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Stochastic Calculus applied in Finance, February 2014

We consider that we are on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t), \mathbb{P})$.
i(*) means exo i is difficult to solve but its result is useful.

1 Prerequisites: conditional expectation, stopping time

0. Recall Borel-Cantelli and Fatou lemmas.

1. Let $\mathcal{G}$ be a sub-σ algebra of $\mathcal{A}$ and an almost surely positive random variable X. Prove that the conditional expectation $E[X/\mathcal{G}]$ is also strictly positive.
Prove that the reciprocal is false given a contra-example (for instance use the trivial σ-algebra $\mathcal{G}$).

2. Let $\mathcal{G} \subset \mathcal{H} \subset \mathcal{A}$ and $X \in L^2(\Omega, \mathcal{A}, \mathbb{P})$. Prove (Pythagore Theorem):

$$E[(X - E[X/\mathcal{G}])^2] = E[(X - E[X/\mathcal{H}])^2] + E[(E[X/\mathcal{H}] - E[X/\mathcal{G}])^2].$$

3. Let O be an open sand in $\mathcal{A}$ and a $\mathcal{F}-$adapted continuous process X. One notes

$$T_0 = \inf\{t \ : \ X_t \in O\}.$$

Prove that T_O is a stopping time.

4. Let be stopping times S and T.
(i) Prove that $S \wedge T$ is a stopping time.
(ii) Prove

$$\mathcal{F}_{S \wedge T} = \mathcal{F}_S \cap \mathcal{F}_T.$$

5. Let be T a stopping time and $A \in \mathcal{A}$. Prove that

$$\begin{aligned} T_A &= T \quad \text{sur} \quad A, \\ &= +\infty \quad \text{sur} \quad A^c, \end{aligned}$$

is a stopping time if and only if $A \in \mathcal{F}_T$.

6. A real random variable X is $\mathcal{F}_T$ measurable if and only if $\forall t \geq 0$, $X\mathbf{1}_{T \leq t}$ is $\mathcal{F}_t$ measurable.

7. Let $X \in L^1$ and a family of σ-algebras $\mathcal{F}^\alpha, \alpha \in \mathcal{A}$. Then the family of conditional expectations $\{E[X/\mathcal{F}^\alpha], \alpha \in \mathcal{A}\}$ is uniformly integrable.

8. let X be a $\mathcal{F}$-progressively measurable process and T a $(\mathcal{F}_t)$ stopping time. Then
(i) the application $\omega \mapsto X_{T(\omega)}(\omega)$ is $\mathcal{F}_T$-measurable
(ii) the process $t \mapsto X_{t \wedge T}$ is $\mathcal{F}$-adapted.

9. If X is an adapted measurable process admitting càd or càg trajectories, it is progressively measurable.

2 Martingales

1. Let X be a martingale, φ a function such that $\forall t$ $\phi(X_t) \in L^1$.
(i) if φ is a convex function, then $\varphi(X)$ is a sub-martingale ; if φ is a concave function $\varphi(X)$is a super-martingale.
(ii) When X is a sub-martingale and φ an increasing convex function such that $\forall t$ $\phi(X_t) \in L^1$, then $\phi(X)$ is a sub-martingale.

2. Martingale convergence: admit the following: let X be a càd super (or sub)-martingale such that $\sup_t E[|X_t|] < \infty$. Then $\lim_{t\to\infty} X_t$ exists almost surely and belongs to $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

And deduce the Corollary : if X is a càd bounded from below super-martingale, then $\lim_{t\to\infty} X_t$ exists almost surely and belongs to $L^1(\Omega, \mathcal{A}, \mathbb{P})$.

3. let X be a martingale. Prove the following are equivalent:
(i) X is uniformly integrable.
(ii) X_t converges almost surely to Y (which belongs to L^1) when t goes to infinity and $\{X_t, t \in \overline{\mathbb{R}^+}\}$ is a martingale.
(iii) X_t converges to Y in L^1 when t goes to infinity.

