce the lifting operators associated with

$\lambda_0 I - A_{\mathbf{z}}$. For all $(\mathbf{g}, h) \in \mathbf{H}^{0,-1/2}_{\Gamma,\,\Omega}$, we denote by $L_{\mathbf{z}}(\mathbf{g}, h) = \mathbf{w}$, and $L_{p,\mathbf{z}}(\mathbf{g}, h) = \pi$ the solution to the equation

$$\lambda_0\mathbf{w} - \nu\Delta\mathbf{w} + (\mathbf{z}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{z} + \nabla\pi = 0 \quad \text{and} \quad \operatorname{div}\mathbf{w} = h \text{ in } \Omega, \quad \mathbf{w} = \mathbf{g} \quad \text{on } \Gamma\,.$$

Following what has been done for the Stokes equations, when $(\mathbf{g}, h) \in C^1([0, T]; \mathbf{H}^{3/2,1}_{\Gamma,\,\Omega})$, we look for the solution $(\mathbf{u}, p)$ of equation (6.1) in the form $(\mathbf{u}, p) = (\mathbf{w}, \pi) + (\mathbf{y}, q)$, where $(\mathbf{w}(t), \pi(t)) = (L_{\mathbf{z}}(\mathbf{g}(t), h(t)), L_{p,\mathbf{z}}(\mathbf{g}(t), h(t)))$, and $(\mathbf{y}, q)$ is the solution of

$$\begin{aligned}
&\frac{\partial\mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{z} + \nabla q = -\frac{\partial\mathbf{w}}{\partial t} + \lambda_0\mathbf{w}, \quad \operatorname{div}\,\mathbf{y} = 0 \quad \text{in } Q,\\
&\mathbf{y} = 0 \ \text{ on } \Sigma, \quad \mathbf{y}(0) = P(\mathbf{u}_0 - \mathbf{w}(0)) \text{ in } \Omega.
\end{aligned}$$

We have

$$\begin{aligned}
\mathbf{y}(t) &= e^{tA_{\mathbf{z}}}P(\mathbf{u}_0 - \mathbf{w}(0)) - \int_0^t e^{(t-s)A_{\mathbf{z}}}P\mathbf{w}'(s)ds + \lambda_0\int_0^t e^{(t-s)A_{\mathbf{z}}}P\mathbf{w}(s)ds\\
&= e^{tA_{\mathbf{z}}}P\mathbf{u}_0 + (\lambda_0 I - A_{\mathbf{z}})\int_0^t e^{(t-s)A_{\mathbf{z}}}P\mathbf{w}(s)ds - P\mathbf{w}(t).
\end{aligned}$$

Thus $P\mathbf{u}$ is defined by

$$P\mathbf{u}(t) = e^{tA_{\mathbf{z}}}P\mathbf{u}_0 + \int_0^t (\lambda_0 I - A_{\mathbf{z}})e^{(t-s)A_{\mathbf{z}}}P\mathbf{w}(s)ds.$$

This leads to the following definition.

Definition 6.2. A function $\mathbf{u} \in L^2(0, T; \mathbf{V}^0(\Omega))$ is a weak solution to equation (6.1) if

$$\begin{aligned}
&P\mathbf{u} \text{ is a weak solution of evolution equation}\\
&P\mathbf{u}' = \tilde{A}_{\mathbf{z}}P\mathbf{u} + (\lambda_0 I - \tilde{A}_{\mathbf{z}})PL_{\mathbf{z}}(\mathbf{g}, h), \qquad P\mathbf{u}(0) = P\mathbf{u}_0, \qquad (6.5)\\
&\text{and}\\
&(I - P)\mathbf{u}(\cdot) = (I - P)L_{\mathbf{z}}(\mathbf{g}, h)(\cdot) \quad \text{in } L^2(0, T; \mathbf{L}^2(\Omega)).
\end{aligned}$$

As in section 2, we can establish the following theorem.

Theorem 6.3. (i) *For all $P\mathbf{u}_0 \in \mathbf{V}^{-1}(\Omega)$, all $(\mathbf{g}, h) \in L^2(0, T; \mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega})$, equation (6.1), admits a unique weak solution in $L^2(0, T; \mathbf{L}^2(\Omega))$ in the sense of Definition 6.2. This solution obeys*

$$\begin{aligned}
&\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}^0_n(\Omega))} + \|P\mathbf{u}\|_{H^1(0,T;(\mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega))')} + \|(I-P)\mathbf{u}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\\
&\le C(\|P\mathbf{u}_0\|_{\mathbf{V}^{-1}(\Omega)} + \|h\|_{L^2(0,T;(H^1(\Omega))')} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{H}^{-1/2}(\Omega))}).
\end{aligned}$$

(ii) *If $(\mathbf{g}, h) \in L^2\big(0, T; \mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big) \cap H^{s/2}\big(0, T; \mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$ with $0 \le s \le 2$, and Ω is of class C^3 when $3/2 < s \le 2$, then*

$$\begin{aligned}
&\|(I-P)\mathbf{u}\|_{L^2(0,T;\mathbf{H}^{s+1/2}(\Omega))} + \|(I-P)\mathbf{u}\|_{H^{s/2}(0,T;\mathbf{H}^{1/2}(\Omega))}\\
&\le C(\|\mathbf{g}\|_{\mathbf{H}^{s,s/2}(\Sigma)} + \|h\|_{L^2(0,T;H^{s-1/2}(\Omega))} + \|h\|_{H^{s/2}(0,T;(H^{1/2}(\Omega))')}).
\end{aligned}$$

(iii) *If $(\mathbf{g}, h) \in L^2\big(0, T; \mathbf{H}^{2,3/2}_{\Gamma,\Omega}\big) \cap H^1\big(0, T; \mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$, $\mathbf{z} \in \mathbf{V}^{3/2\vee(s-1/2)}(\Omega)$, Ω satisfies (5.6), $P\mathbf{u}_0 \in \mathbf{V}^{0\vee(s-1/2)}_n(\Omega)$, with $s \in [0, 1[\cap]1, 3[$, and if $\mathbf{u}_0$ and $(\mathbf{g}(0), h(0))$*

satisfy the compatibility condition (4.5) *when* $1 < s < 3$, *then*

$$\begin{aligned}&\|P\mathbf{u}\|_{\mathbf{V}^{s+1/2,s/2+1/4}(Q)}\\&\leq C\left(\|P\mathbf{u}_0\|_{\mathbf{V}_n^{0\vee(s-1/2)}(\Omega)}+\|(\mathbf{g},h)\|_{L^2\left(0,T;\mathbf{H}^{2,3/2}_{\Gamma,\Omega}\right)\cap H^1\left(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\right)}\right).\end{aligned} \tag{6.6}$$

Proof. We refer to [19, Theorem 4.1] for a very similar proof in the case when $h = 0$. □

Proposition 6.4. *For all* $P\mathbf{u}_0 \in \mathbf{V}^0_n(\Omega)$ *and all* $(\mathbf{g}, h) \in L^2(0, T; \mathbf{H}^{0,-1/2}_{\Gamma,\,\Omega})$, *problem* (6.3) *admits a unique weak solution* $P\mathbf{u}$ *in* $L^2(0, T; \mathbf{V}^0_n(\Omega))$ *and it satisfies*

$$\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}^0_n(\Omega))} \leq C\left(\|P\mathbf{u}_0\|_{\mathbf{V}^0_n(\Omega)} + \|(\mathbf{g},h)\|_{L^2(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\,\Omega})}\right).$$

Proof. For all $\mathbf{v} \in L^2(0, T; \mathbf{V}^0_n(\Omega))$, $\mathbf{z}\otimes\mathbf{v}$ and $\mathbf{v}\otimes\mathbf{z}$ belong to $L^2(0, T; (\mathbf{L}^{3/2}(\Omega))^N)$. Thus, if $\mathbf{v} \in L^2(0, T; \mathbf{V}^0_n(\Omega))$, the evolution equation

$$\begin{aligned}&\mathbf{y}' = \tilde{A}\mathbf{y} + (-\tilde{A})PL(\mathbf{g}, h) + P(\operatorname{div}(\mathbf{z} \otimes ((I - P)L(\mathbf{g}, h))))\\&\qquad +P(\operatorname{div}(((I - P)L(\mathbf{g}, h)) \otimes \mathbf{z})) + P(\operatorname{div}(\mathbf{z} \otimes \mathbf{v})) + P(\operatorname{div}(\mathbf{v} \otimes \mathbf{z})),\\&P\mathbf{y}(0) = P\mathbf{u}_0,\end{aligned}$$

admits a unique solution $\mathbf{y}_\mathbf{v}$ in $L^2(0, T; \mathbf{V}^0_n(\Omega))$. As in [18, Proposition 2.7] we can show that for $T^* > 0$ small enough, the mapping

$$\mathbf{v} \longmapsto \mathbf{y}_\mathbf{v}$$

is a contraction in $L^2(0, T^*; \mathbf{V}^0_n(\Omega))$. Thus we have proved the existence of a unique local solution to equation (6.3). As in [18] we can iterate this process to prove the existence of a unique global in time solution in $L^2(0, T; \mathbf{V}^0(\Omega))$ to equation (6.3). The estimate of $P\mathbf{u}$ in $L^2(0, T; \mathbf{V}^0_n(\Omega))$ can be derived as in [18]. The estimate of $(I - P)\mathbf{u} = (I - P)L_\mathbf{z}(\mathbf{g}, h)$ follows from the continuity of the operator $(I - P)L_\mathbf{z}$. The proof is complete. □