Indication: $(i) \to (iii) \to (ii) \to (i)$

4. let be $(X_t)_{t\geq 0}$ a positive right continuous upper-martingale and

$$T = \inf\{t > 0 \ : X_t = 0\}.$$

(i) Prove that almost surely $\forall t \geq T,\ X_t = 0$. (First prove $\mathbf{E}(X_t \mathbf{1}_{T\leq t}) = 0$.)
(ii) Prove that almost surely $X_\infty = \lim_{t\to\infty} X_t$ exists. Deduce:

$$\{X_\infty > 0\} \subset \{\forall t,\ \ X_t > 0\} = \{T = +\infty\}.$$

Give a contra-example using

$$\{X_\infty > 0\} \neq \{T = +\infty\}.$$

5. If $M \in \mathcal{M}_{loc}$ is such that $E[M_t^*] < \infty \forall t$, then M is a 'true' martingale.
Moreover suppose $E[M^*] < \infty$, then M is uniformly integrable.

6. If X is a closed martingale with Z, meaning Z is interable and $\forall t$, $X_t = E[Z/\mathcal{F}_t]$, prove that it also closed with $\lim_{t\to\infty} X_t$ denoted as X_∞ equal to $E[Z/ \vee_{t\geq 0} \mathcal{F}_t]$.

3 Brownian motion

1. Prove that the real Brownian motion is a centered continuous Gaussian process with covariance function $\rho(s,t) = s \wedge$.
Conversely a centered continuous Gaussian process with covariance function $\rho(s,t) = s\wedge$ is a real Brownian motion.

2. Prove that the Brownian motion is martingale w.r.t. its proper filtration, i.e. $\mathcal{F}_t = \sigma(B_s, s \leq t)$.
Prove that it is also a Markov process.

3. let be $\mathcal{G}_t = \sigma(B_s, s \leq t) \vee \mathcal{N}, t \geq 0$. Prove this filtration is càd, meaning $\mathcal{G}_{t^+} = \cap_{s>t}\mathcal{G}_s$.
Indication: use
1. the $\mathcal{G}_{t^+}$-conditional characteristic of the vector (B_u, B_z), $z, u > t$ is the limit of $\mathcal{G}_w$-conditional characteristic function of the vector (B_u, B_z), when w decreases to t,
2. this limit is equal to the $\mathcal{G}_t$-conditional characteristic of the vector (B_u, B_z), $z, u > t$,
3. thus for any integrable Y $E[Y/\mathcal{G}_{t^+}] = E[Y/\mathcal{G}_t]$. So any $\mathcal{G}_{t^+}$-measurable is $\mathcal{G}_t$-measurable and conclude.

4(*). On considère l'ensemble des zéros du mouvement brownien : $\mathcal{X} = \{(t,\omega) \in R^+ \times \Omega : B_t(\omega) = 0\}$ and les sections de celui-ci par trajectoire $\omega \in \Omega$: $\mathcal{X}_\omega = \{t \in R^+ : B_t(\omega) = 0\}$.

Prove that P-presque sûrement en ω on a :
(i) la mesure de Lebesgue de $\mathcal{X}_\omega$ est nulle ,
(ii) $\mathcal{X}_\omega$ est fermé non borné (preuve un peu difficile...),
(iii) $t = 0$ est un point d'accumulation de $\mathcal{X}_\omega$,
(iv) $\mathcal{X}_\omega$ n'a pas de point isolé, donc est dense dans lui-même.

5(*). Théorème de Paley-Wiener-Zygmund 1933, *preuve pages 110-111, du Karatzas-Schreve.* Pour presque tout ω, l'application $t \mapsto B_t(\omega)$ n'est pas différentiable. Plus précisément, l'événement

$$\mathbb{P}\{\omega \in \Omega : \ \forall \mathrm{t}, \ \overline{\lim}_{\mathrm{h}\to 0^+} \frac{(\mathrm{B_{t+h}} - \mathrm{B_t})(\omega)}{\mathrm{h}} = +\infty \text{ and } \underline{\lim}_{\mathrm{h}\to 0^+} \frac{(\mathrm{B_{t+h}} - \mathrm{B_t})(\omega)}{\mathrm{h}} = -\infty\} = 1.$$

6. Let be (B_t) a real Brownian motion.
a) Prove that the sequence $\frac{B_n}{n}$ goes to 0 almost surely.
b) Use that B is a martingale and a Doob inequality to deduce the majoration

$$E[\sup_{\sigma \leq t \leq \tau} (\frac{B_t}{t})^2] \leq \frac{4\tau}{\sigma^2}.$$

c) Let be $\tau = 2\sigma = 2^{n+1}$, give a bound for $\mathbb{P}\{\sup_{2^n \leq t \leq 2^{n+1}} |\frac{B_t}{t}| > \varepsilon\}$ that proves the convergence of this sequence, then apply Borel Cantelli lemma.
d) Deduce $\lim_{t\to\infty} \frac{B_t}{t} = 0$ almost surely. (meaning the large numbers law, cf. problem 9.3, correction pages 124-125, in Karatzas-Schreve.)

7. Let be $Y_t = t.B_{1/t}$; $Y_0 = 0$ and $\mathcal{F}_t^Y$ the natural filtration associated to the process Y. Prove that $(Y_t, \mathcal{F}_t^Y)$ is a Brownian motion (use the criterium in 1 and exercise 6 above).

4 Stochastic integral

In this section and the following let be M square integrable martingale on the filtered probability space $(\Omega, \mathcal{F}_t, P)$ such that $d\langle M\rangle_t$ is absolutely continuous w.r.t. Lebesgue measure dt: $\exists$ an integrable measurable positive function on any $[0,t]$ such that $d\langle M\rangle_t = f(t)dt$.
1. Let be $\mathcal{L}_T(M)$ the set of adapted processes X on $[0,T]$ such that:

$$[X]_T^2 = E[\int_0^T X_s^2 d < M >_s] < +\infty.$$

Prove that $\mathcal{L}_T(M)$ is a metric space w.r.t. the distance d: $d(X,Y) = \sqrt{[X-Y]_T^2}$.
Actually it is a semi-norm which defines an equivalence relation $X \sim Y$ if $d(X,Y) = 0$.
2. Prove the equivalence

$$\sum_{j\geq 1} 2^{-j} \inf(1, [X - X_n]_j) \to 0 \iff \forall T,\ [X - X_n]_T \to 0.$$

3. Let be $\mathcal{S}$ the set of simple processes for which is defined the stochastic integral w.r.t. M :

$$I_t(X) = \sum_{j=0}^{J-1} X_j (M_{t_{j+1}} - M_{t_j}) + X_J (M_t - M_{t_J}) \text{ on the event } \{t_J \leq t \leq t_{J+1}\}.$$

Prove that I_t satisfies the following:

(i) I_t is a linear application on $\mathcal{S}$.

(ii) $I_t(X)$ is $\mathcal{F}_t$-measurable and square integrable.

(iii) $E[I_t(X)] = 0$.

(iv) $I_t(X)$ is a continuous martingale.

(v) $E[(I_t(X) - I_s(X))^2/\mathcal{F}_s] = E[I_t^2(X) - I_s^2(X)/\mathcal{F}_s] = E[\int_s^t X_u^2 d < M >_u /\mathcal{F}_s]$.

(vi) $E[I_t(X)]^2 = E[\int_0^t X_s^2 d < M >_s] = [X]_t^2$.

(vii) $< I_.(X) >_t = \int_0^t X_s^2 d < M >_s$.

Indication: actually, (vi) and (vii) are consequence of (v).