Theorem 6.5. *A function* $\mathbf{u} \in L^2(0, T; \mathbf{L}^2(\Omega))$ *is a weak solution to equation (6.1), in the sense of definition 6.2, if and only if* $\mathbf{u}$ *is the weak solution to problem* (6.3)*–*(6.4)*.*

Proof. This equivalence can be easily shown in the case when $\mathbf{u}_0 \in \mathbf{V}^1_0(\Omega)$ and $(\mathbf{g}, h) \in C^1_c(0, T; \mathbf{H}^{3/2,1}_{\Gamma,\Omega})$. Due to the estimates in Proposition 6.4 and in Theorem 6.3(i), the equivalence follows from a density argument. □

6.2. **Linearized Navier-Stokes equations around an instationary state.** In this section, we want to study the linearized Navier-Stokes equations around an instationary state $\mathbf{z}$, with homogeneous boundary conditions:

$$\begin{aligned}&\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{z}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{z} + \nabla p = \mathbf{f},\ \operatorname{div}\mathbf{u} = 0 \quad \text{in } Q,\\&\mathbf{u} = 0 \ \text{ on } \Sigma, \quad \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega,\end{aligned} \tag{6.7}$$

in the case where $\mathbf{z}$ belongs to $L^2(0, T; \mathbf{V}^1(\Omega)) \cap L^\infty(0, T; \mathbf{V}^s(\Omega))$ with $1/2 < s < 5/2$, and $\mathbf{f}$ belongs to $L^2(0, T; \mathbf{H}^{-1}(\Omega))$. In this section we only treat the case when $N = 3$, the adaptation to the case when $N = 2$ can be easily done.

We prefer to rewrite the term $(\mathbf{u}\cdot\nabla)\mathbf{z}$ in the form $\mathrm{div}(\mathbf{z}\otimes\mathbf{u})$. Thus, for almost all $t\in(0,T)$, we define the operators $A_{\mathbf{z}}(t)\in\mathcal{L}(\mathbf{V}^1_0(\Omega),\mathbf{V}^{-1}(\Omega))$ and $A^*_{\mathbf{z}}(t)\in\mathcal{L}(\mathbf{V}^1_0(\Omega),\mathbf{V}^{-1}(\Omega))$ by

$$\begin{aligned}&\big\langle A_{\mathbf{z}}(t)\mathbf{u},\mathbf{v}\big\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}^1_0(\Omega)}\\ &\qquad=\int_\Omega\big(-\nu\nabla\mathbf{u}\cdot\nabla\mathbf{v}-((\mathbf{z}(t)\cdot\nabla)\mathbf{u})\cdot\mathbf{v}+((\mathbf{u}\cdot\nabla)\mathbf{v})\cdot\mathbf{z}(t)\big)dx,\\ &\big\langle A^*_{\mathbf{z}}(t)\mathbf{u},\mathbf{v}\big\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}^1_0(\Omega)}\\ &\qquad=\int_\Omega\big(-\nu\nabla\mathbf{u}\cdot\nabla\mathbf{v}+((\mathbf{z}(t)\cdot\nabla)\mathbf{u})\cdot\mathbf{v}+((\mathbf{v}\cdot\nabla)\mathbf{u})\cdot\mathbf{z}(t)\big)dx,\end{aligned}$$

for all $\mathbf{u}\in\mathbf{V}^1_0(\Omega)$ and all $\mathbf{v}\in\mathbf{V}^1_0(\Omega)$.

Let us still denote by P the continuous extension of the Helmholtz projector to $\mathbf{H}^{-1}(\Omega)$, that is the bounded operator from $\mathbf{H}^{-1}(\Omega)$ onto $\mathbf{V}^{-1}(\Omega)$ defined by $\langle P\mathbf{f},\mathbf{\Phi}\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}^1_0(\Omega)}=\langle\mathbf{f},\mathbf{\Phi}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)}$ for all $\mathbf{f}\in\mathbf{H}^{-1}(\Omega)$, and all $\mathbf{\Phi}\in\mathbf{V}^1_0(\Omega)$ (see e.g. [26, page xxiii] or [3, Appendix A.1]). Equation (6.7) can be rewritten in the form

$$\mathbf{u}'=A_{\mathbf{z}}(t)\mathbf{u}+P\mathbf{f},\qquad\mathbf{u}(0)=\mathbf{u}_0.$$

Lemma 6.6. *There exist $\lambda_0>0$ and $M>0$ such that*

$$\Big|\big\langle A_{\mathbf{z}}(t)\mathbf{u},\mathbf{v}\big\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}^1_0(\Omega)}\Big|\le M\|\mathbf{u}\|_{\mathbf{V}^1_0(\Omega)}\|\mathbf{v}\|_{\mathbf{V}^1_0(\Omega)}$$

and

$$\big\langle\lambda_0\mathbf{u}-A_{\mathbf{z}}(t)\mathbf{u},\mathbf{u}\big\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}^1_0(\Omega)}\ge\frac{\nu}{2}\|\mathbf{u}\|^2_{\mathbf{V}^1_0(\Omega)},$$

for all $\mathbf{u}\in\mathbf{V}^1_0(\Omega)$, all $\mathbf{v}\in\mathbf{V}^1_0(\Omega)$ and almost all $t\in(0,T)$.

Proof. For all $\mathbf{u}\in\mathbf{V}^1_0(\Omega)$, almost all $t\in(0,T)$, and $\lambda_0>0$, we have:

$$\begin{aligned}&\langle\lambda_0\mathbf{u}-A_{\mathbf{z}}(t)\mathbf{u},\mathbf{u}\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}^1_0(\Omega)}\\ &=\int_\Omega\Big(\lambda_0|\mathbf{u}|^2+\nu|\nabla\mathbf{u}|^2+((\mathbf{z}(t)\cdot\nabla)\mathbf{u})\cdot\mathbf{u}-((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{z}(t)\Big)\,dx\\ &=\int_\Omega\Big(\lambda_0|\mathbf{u}|^2+\nu|\nabla\mathbf{u}|^2-((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{z}(t)\Big)\,dx\\ &\ge\int_\Omega\big(\lambda_0|\mathbf{u}|^2+\nu|\nabla\mathbf{u}|^2\big)\,dx-\|\mathbf{u}\|_{\mathbf{L}^{3/s}(\Omega)}\|\mathbf{u}\|_{\mathbf{V}^1_0(\Omega)}\|\mathbf{z}\|_{L^\infty(0,T;\mathbf{V}^s(\Omega))}\\ &\ge\int_\Omega\big(\lambda_0|\mathbf{u}|^2+\nu|\nabla\mathbf{u}|^2\big)\,dx-C\|\mathbf{u}\|^{(2s-1)/2}_{\mathbf{V}^0_n(\Omega)}\|\mathbf{u}\|^{(5-2s)/2}_{\mathbf{V}^1_0(\Omega)}\|\mathbf{z}\|_{L^\infty(0,T;\mathbf{V}^s(\Omega))}\\ &\ge\int_\Omega\big(\lambda_0|\mathbf{u}|^2+\nu|\nabla\mathbf{u}|^2\big)\,dx-\frac{\nu}{2}\|\mathbf{u}\|^2_{\mathbf{V}^1_0(\Omega)}-\tilde C\|\mathbf{z}\|^{4/(2s-1)}_{L^\infty(0,T;\mathbf{V}^s(\Omega))}\|\mathbf{u}\|^2_{\mathbf{V}^0_n(\Omega)}\,,\end{aligned}$$

for some $\tilde C>0$. It is sufficient to choose $\lambda_0=\frac{\nu}{2}+\tilde C\|\mathbf{z}\|^{4/(2s-1)}_{L^\infty(0,T;\mathbf{V}^s(\Omega))}$.

For all $\mathbf{u}\in\mathbf{V}^1_0(\Omega)$, all $\mathbf{v}\in\mathbf{V}^1_0(\Omega)$, and almost all $t\in(0,T)$, we have:

$$\begin{aligned}&\Big|\big\langle A_{\mathbf{z}}(t)\mathbf{u},\mathbf{v}\big\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}^1_0(\Omega)}\Big|\\ &\le\nu\|\mathbf{u}\|_{\mathbf{V}^1_0(\Omega)}\|\mathbf{v}\|_{\mathbf{V}^1_0(\Omega)}+\|\mathbf{z}\|_{L^\infty(0,T;\mathbf{V}^s(\Omega))}\|\mathbf{u}\|_{\mathbf{V}^1_0(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^{3/s}(\Omega)}\\ &\qquad\qquad+\|\mathbf{u}\|_{\mathbf{L}^{3/s}(\Omega)}\|\mathbf{v}\|_{\mathbf{V}^1_0(\Omega)}\|\mathbf{z}\|_{L^\infty(0,T;\mathbf{V}^s(\Omega))}\\ &\le\nu\|\mathbf{u}\|_{\mathbf{V}^1_0(\Omega)}\|\mathbf{v}\|_{\mathbf{V}^1_0(\Omega)}+C\|\mathbf{z}\|_{L^\infty(0,T;\mathbf{V}^s(\Omega))}\|\mathbf{u}\|_{\mathbf{V}^1_0(\Omega)}\|\mathbf{v}\|_{\mathbf{V}^1_0(\Omega)}\,.\end{aligned}$$

The proof is complete. □

Theorem 6.7. *Assume that* $\mathbf{z}$ *belongs to* $L^2(0,T;\mathbf{V}^1(\Omega))\cap L^\infty(0,T;\mathbf{V}^s(\Omega))$, *with* $1/2<s<5/2$, *and that* $N=3$. *For all* $\mathbf{u}_0\in\mathbf{V}^0_n(\Omega)$ *and all* $\mathbf{f}\in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ *equation (6.7), admits a unique weak solution* $\mathbf{u}$ *in* $W(0,T;\mathbf{V}^1_0(\Omega),\mathbf{V}^{-1}(\Omega))$.

Proof. The theorem is a direct consequence of Lemma 6.6 and of a theorem by J.-L. Lions (see e.g. [6, Chapter 18, Section 3.2, Theorems 1 and 2]). □

With $A_{\mathbf{z}}(t)$ and $A^*_{\mathbf{z}}(t)$, we can associate two unbounded operators in $\mathbf{V}^0_n(\Omega)$, still denoted by $A_{\mathbf{z}}(t)$ and $A^*_{\mathbf{z}}(t)$ for simplicity, and defined by

$$D(A_{\mathbf{z}}(t))=\Big\{\mathbf{u}\in\mathbf{V}^1_0(\Omega)\mid \nu P\Delta\mathbf{u}-P((\mathbf{z}(t)\cdot\nabla)\mathbf{u})-P((\mathbf{u}\cdot\nabla)\mathbf{z}(t))\in\mathbf{V}^0_n(\Omega)\Big\},$$

$$D(A^*_{\mathbf{z}}(t))=\Big\{\mathbf{u}\in\mathbf{V}^1_0(\Omega)\mid \nu P\Delta\mathbf{u}+P((\mathbf{z}(t)\cdot\nabla)\mathbf{u})-P((\nabla\mathbf{z}(t))^T\mathbf{u})\in\mathbf{V}^0_n(\Omega)\Big\},$$

$$A_{\mathbf{z}}(t)\mathbf{u}=\nu P\Delta\mathbf{u}-P((\mathbf{z}(t)\cdot\nabla)\mathbf{u})-P((\mathbf{u}\cdot\nabla)\mathbf{z}(t)),$$

and

$$A^*_{\mathbf{z}}(t)\mathbf{u}=\nu P\Delta\mathbf{u}+P((\mathbf{z}(t)\cdot\nabla)\mathbf{u})-P((\nabla\mathbf{z}(t))^T\mathbf{u})\,.$$

7. **The Navier-Stokes equation.** In this section, we want to study the equation

$$\begin{aligned}&\frac{\partial\mathbf{u}}{\partial t}-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=0,\quad \operatorname{div}\mathbf{u}=h\quad\text{ in } Q,\\ &\mathbf{u}=\mathbf{g}\ \text{ on }\Sigma,\quad P\mathbf{u}(0)=\mathbf{u}_0 \text{ in }\Omega,\end{aligned}\tag{7.1}$$

where $(\mathbf{g},h)$ belongs to $L^2\big(0,T;\mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big)\cap H^s\big(0,T;\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$ with $s>1/2$, and $\mathbf{u}_0\in\mathbf{V}^0_n(\Omega)$ and $N=3$. Let $\mathbf{z}$ be the solution to equation

$$\begin{aligned}&P\mathbf{z}'=\tilde AP\mathbf{z}+(-\tilde A)PL(\mathbf{g},h),\qquad P\mathbf{z}(0)=PL(\mathbf{g}(0),h(0)),\\ &(I-P)\mathbf{z}(\cdot)=(I-P)L(\mathbf{g}(\cdot),h(\cdot))\quad\text{in } L^2(0,T;\mathbf{L}^2(\Omega)).\end{aligned}\tag{7.2}$$

We look for a solution $\mathbf{u}$ to equation (7.1) in the form $\mathbf{u}=\mathbf{z}+\mathbf{y}$, where $\mathbf{y}$ is the solution of

$$\begin{aligned}&\frac{\partial\mathbf{y}}{\partial t}-\nu\Delta\mathbf{y}+(\mathbf{z}\cdot\nabla)\mathbf{y}+(\mathbf{y}\cdot\nabla)\mathbf{z}+(\mathbf{y}\cdot\nabla)\mathbf{y}+(\mathbf{z}\cdot\nabla)\mathbf{z}+\nabla q=0,\\ &\operatorname{div}\mathbf{y}=0\quad\text{ in } Q,\\ &\mathbf{y}=0\ \text{ on }\Sigma,\quad \mathbf{y}(0)=\mathbf{u}_0-PL(\mathbf{g}(0),h(0)) \text{ in }\Omega.\end{aligned}\tag{7.3}$$

Since $\mathbf{g}\in\mathbf{H}^{s,s}(\Sigma)\hookrightarrow C([0,T];\mathbf{H}^{s-1/2}(\Gamma))$, and $h\in H^s(0,T;(H^{1/2}(\Omega))')\cap L^2(0,T;H^{s-1/2}(\Omega))\hookrightarrow C([0,T];H^{s-1}(\Omega))$, we have that $PL(\mathbf{g}(0),h(0))\in\mathbf{H}^s(\Omega)$.

According to Corollary 5.9, $\mathbf{z}$ belongs to $L^2(0,T;\mathbf{H}^{s+1/2}(\Omega))\cap C([0,T];\mathbf{H}^s(\Omega))$.

With the notation introduced in section 6.2, we can rewrite equation (7.3) in the form

$$\mathbf{y}'=A_{\mathbf{z}}(t)\mathbf{y}-P((\mathbf{y}\cdot\nabla)\mathbf{y})-P((\mathbf{z}\cdot\nabla)\mathbf{z}),\qquad \mathbf{y}(0)=\mathbf{u}_0-PL(\mathbf{g}(0),h(0)).\tag{7.4}$$

Since $\mathbf{z}$ belongs to $L^2(0,T;\mathbf{H}^{s+1/2}(\Omega))\cap C([0,T];\mathbf{H}^s(\Omega))$, it is clear that $P((\mathbf{z}\cdot\nabla)\mathbf{z})$ belongs to $L^2(0,T;\mathbf{H}^{-1}(\Omega))$. Thus equation (7.4) is very similar to the three dimensional Navier-Stokes equation with a source term belonging to $L^2(0,T;\mathbf{V}^{-1}(\Omega))$. The only difference is that the Stokes operator A is now replaced by $A_{\mathbf{z}}(t)$. Let us denote by $C_w([0,T];\mathbf{V}^0_n(\Omega))$ the subspace in $L^\infty(0,T;\mathbf{V}^0_n(\Omega))$ of functions which are continuous from $[0,T]$ into $\mathbf{V}^0_n(\Omega)$ equipped with its weak topology.

Theorem 7.1. *Assume that $N = 3$. For all $\mathbf{u}_0 \in \mathbf{V}^0_n(\Omega)$, all $(\mathbf{g}, h) \in L^2\big(0, T; \mathbf{H}^{s,s-1/2}_{\Gamma,\Omega}\big) \cap H^s\big(0, T; \mathbf{H}^{0,-1/2}_{\Gamma,\Omega}\big)$ with $s > 1/2$, equation (7.3) admits at least one weak solution in $C_w([0, T]; \mathbf{V}^0_n(\Omega)) \cap L^2(0, T; \mathbf{V}^1_0(\Omega))$.*

Proof. Let λ_0 be the exponent appearing in Lemma 6.6. A function $\mathbf{y} \in C_w([0, T]; \mathbf{V}^0_n(\Omega)) \cap L^2(0, T; \mathbf{V}^1_0(\Omega))$ is a weak solution to (7.4) if and only if $\hat{\mathbf{y}}(t) = e^{-\lambda_0 t}\mathbf{y}(t)$ is the solution in $C_w([0, T]; \mathbf{V}^0_n(\Omega)) \cap L^2(0, T; \mathbf{V}^1_0(\Omega))$ to

$$\begin{aligned} &\hat{\mathbf{y}}' = A_{\mathbf{z}}(t)\hat{\mathbf{y}} - \lambda_0\hat{\mathbf{y}} - P(e^{\lambda_0 t}(\hat{\mathbf{y}} \cdot \nabla)\hat{\mathbf{y}}) - P(e^{-\lambda_0 t}(\mathbf{z} \cdot \nabla)\mathbf{z}), \\ &\hat{\mathbf{y}}(0) = \mathbf{u}_0 - PL(\mathbf{g}(0), h(0)). \end{aligned} \tag{7.5}$$

Due to Lemma 6.6, the existence in $L^\infty(0, \infty; \mathbf{V}^0_n(\Omega)) \cap L^2(0, T; \mathbf{V}^1_0(\Omega))$ of a function $\hat{\mathbf{y}}$ satisfying the weak formulation of equation (7.5) may be proved as in the case of the Navier-Stokes equations (see e.g. [24, Chapter 3, Theorem 3.1]). Moreover we have

$$\begin{aligned} &\frac{1}{2}\|\hat{\mathbf{y}}(t)\|^2_{\mathbf{V}^0_n(\Omega)} + \frac{1}{2}\int_0^t \|\hat{\mathbf{y}}(\tau)\|^2_{\mathbf{V}^1_0(\Omega)}\, d\tau \\ &\leq \frac{1}{2}\|\mathbf{u}_0 - PL(\mathbf{g}(0), h(0))\|^2_{\mathbf{V}^0_n(\Omega)} - \big\langle P(e^{-\lambda_0(\cdot)}(\mathbf{z} \cdot \nabla)\mathbf{z}), \hat{\mathbf{y}}\big\rangle_{L^2(0,t;\mathbf{V}^{-1}(\Omega)), L^2(0,t;\mathbf{V}^1_0(\Omega))}. \end{aligned}$$

Thus $\hat{\mathbf{y}}$ obeys the estimate

$$\begin{aligned} &\|\hat{\mathbf{y}}(t)\|^2_{\mathbf{V}^0_n(\Omega)} + \|\hat{\mathbf{y}}\|^2_{L^2(0,t;\mathbf{V}^1_0(\Omega))} \\ &\leq C(\|P((\mathbf{z} \cdot \nabla)\mathbf{z})\|^2_{L^2(0,t;\mathbf{V}^{-1}(\Omega))} + \|\mathbf{u}_0 - PL(\mathbf{g}(0), h(0))\|^2_{\mathbf{V}^0_n(\Omega)}), \end{aligned} \tag{7.6}$$

for all $0 < t < T$, where C is independent of t and T. Moreover $\mathrm{div}(\mathbf{z} \otimes \mathbf{y})$ and $\mathrm{div}(\mathbf{y} \otimes \mathbf{z})$ belong to $L^2(0, T; \mathbf{H}^{-1}(\Omega))$, and $(\mathbf{y} \cdot \nabla)\mathbf{y}$ belongs to $L^2(0, T; \mathbf{L}^1(\Omega))$. We have $\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega) \hookrightarrow C_0(\Omega)$ with a dense imbedding. Hence $\mathcal{M}_b(\Omega) \hookrightarrow (\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega))'$, and $(\mathbf{y} \cdot \nabla)\mathbf{y}$ which belongs to $L^2(0, T; \mathbf{L}^1(\Omega))$ can be identified with an element in $L^2(0, T; (\mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega))')$. Thus

$$\mathbf{f} = (\mathbf{z} \cdot \nabla)\mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{z} + (\mathbf{y} \cdot \nabla)\mathbf{y} \in L^2(0, T; (\mathbf{H}^2 \cap \mathbf{H}^1_0(\Omega))')\,.$$

Defining $P\mathbf{f}$ in $L^2(0, T; (\mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega))')$ by

$$\langle P\mathbf{f}(t), \mathbf{\Phi}\rangle_{(\mathbf{V}^2 \cap \mathbf{V}^1_0(\Omega))', \mathbf{V}^2 \cap \mathbf{V}^1_0(\Omega)} = \langle \mathbf{f}(t), \mathbf{\Phi}\rangle_{(\mathbf{H}^2 \cap \mathbf{H}^1_0(\Omega))', \mathbf{H}^2 \cap \mathbf{H}^1_0(\Omega)} \quad \forall \mathbf{\Phi} \in \mathbf{V}^2 \cap \mathbf{V}^1_0(\Omega),$$

with equation (7.3) we can prove that $\mathbf{y}' \in L^2(0, T; (\mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega))')$. Since $\mathbf{y} \in L^\infty(0, T; \mathbf{V}^0_n(\Omega))$, we can claim that $\mathbf{y} \in C_w([0, T]; \mathbf{V}^0_n(\Omega))$, and the proof is complete. □

8. **Appendix 1.**

Lemma 8.1. *For all $\mathbf{\Phi} \in \mathbf{L}^2(\Omega)$, $(\mathbf{h}, g) \in \mathbf{H}^{3/2,1}_{\Gamma,\Omega}$, the equation:*

$$-\nu\Delta\mathbf{y} + \nabla q = \mathbf{\Phi} \quad \text{and} \quad \mathrm{div}\,\mathbf{y} = g \text{ in } \Omega, \quad \mathbf{y} = \mathbf{h} \quad \text{on } \Gamma, \tag{8.1}$$

admits a unique solution $(\mathbf{y}, q)$ in $\mathbf{H}^1(\Omega) \times \mathcal{H}^0(\Omega)$. It satisfies the estimate:

$$\|\mathbf{y}\|_{\mathbf{H}^2(\Omega)} + \|q\|_{\mathcal{H}^1(\Omega)} \leq C\big(\|\mathbf{\Phi}\|_{\mathbf{L}^2(\Omega)} + \|(\mathbf{h}, g)\|_{\mathbf{H}^{3/2,1}_{\Gamma,\Omega}}\big).$$

This result is stated in [11, Exercice 6.2, Chapter 4].

Lemma 8.2. *For all* $\mathbf{f} \in \mathbf{L}^2(\Omega)$ *and all* $(\mathbf{g}, h) \in \mathbf{H}^{3/2,1}_{\Gamma,\Omega}$, *the solution* $(\mathbf{w}, \rho) \in \mathbf{H}^2(\Omega) \times \mathcal{H}^1(\Omega)$ *to equation:*

$$-\nu\Delta \mathbf{w} + \nabla\rho = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{w} = h \text{ in } \Omega, \quad \mathbf{w} = \mathbf{g} \quad \text{on } \Gamma,$$

obeys the estimate:

$$\begin{aligned}&\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} + \|\rho\|_{(\mathcal{H}^1(\Omega))'} + \Big\|\Big(\nu\frac{\partial \mathbf{w}}{\partial \mathbf{n}} - \rho\mathbf{n} - k(\mathbf{w},\rho)\mathbf{n}, \rho + k(\mathbf{w},\rho)\Big)\Big\|_{\mathbf{H}^{-3/2,-1}_{\Gamma,\ \Omega}}\\ &\le C\big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'} + \|(\mathbf{g},h)\|_{\mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}}\big),\end{aligned}$$

where $k(\mathbf{w},\rho)$ *is the constant corresponding to* $(\mathbf{w},\rho)$*, and defined in* (2.1).

Proof. Let $\mathbf{\Phi}$ be in $\mathbf{L}^2(\Omega)$, $(\mathbf{h}, g)$ be in $\mathbf{H}^{3/2,1}_{\Gamma,\Omega}$, and let $(\mathbf{y}, q)$ be the solution to equation (8.1). The solutions $(\mathbf{y}, q)$ and $(\mathbf{w},\rho)$ obey the Green formula:

$$\begin{aligned}&\int_\Omega \mathbf{w}\,\mathbf{\Phi} = \int_\Omega \mathbf{f}\,\mathbf{y} + \int_\Gamma \Big(\nu\frac{\partial \mathbf{w}}{\partial \mathbf{n}} - \rho\mathbf{n}\Big)\,\mathbf{h} - \int_\Omega q\,h + \int_\Gamma \Big(-\nu\frac{\partial \mathbf{y}}{\partial \mathbf{n}} + q\mathbf{n}\Big)\,\mathbf{g} + \int_\Omega \rho\, g\\ &= \int_\Omega \mathbf{f}\,\mathbf{y} + \int_\Gamma \Big(\nu\frac{\partial \mathbf{w}}{\partial \mathbf{n}} - \rho\mathbf{n} - k(\mathbf{w},\rho)\mathbf{n}\Big)\,\mathbf{h} - \int_\Omega (q + k(\mathbf{y},q))\,h\\ &\quad + \int_\Gamma \Big(-\nu\frac{\partial \mathbf{y}}{\partial \mathbf{n}} + q\mathbf{n} + k(\mathbf{y},q)\mathbf{n}\Big)\,\mathbf{g} + \int_\Omega (\rho + k(\mathbf{w},\rho))\,g.\end{aligned}$$

Setting $(\mathbf{h}, g) = 0$, with Lemma 8.1 we obtain

$$\begin{aligned}&\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} = \sup_{\|\mathbf{\Phi}\|_{\mathbf{L}^2(\Omega)}=1}\int_\Omega \mathbf{w}\,\mathbf{\Phi}\\ &\le \sup_{\|\mathbf{\Phi}\|_{\mathbf{L}^2(\Omega)}=1}\Big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'}\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)}\\ &\qquad + \Big\|\Big(\nu\frac{\partial \mathbf{y}}{\partial \mathbf{n}} - q\mathbf{n} - k(\mathbf{y},q)\mathbf{n}, q + k(\mathbf{y},q)\Big)\Big\|_{\mathbf{H}^{1/2,1}_{\Gamma,\Omega}}\|(\mathbf{g},h)\|_{\mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}}\Big)\\ &\le C(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'} + \|(\mathbf{g},h)\|_{\mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}}).\end{aligned}$$

Setting $\mathbf{\Phi} = 0$, we obtain

$$\begin{aligned}&\Big\|\Big(\nu\frac{\partial \mathbf{w}}{\partial \mathbf{n}} - \rho\mathbf{n} - k(\mathbf{w},\rho)\mathbf{n}, \rho + k(\mathbf{w},\rho)\Big)\Big\|_{\mathbf{H}^{-3/2,-1}_{\Gamma,\ \Omega}}\\ &= \sup_{\|(\mathbf{h},g)\|_{\mathbf{H}^{3/2,1}_{\Gamma,\Omega}}=1}\int_\Gamma \Big(\nu\frac{\partial \mathbf{w}}{\partial \mathbf{n}} - \rho\mathbf{n} - k(\mathbf{w},\rho)\mathbf{n}\Big)\,\mathbf{h} + \int_\Omega (\rho + k(\mathbf{w},\rho))\,g\\ &\le \sup_{\|(\mathbf{h},g)\|_{\mathbf{H}^{3/2,1}_{\Gamma,\Omega}}=1}\Big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'}\|\mathbf{y}\|_{\mathbf{H}^2(\Omega)}\\ &\qquad + \Big\|\Big(\nu\frac{\partial \mathbf{y}}{\partial \mathbf{n}} - q\mathbf{n} - k(\mathbf{y},q)\mathbf{n}, q + k(\mathbf{y},q)\Big)\Big\|_{\mathbf{H}^{1/2,1}_{\Gamma,\Omega}}\|(\mathbf{g},h)\|_{\mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}}\Big)\\ &\le C(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'} + \|(\mathbf{g},h)\|_{\mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}}).\end{aligned}$$

The proof is complete. □

We want to define $(\mathbf{w},\rho)$ and $(\nu\frac{\partial \mathbf{w}}{\partial \mathbf{n}} - \rho\mathbf{n} - k(\mathbf{w},\rho)\mathbf{n}, \rho + k(\mathbf{w},\rho))$ in the case where $\mathbf{f} \in (\mathbf{H}^2(\Omega))'$ and $(\mathbf{g}, h) \in \mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}$. For all $\mathbf{f} \in (\mathbf{H}^2(\Omega))'$ and all $(\mathbf{g}, h) \in \mathbf{H}^{-3/2,-1}_{\Gamma,\ \Omega}$, we consider the variational problem

$$
\left\{
\begin{aligned}
& \textit{determine } (\mathbf{w}, \mathbf{\Psi}, \tilde{\rho}) \in \mathbf{L}^2(\Omega) \times \mathbf{H}^{-3/2,-1}_{\Gamma,\ \Omega} \textit{ such that} \\
& \int_\Omega \mathbf{w}\,\mathbf{\Phi} = \Big\langle \mathbf{f}, \mathbf{y} \Big\rangle_{(\mathbf{H}^2(\Omega))', \mathbf{H}^2(\Omega)} \\
& \qquad -\Big\langle (\mathbf{g}, h), \Big(\nu \frac{\partial \mathbf{y}}{\partial \mathbf{n}} + q\mathbf{n} - k(\mathbf{y}, q)\mathbf{n}, q + k(\mathbf{y}, q)\Big)\Big\rangle_{\mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}, \mathbf{H}^{1/2,1}_{\Gamma,\Omega}} \\
& \qquad +\Big\langle (\mathbf{\Psi}, \tilde{\rho}), (\mathbf{h}, g) \Big\rangle_{\mathbf{H}^{-3/2,-1}_{\Gamma,\ \Omega}, \mathbf{H}^{3/2,1}_{\Gamma,\Omega}} \quad \forall (\mathbf{\Phi}, \mathbf{h}, g) \in \mathbf{L}^2(\Omega) \times \mathbf{H}^{3/2,1}_{\Gamma,\Omega},
\end{aligned}
\right.
\tag{8.2}
$$

where $(\mathbf{y}, q)$ is solution of equation (8.1).

Theorem 8.3. *For all* $(\mathbf{f}, \mathbf{g}, h) \in (\mathbf{H}^2(\Omega))' \times \mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}$*, the variational problem (8.2) admits a unique solution* $(\mathbf{w}, \Psi, \rho) \in \mathbf{L}^2(\Omega) \times \mathbf{H}^{-3/2,-1}_{\Gamma,\ \Omega}$ *satisfying*

$$
\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} + \|(\Psi, \rho)\|_{\mathbf{H}^{-3/2,-1}_{\Gamma,\ \Omega}} \le C\Big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'} + \|(\mathbf{g}, h)\|_{\mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}}\Big).
$$

Proof. (i) Let us first prove the uniqueness. If $\mathbf{f} = 0$, $\mathbf{g} = 0$, $h = 0$ and if $(\mathbf{w}, \Psi, \rho)$ is a corresponding solution to problem (8.2), choosing $(\mathbf{\Phi}, \mathbf{h}, g) = (\mathbf{w}, 0, 0)$ in (8.2), we obtain $\mathbf{w} = 0$. Choosing $(\mathbf{\Phi}, \mathbf{h}, g) = (0, \mathbf{h}, g)$ in (8.2), with any $(\mathbf{h}, g)$ in $\mathbf{H}^{3/2,1}_{\Gamma,\Omega}$, we obtain $(\Psi, \rho) = (0, 0)$.

(ii) The existence result relies on a density argument. Let $(\mathbf{f}, \mathbf{g}, h)$ be in $(\mathbf{H}^2(\Omega))' \times \mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}$. The space $\mathbf{L}^2(\Omega) \times \mathbf{H}^{3/2,1}_{\Gamma,\Omega}$ being dense in $(\mathbf{H}^2(\Omega))' \times \mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}$, there exists a sequence $(\mathbf{f}_n, \mathbf{g}_n, h_n)_n \subset \mathbf{L}^2(\Omega) \times \mathbf{H}^{3/2,1}_{\Gamma,\Omega}$ converging to $(\mathbf{f}, \mathbf{g}, h)$ in $(\mathbf{H}^2(\Omega))' \times \mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}$. Let $(\mathbf{w}_n, \rho_n)$ be the solution to the equation

$$
-\nu\Delta\mathbf{w}_n + \nabla q_n = \mathbf{f}_n \quad \text{and} \quad \operatorname{div} \mathbf{w}_n = h_n \text{ in } \Omega, \quad \mathbf{w}_n = \mathbf{g}_n \quad \text{on } \Gamma.
$$

We can easily verify that $(\mathbf{w}_n, \mathbf{\Psi}_n, \tilde{\rho}_n)$, with $\mathbf{\Psi}_n = \nu\frac{\partial \mathbf{w}_n}{\partial \mathbf{n}} - \rho_n\mathbf{n} - k(\mathbf{w}_n, \rho_n)\mathbf{n}$ and $\tilde{\rho}_n = \rho_n + k(\mathbf{w}_n, \rho_n)$, is solution to problem (8.2) corresponding to $(\mathbf{f}_n, \mathbf{g}_n, h_n)$. From Lemma 8.2, we deduce that $(\mathbf{w}_n, \mathbf{\Psi}_n, \tilde{\rho}_n)_n$ converges to some $(\mathbf{w}, \mathbf{\Psi}, \tilde{\rho})$ in $\mathbf{V}^0(\Omega) \times \mathbf{H}^{-3/2,-1}_{\Gamma,\ \Omega}$. To show that $(\mathbf{w}, \mathbf{\Psi}, \tilde{\rho})$ is solution to problem (8.2) corresponding to $(\mathbf{f}, \mathbf{g}, h)$, it is sufficient to pass to the limit in the identity

$$
\begin{aligned}
& \int_\Omega \mathbf{w}_n\,\mathbf{\Phi} = \\
& \int_\Omega \mathbf{f}_n\,\mathbf{y} - \int_\Omega q\,h_n - \int_\Omega \rho_n\,g + \int_\Gamma \Big(\nu\frac{\partial \mathbf{w}_n}{\partial \mathbf{n}} - \rho_n\mathbf{n}\Big)\,\mathbf{h} - \int_\Gamma \Big(\nu\frac{\partial \mathbf{y}}{\partial \mathbf{n}} - q\mathbf{n}\Big)\,\mathbf{g}_n \\
& = \int_\Omega \mathbf{f}_n\,\mathbf{y} - \int_\Omega q\,h_n - \int_\Omega (\rho_n + k(\mathbf{w}_n, \rho_n))\,g \\
& \qquad\qquad + \int_\Gamma \Big(\nu\frac{\partial \mathbf{w}_n}{\partial \mathbf{n}} - \rho_n\mathbf{n} - k(\mathbf{w}_n, \rho_n)\mathbf{n}\Big)\,\mathbf{h} - \int_\Gamma \Big(\nu\frac{\partial \mathbf{y}}{\partial \mathbf{n}} - q\mathbf{n}\Big)\,\mathbf{g}_n.
\end{aligned}
$$

The proof is complete. □

Let us recall that, for $(\mathbf{g}, h)$ in $\mathbf{H}^{3/2,1}_{\Gamma,\Omega}$, $(L(\mathbf{g}, h), L_p(\mathbf{g}, h)) = (\mathbf{w}, \rho)$ is the unique solution in $\mathbf{H}^1(\Omega) \times \mathcal{H}^0(\Omega)$ to the equation

$$
-\nu\Delta\mathbf{w} + \nabla\rho = 0 \quad \text{and} \quad \operatorname{div} \mathbf{w} = h \text{ in } \Omega, \quad \mathbf{w} = \mathbf{g} \quad \text{on } \Gamma.
$$

From Theorem 8.3 we deduce the following corollary.

Corollary 8.4. *The operator L is linear and continuous from $\mathbf{H}^{s+1/2,s}_{\Gamma,\Omega}$ into $\mathbf{H}^{s+1}(\Omega)$ for all $-1 \le s \le 1$, and the operator L_p is linear and continuous from $\mathbf{H}^{s+1/2,s}_{\Gamma,\Omega}$ into $\mathcal{H}^s(\Omega)$ for all $-1 \le s \le 1$.*

If in addition Ω is of class C^3 the above results are still true for $-1 \le s \le 3/2$.

Proof. Let us prove the result when Ω is of class C^2 and $-1 \le s \le 1$. The other case can be treated similarly. Due to Lemma 8.1, L is continuous from $\mathbf{H}^{3/2,1}_{\Gamma,\Omega}$ into $\mathbf{H}^2(\Omega)$, and L_p is continuous from $\mathbf{H}^{3/2,1}_{\Gamma,\Omega}$ into $\mathcal{H}^1(\Omega)$. From Theorem 8.3 it follows that L can be extended to a bounded operator from $\mathbf{H}^{-1/2,-1}_{\Gamma,\Omega}$ into $\mathbf{L}^2(\Omega)$, and L_p can be extended to a bounded operator from $\mathbf{H}^{-1/2,-1}_{\Gamma,\Omega}$ into $\mathcal{H}^{-1}(\Omega)$. The result follows by interpolation. □

We define $L^* \in \mathcal{L}(\mathbf{L}^2(\Omega), \mathbf{H}^{0,1/2}_{\Gamma,\Omega})$ as the adjoint of $L \in \mathcal{L}(\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}, \mathbf{L}^2(\Omega))$.

Lemma 8.5. *For all $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $L^*\mathbf{f}$ is defined by*

$$L^*\mathbf{f} = \Big(-\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} + \pi\mathbf{n} + k(\mathbf{v},\pi)\mathbf{n}, -\pi - k(\mathbf{v},\pi)\Big),$$

where $(\mathbf{v},\pi) \in \mathbf{H}^2(\Omega)\times\mathcal{H}^1(\Omega)$ is the solution to

$$-\nu\Delta\mathbf{v} + \nabla\pi = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v} = 0 \quad \text{on } \Gamma. \tag{8.3}$$

The operator L^* is bounded from $\mathbf{H}^s(\Omega)$ into $\mathbf{H}^{s+1/2,s+1}_{\Gamma,\Omega}$ for all $0 \le s \le 2$. Moreover, for all $\mathbf{\Phi} \in \mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega)$, we have:

$$L^*(-A)\mathbf{\Phi} = \Big(-\nu\frac{\partial \mathbf{\Phi}}{\partial \mathbf{n}} + \psi\mathbf{n} + k(\mathbf{\Phi},\psi)\mathbf{n}, -\psi - k(\mathbf{\Phi},\psi)\Big),$$

where $\psi \in \mathcal{H}^1(\Omega)$ is determined by

$$\nabla\psi = \nu(I-P)\Delta\mathbf{\Phi}\,.$$

Proof. (i) For all $\mathbf{f} \in \mathbf{L}^2(\Omega)$, and all $(\mathbf{g},h) \in \mathbf{H}^{0,-1/2}_{\Gamma,\Omega}$, the solution $(\mathbf{v},\pi)$ to equation (8.3) and $\mathbf{w} = L(\mathbf{g},h)$ obey:

$$\begin{aligned}\int_\Omega L(\mathbf{g},h)\cdot\mathbf{f} &= \int_\Gamma \Big(-\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} + \pi\mathbf{n}\Big)\mathbf{g} - \int_\Omega h\,\pi \\ &= \Big\langle \Big(-\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} + \pi\mathbf{n} + k(\mathbf{v},\pi)\mathbf{n}, -\pi - k(\mathbf{v},\pi)\Big), (\mathbf{g},h)\Big\rangle_{\mathbf{H}^{0,-1/2}_{\Gamma,\Omega},\mathbf{H}^{0,1/2}_{\Gamma,\Omega}}.\end{aligned}$$

Thus $L^*\mathbf{f}$ is well defined as indicated in the statement of the lemma. Due to regularity results for the Stokes equations we have

$$\|L^*\mathbf{f}\|_{\mathbf{H}^{s+1/2,s+1}_{\Gamma,\Omega}} = \Big\|\Big(\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}}\mathbf{g} - \pi\mathbf{n} - k(\mathbf{v},\pi)\mathbf{n}, \pi + k(\mathbf{v},\pi)\Big)\Big\|_{\mathbf{H}^{s+1/2,s+1}_{\Gamma,\Omega}} \le C\|\mathbf{f}\|_{\mathbf{H}^s(\Omega)},$$

if $0 \le s \le 2$. The first part of the lemma is proved.

(ii) From the first part of the proof it follows that

$$L^*(-A)\mathbf{\Phi} = \Big(-\nu\frac{\partial \mathbf{\Phi}}{\partial \mathbf{n}} + \psi\mathbf{n} + k(\mathbf{\Phi},\psi)\mathbf{n}, -\psi - k(\mathbf{\Phi},\psi)\Big)\,,$$

where $(\hat{\mathbf{\Phi}},\psi)$ is the solution of the equation

$$-\nu\Delta\hat{\mathbf{\Phi}} + \nabla\psi = (-A)\mathbf{\Phi} \quad \text{and} \quad \operatorname{div}\hat{\mathbf{\Phi}} = 0 \text{ in } \Omega, \quad \hat{\mathbf{\Phi}} = 0 \quad \text{on } \Gamma.$$

This equation is equivalent to

$$(-A)\hat{\mathbf{\Phi}} = (-A)\mathbf{\Phi} \quad \text{and} \quad \nabla\psi = \nu(I-P)\Delta\hat{\mathbf{\Phi}}.$$

Thus $\hat{\mathbf{\Phi}} = \mathbf{\Phi}$ and $\nabla\psi = \nu(I-P)\Delta\mathbf{\Phi}$. The proof is complete. $\square$

9. **Appendix 2.** Throughout this appendix we assume that $\lambda_0 > 0$ satisfies (6.2), and that $\mathbf{z}$ belongs at least to $\mathbf{V}^1(\Omega)$, or is more regular than that.

Lemma 9.1. [19, Lemma 7.1] *For all $\mathbf{\Phi} \in \mathbf{L}^2(\Omega)$, $(\mathbf{h}, g) \in \mathbf{H}^{3/2,1}_{\Gamma,\Omega}$, the equation:*

$$\begin{aligned} &\lambda_0\mathbf{v} - \nu\Delta\mathbf{v} + (\mathbf{z}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{z} + \nabla\pi = \mathbf{\Phi} \quad \text{and} \quad \text{div } \mathbf{v} = g \text{ in } \Omega, \\ &\mathbf{v} = \mathbf{h} \quad \text{on } \Gamma, \end{aligned} \tag{9.1}$$

admits a unique solution $(\mathbf{v},\pi)$ in $\mathbf{H}^1(\Omega)\times\mathcal{H}^0(\Omega)$. Moreover the following estimate holds:

$$\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)} + \|\pi\|_{\mathcal{H}^1(\Omega)} \leq C\big(\|\mathbf{\Phi}\|_{\mathbf{L}^2(\Omega)} + \|(\mathbf{h},g)\|_{\mathbf{H}^{3/2,1}_{\Gamma,\Omega}}\big). \tag{9.2}$$

If in addition Ω is of class C^3, $\mathbf{z} \in \mathbf{V}^{3/2}(\Omega)$, and $(\mathbf{\Phi},\mathbf{h},g) \in \mathbf{H}^{1/2}(\Omega)\times\mathbf{H}^{2,3/2}_{\Gamma,\Omega}$, then

$$\|\mathbf{v}\|_{\mathbf{V}^{5/2}(\Omega)} + \|\pi\|_{H^{3/2}(\Omega)/\mathbb{R}} \leq C\Big(\|\mathbf{\Phi}\|_{\mathbf{H}^{1/2}(\Omega)} + \|(\mathbf{h},g)\|_{\mathbf{H}^{2,3/2}_{\Gamma,\Omega}}\Big).$$

The above results are also true if we replace equation (9.1) by the following one

$$\begin{aligned} &\lambda_0\mathbf{v} - \nu\Delta\mathbf{v} - (\mathbf{z}\cdot\nabla)\mathbf{v} + (\nabla\mathbf{z})^T\mathbf{v} + \nabla\pi = \mathbf{\Phi} \quad \text{and} \quad \text{div } \mathbf{v} = g \text{ in } \Omega, \\ &\mathbf{v} = \mathbf{h} \quad \text{on } \Gamma. \end{aligned} \tag{9.3}$$

Lemma 9.2. *For all $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and all $(\mathbf{g}, h) \in \mathbf{H}^{3/2,1}_{\Gamma,\Omega}$, the solution $(\mathbf{w},\rho)$ to equation:*

$$\begin{aligned} &\lambda_0\mathbf{w} - \nu\Delta\mathbf{w} + (\mathbf{z}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{z} + \nabla\rho = \mathbf{f} \quad \text{and} \quad \text{div}\,\mathbf{w} = h \text{ in } \Omega, \\ &\mathbf{w} = \mathbf{g} \quad \text{on } \Gamma, \end{aligned} \tag{9.4}$$

obeys the estimate:

$$\begin{aligned} &\|\mathbf{w}\|_{\mathbf{V}^{1/2}(\Omega)} + \Big\|\Big(\nu\tfrac{\partial\mathbf{w}}{\partial\mathbf{n}} - \rho\mathbf{n} - k(\mathbf{w},\rho)\mathbf{n}, \rho + k(\mathbf{w},\rho)\Big)\Big\|_{\mathbf{H}^{-1,-1/2}_{\Gamma,\ \Omega}} \\ &\leq C\big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'} + \|(\mathbf{g},h)\|_{\mathbf{H}^{0,-1/2}_{\Gamma,\ \Omega}}\big), \end{aligned}$$

where $k(\mathbf{w},\rho)$ is the constant corresponding to $(\mathbf{w},\rho)$, and defined in (2.1).

If in addition $\mathbf{z} \in \mathbf{V}^{3/2}(\Omega)$ then we also have:

$$\begin{aligned} &\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} + \Big\|\Big(\nu\tfrac{\partial\mathbf{w}}{\partial\mathbf{n}} - \rho\mathbf{n} - k(\mathbf{w},\rho)\mathbf{n}, \rho + k(\mathbf{w},\rho)\Big)\Big\|_{\mathbf{H}^{-3/2,-1}_{\Gamma,\ \Omega}} \\ &\leq C\Big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'} + \|(\mathbf{g},h)\|_{\mathbf{H}^{-1/2,-1}_{\Gamma,\ \Omega}}\Big). \end{aligned}$$

Proof. If $(\mathbf{v},\pi)$ is the solution to equation (9.3), and $(\mathbf{w},\rho)$ the solution to equation (9.4), then we have

$$\int_\Omega \mathbf{w}\,\mathbf{\Phi} = \int_\Omega \mathbf{f}\,\mathbf{v} + \int_\Gamma \Big(\nu\frac{\partial\mathbf{w}}{\partial\mathbf{n}} - \rho\mathbf{n}\Big)\,\mathbf{h} - \int_\Omega \pi\,h + \int_\Gamma \Big(-\nu\frac{\partial\mathbf{v}}{\partial\mathbf{n}} + \pi\mathbf{n} - \mathbf{z}\cdot\mathbf{n}\,\mathbf{h}\Big)\,\mathbf{g} + \int_\Omega \rho\,g.$$

Thus the proof can be performed as in the one of Lemma 8.2. The assumption $\mathbf{z} \in \mathbf{V}^{3/2}(\Omega)$ is needed to estimate $\mathbf{z}\cdot\mathbf{n}\,\mathbf{h}$ in $\mathbf{H}^{1/2}(\Gamma)$ when $(\mathbf{g},h)$ belongs to $\mathbf{H}^{0,-1/2}_{\Gamma,\ \Omega}$. $\square$

We want to define $(\mathbf{w}, q)$ and $\frac{\partial \mathbf{w}}{\partial \mathbf{n}} - q\mathbf{n} - k(\mathbf{w}, q)\mathbf{n}$ in the case where $\mathbf{f} \in (\mathbf{H}^2(\Omega))'$ and $(\mathbf{g}, h) \in \mathbf{H}^{-3/2,-1}_{\Gamma,\,\Omega}$. As in Appendix 1, for all $\mathbf{f} \in (\mathbf{H}^2(\Omega))'$ and all $(\mathbf{g}, h) \in \mathbf{H}^{-3/2,-1}_{\Gamma,\,\Omega}$, we consider the variational problem

$$\left\{\begin{array}{l} \textit{determine } (\mathbf{w}, \mathbf{\Psi}, \tilde\rho) \in \mathbf{L}^2(\Omega) \times \mathbf{H}^{-3/2,-1}_{\Gamma,\,\Omega} \textit{ such that} \\ \displaystyle\int_\Omega \mathbf{w}\,\mathbf{\Phi} = \Big\langle \mathbf{f}, \mathbf{v} \Big\rangle_{(\mathbf{H}^2(\Omega))', \mathbf{H}^2(\Omega)} \\ \quad -\Big\langle (\mathbf{g}, h), \Big(\nu \dfrac{\partial \mathbf{v}}{\partial \mathbf{n}} - \pi\mathbf{n} + \mathbf{z}\cdot\mathbf{n}\,\mathbf{h} - k_{\mathbf{z}}(\mathbf{v}, \pi)\mathbf{n}, \pi + k_{\mathbf{z}}(\mathbf{v}, \pi)\Big)\Big\rangle_{\mathbf{H}^{-3/2,-1}_{\Gamma,\,\Omega}, \mathbf{H}^{3/2,1}_{\Gamma,\Omega}} \\ \quad +\Big\langle (\mathbf{\Psi}, \tilde\rho), (\mathbf{h}, g)\Big\rangle_{\mathbf{H}^{-3/2,-1}_{\Gamma,\,\Omega}, \mathbf{H}^{3/2,1}_{\Gamma,\Omega}} \qquad \forall (\mathbf{\Phi}, \mathbf{h}, g) \in \mathbf{L}^2(\Omega) \times \mathbf{H}^{3/2,1}_{\Gamma,\Omega}, \end{array}\right. \tag{9.5}$$

where $(\mathbf{v}, \pi)$ is solution of the equation (9.3) and

$$k_{\mathbf{z}}(\mathbf{v}, \pi) = \frac{1}{|\Gamma| + |\Omega|}\left(\int_\Gamma \left(\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \cdot \mathbf{n} - \pi + \mathbf{z}\cdot\mathbf{n}\,\mathbf{h}\cdot\mathbf{n}\right) - \int_\Omega \pi\right).$$

Theorem 9.3. *For all $(\mathbf{f}, \mathbf{g}, h) \in (\mathbf{H}^{3/2}(\Omega))' \times \mathbf{H}^{0,-1/2}_{\Gamma,\,\Omega}$, the variational problem (9.5) admits a unique solution $(\mathbf{w}, \mathbf{\Psi}, \tilde\rho) \in \mathbf{H}^{1/2}(\Omega) \times \mathbf{H}^{0,-1/2}_{\Gamma,\,\Omega}$ satisfying*

$$\|\mathbf{w}\|_{\mathbf{H}^{1/2}(\Omega)} + \|(\mathbf{\Psi}, \tilde\rho)\|_{\mathbf{H}^{-1,-1/2}_{\Gamma,\,\Omega}} \le C\big(\|\mathbf{f}\|_{(\mathbf{H}^{3/2}(\Omega))'} + \|(\mathbf{g}, h)\|_{\mathbf{H}^{-1,-1/2}_{\Gamma,\,\Omega}}\big).$$

If in addition $\mathbf{z} \in \mathbf{V}^{3/2}(\Omega)$ then, for all $(\mathbf{f}, \mathbf{g}, h) \in (\mathbf{H}^2(\Omega))' \times \mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}$, the variational problem (9.5) admits a unique solution $(\mathbf{w}, \mathbf{\Psi}, \tilde\rho) \in \mathbf{L}^2(\Omega) \times \mathbf{H}^{-3/2,-1}_{\Gamma,\,\Omega}$ satisfying

$$\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} + \|(\mathbf{\Psi}, \tilde\rho)\|_{\mathbf{H}^{-3/2,-1}_{\Gamma,\,\Omega}} \le C\Big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'} + \|(\mathbf{g}, h)\|_{\mathbf{H}^{-1/2,-1}_{\Gamma,\,\Omega}}\Big).$$

Proof. The proof is similar to that of Theorem 8.3. □

From Theorem 9.3 we deduce the following corollary.

Corollary 9.4. *The operator $L_{\mathbf{z}}$ is linear and continuous from $\mathbf{H}^{s+1/2,s}_{\Gamma,\Omega}$ into $\mathbf{H}^{s+1}(\Omega)$ for all $-1 \le s \le 1$, and the operator $L_{\mathbf{z},p}$ is linear and continuous from $\mathbf{H}^{s+1/2,s}_{\Gamma,\Omega}$ into $\mathcal{H}^s(\Omega)$ for all $-1 \le s \le 1$.*

If in addition Ω is of class C^3 the above results are still true for $-1 \le s \le 3/2$.
If Ω is of class C^3 and if $\mathbf{z} \in \mathbf{V}^{3/2}(\Omega)$, then the above results are still true for $-1/2 \le s \le 2$.

Proof. See the proof of Corollary 8.4. □

We define $L^*_{\mathbf{z}} \in \mathcal{L}(\mathbf{L}^2(\Omega); \mathbf{H}^{0,1/2}_{\Gamma,\Omega})$ as the adjoint of $L_{\mathbf{z}} \in \mathcal{L}(\mathbf{H}^{0,-1/2}_{\Gamma,\Omega}; \mathbf{L}^2(\Omega))$.

Lemma 9.5. *For all $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $L^*_{\mathbf{z}}\mathbf{f}$ is defined by*

$$L^*_{\mathbf{z}}\mathbf{f} = \Big(-\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} + \pi\mathbf{n} + k(\mathbf{v}, \pi)\mathbf{n}, -\pi - k(\mathbf{v}, \pi)\Big),$$

where $(\mathbf{v}, \pi)$ is the solution to

$$\lambda_0\mathbf{v} - \nu\Delta\mathbf{v} - (\mathbf{z}\cdot\nabla)\mathbf{v} + (\nabla\mathbf{z})^T\mathbf{v} + \nabla\pi = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v} = 0 \quad \text{on } \Gamma.$$

The operator $L_{\mathbf{z}}^*$ *is bounded from* $\mathbf{H}^s(\Omega)$ *into* $\mathbf{H}^{s+1/2,s+1}_{\Gamma,\Omega}$ *for all* $0 \le s \le 2$. *Moreover, for all* $\mathbf{\Phi} \in \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega)$, *we have:*

$$L_{\mathbf{z}}^*(\lambda_0 I - A_{\mathbf{z}}^*)\mathbf{\Phi} = \Big(-\nu \frac{\partial \mathbf{\Phi}}{\partial \mathbf{n}} + \psi \mathbf{n} + k(\mathbf{\Phi},\psi)\mathbf{n}, -\psi - k(\mathbf{\Phi},\psi)\Big),$$

where $\psi \in \mathcal{H}^1(\Omega)$ *is determined by*

$$\nabla\psi = (I-P)\big(\nu\Delta\mathbf{\Phi} + (\mathbf{z}\cdot\nabla)\mathbf{\Phi} - (\nabla\mathbf{z})^T\mathbf{\Phi}\big).$$

Proof. (i) For all $\mathbf{f} \in \mathbf{L}^2(\Omega)$, and all $(\mathbf{g}, h) \in \mathbf{H}^{0,-1/2}_{\Gamma,\Omega}$, the pairs $(L_{\mathbf{z}} g, L_{p,\mathbf{z}} g) = (\mathbf{w},\rho)$ and $(\mathbf{v},\pi)$ obey:

$$\begin{aligned}\int_\Omega L(\mathbf{g},h)\cdot\mathbf{f} &= \int_\Gamma \Big(-\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} + \pi\mathbf{n}\Big)\mathbf{g} - \int_\Omega h\,\pi \\ &= \left\langle \left(-\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} + \pi\mathbf{n} + k(\mathbf{v},\pi)\mathbf{n}, -\pi - k(\mathbf{v},\pi)\right), (\mathbf{g},h)\right\rangle_{\mathbf{H}^{0,-1/2}_{\Gamma,\Omega},\mathbf{H}^{0,1/2}_{\Gamma,\Omega}}.\end{aligned}$$

This identity gives the expression of $L_{\mathbf{z}}^*$. As in the proof of Lemma 8.5, we can easily show that $L_{\mathbf{z}}^*$ is bounded from $\mathbf{H}^s(\Omega)$ into $\mathbf{H}^{s+1/2,s+1}_{\Gamma,\Omega}$ for all $0 \le s \le 2$.
(ii) From the first part of the proof it follows that

$$L_{\mathbf{z}}^*(\lambda_0 I - A_{\mathbf{z}}^*)\mathbf{\Phi} = \Big(-\nu \frac{\partial \mathbf{\Phi}}{\partial \mathbf{n}} + \psi \mathbf{n} + k(\mathbf{\Phi},\psi)\mathbf{n}, -\psi - k(\mathbf{\Phi},\psi)\Big),$$

where $(\hat{\mathbf{\Phi}},\psi)$ is the solution of the equation

$$\lambda_0\hat{\mathbf{\Phi}} - \nu\Delta\hat{\mathbf{\Phi}} - (\mathbf{z}\cdot\nabla)\hat{\mathbf{\Phi}} - (\nabla\mathbf{z})^T\hat{\mathbf{\Phi}} + \nabla\psi = (\lambda_0 I - A_{\mathbf{z}}^*)\mathbf{\Phi} \quad \text{and} \quad \operatorname{div}\hat{\mathbf{\Phi}} = 0 \text{ in } \Omega,$$
$$\hat{\mathbf{\Phi}} = 0 \quad \text{on } \Gamma.$$

This equation is equivalent to

$$(\lambda_0 I - A_{\mathbf{z}})\hat{\mathbf{\Phi}} = (\lambda_0 I - A_{\mathbf{z}})\mathbf{\Phi} \quad \text{and} \quad \nabla\psi = (I-P)(\nu\Delta\hat{\mathbf{\Phi}} + (\mathbf{z}\cdot\nabla)\hat{\mathbf{\Phi}} - (\nabla\mathbf{z})^T\hat{\mathbf{\Phi}}).$$

Thus $\hat{\mathbf{\Phi}} = \mathbf{\Phi}$ and $\nabla\psi = (I-P)(\nu\Delta\mathbf{\Phi} + (\mathbf{z}\cdot\nabla)\mathbf{\Phi} - (\nabla\mathbf{z})^T\mathbf{\Phi})$. The proof is complete. □

REFERENCES

[1] H. Amann, Nonhomogeneous Navier-Stokes equations with Integrable Low-regularity data, in Nonlinear Problems in Mathematical Physics and Related Topics II, Edited by Birman et al., Kluwer Academic/Plenum Publishers, New York, 2002, 1-28.

[2] H. Amann, Navier-Stokes equations with nonhomogeneous Dirichlet data, J. Nonlinear Math. Physics, Vol. 10, Supplement 1 (2003), 1-11.

[3] V. Barbu, I. Lasiecka, R. Triggiani, Tangential boundary stabilization of Navier-Stokes equations, Mem. Amer. Math. Soc. 181 (2006), no. 852.

[4] A. Bensoussan, G. Da Prato, M. C. Delfour, S. K. Mitter, Representation and Control of Infinite Dimensional Systems, Vol. 1, Birkhäuser, 1992.

[5] E. Casas, M. Mateos, J.-P. Raymond, Penalization of Dirichlet optimal control problems, ESAIM COCV, 15 (2009), 782-809.

[6] R. Dautray, J.-L. Lions, Analyse Mathématique et Calcul Numérique, Vol. 8, Masson, Paris, 1988.

[7] R. Farwig, G. P. Galdi, H. Sohr, A new class of weak solutions of the Navier-Stokes equations with nonhomogeneous data, J. Math. Fluid Mech., Vol. (2005), 1-22.

[8] R. Farwig, H. Kozono, H. Sohr, Very weak and strong solutions to the instationary Navier-Stokes system, in Topics on partial differential equations, P. Kaplický and Š. Nečasová ed., Matfyzpress, Praha, 2007.

[9] A. V. Fursikov, M. D. Gunzburger, L. S. Hou, Trace theorems for three-dimensional time-dependent solenoidal vector fields and their applications, Transactions of the American Mathematical Society, Vol. 354 (2001), 1079-1116.

[10] A. V. Fursikov, M. D. Gunzburger, L. S. Hou, Inhomogeneous boundary value problems for the three-dimensional evolutionary Navier-Stokes equations, J. Math. Fluid Mech., 4 (2002), 45-75.

[11] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations, Volume 1, Springer-Verlag, 1994.

[12] M. Gonzalez-Burgos, S. Guerrero and J.-P. Puel, Local exact controllability to the trajectories of the Boussinesq system via a fictitious control on the divergence equation, Communications on Pure and Applied Analysis (2009), no. 1, 311–333.

[13] P. Grisvard, Commutativité de deux foncteurs d'interpolation et applications, J. Math. pures et Appl., Vol. 45 (1966), 143-206, and 207-290.

[14] G. Grubb, V. A. Solonnikov, Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods, Math. Scand., Vol. 69 (1991), 217-290.

[15] G. Grubb, Nonhomogeneous Dirichlet Navier-Stokes problems in low regularity L_p Sobolev spaces, J. Math. Fluid Mech., Vol. 3 (2001), 57-81.

[16] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes, Vol. 1, Dunod, Paris 1968.

[17] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes, Vol. 2, Dunod, Paris 1968.

[18] P. A. Nguyen, J.-P. Raymond, Control problems for convection-diffusion equations with control localized on manifolds, ESAIM Control, Optimisation and Calculus of Variations, Vol. 6 (2001), 467-488.

[19] J.-P. Raymond, Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions, Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (2007), 921-951.

[20] J.-P. Raymond, J.-P. Raymond, Feedback boundary stabilization of the two dimensional Navier-Stokes equations, SIAM J. Control Optim. 45 (2006), 790-828.

[21] J.-P. Raymond, Feedback stabilization of a fluid-structure model, submitted to SIAM J. Control Optim..

[22] J.-P. Raymond, M. Vanninathan, Null Controllability in a Fluid - Solid Structure Model, J. Diff. Eq. 248 (2010), 1826-1865.

[23] V. A. Solonnikov, Estimates of the solutions of a nonstationary linearized system of Navier-Stokes equations, Am. Math. Soc. Transl., Ser. 2, Vol. 75 (1968), 1-116.

[24] R. Temam, Navier-Stokes equations, North-Holland, 1984.

[25] G. M. Troianiello, Elliptic differential equations and obstacle problems, Plenum Press, New-York, London, 1987.

[26] W. von Wahl, The equations of Navier-Stokes and abstract parabolic equations, Vieweg and Sohn, Braunschweig, Wiesbaden, 1985.

E-mail address: `raymond@math.univ-toulouse.fr`