4. Prove that stochastic integral is associative, meaning: if H is stochastically integrable w.r.t. the martingale M, giving the integral $H.M$, and if G is stochastically integrable w.r.t. the martingale $H.M$, then $G.H$ is stochastically integrable w.r.t. the martingale M and:

$$G.(H.M) = (G.H).M.$$

5. Let be M a continuous martingale and $X \in \mathcal{L}(M)$. let be $s < t$ and Z a $\mathcal{F}_s$-measurable bounded random variable. Compute $E[\int_s^t Z X_u dM_u - Z \int_s^t X_u dM_u]^2$ and prove:

$$\int_s^t Z X_u dM_u = Z \int_s^t X_u dM_u.$$

6. Let be T a stopping time, two processes X and Y such that $X^T = Y^T$, two martingales M and N such that $M^T = N^T$. Suppose $X \in \mathcal{L}(M)$ and $Y \in \mathcal{L}(N)$. Prove that $I_M(X)^T = I_N(Y)^T$. (Use that for any square integrable martingale: $M_t = 0\ p.s. \iff < M >_t = 0\ p.s.$)

7. Let M and N square integrable continuous martingales, and processes $X \in \mathcal{L}_\infty(M)$, $Y \in \mathcal{L}_\infty(N)$. Prove that
(i) $X.M$ and $Y.N$ are uniformly integrable, with terminal value $\int_0^\infty X_s dM_s$ and $\int_0^\infty Y_s dN_s$.
(ii) $\lim_{t\to\infty} \langle X.M, Y.N\rangle_t$ exists almost surely.
This is a direct application of Kunita-Watanabe's inequality.
(iii)$E[X.M_\infty Y.N_\infty] = E[\int_0^\infty X_s Y_s d\langle M, N\rangle_s]$.

Use the following theorem: if M is a continuous local martingale such that $E[\langle M\rangle_\infty] < \infty$, then it is uniformly integrable and converges almost surely when $t \to \infty$. Moreover $E[\langle M\rangle_\infty] = E[M_\infty^2]$.

8. Let be M and N two local continuous martingales and real numbers a and b, $X \in \mathcal{L}_\infty(M) \cap \mathcal{L}_\infty(N)$. Prove that the stochastic integration with respect to the local continuous martingales is a linear application, meaning $X.(aM + bN) = aX.M + bX.N$

9. Let be M a local continuous martingale and $X \in \mathcal{L}_\infty(M)$. Prove there exists a sequence of simple processes (X^n) such that $\forall T > 0$, $\mathbb{P}$-almost surely:

$$\lim_{n\to\infty} \int_0^T |X_s^n - X_s|^2 d\langle M\rangle_s = 0,$$

and

$$\lim_{n\to\infty} \sup_{0\le t\le T} |I_t(X^n) - I_t(X)| = 0.$$

10. Let W a standard Brownian motion, ε a number in $[0, 1]$, and $\Pi = (t_0, \cdots, t_m)$ a partition of $[0, 1]$ with $0 = t_0 < \cdots < t_m = t$). Consider the approximating sum :

$$S_\varepsilon(\Pi) = \sum_{i=0}^{m-1} [(1-\varepsilon)W_{t_i} + \varepsilon W_{t_{i+1}}](W_{t_{i+1}} - W_{t_i})$$

for the stochastic integral $\int_0^t W_s dW_s$. Show that :

$$\lim_{|\Pi|\to 0} S_\varepsilon(\Pi) = \frac{1}{2}W_t^2 + (\varepsilon - \frac{1}{2})t,$$

where the limit is in probability. The right hand of the last limit is a martingale if and only if $\varepsilon = 0$, so that W is evaluated at the left-hand endpoint of each interval $[t_i, t_{i+1}]$ in the approximating sum ; this corresponds to the Ito integral.

With $\varepsilon = \frac{1}{2}$ we obtain the Stratonovitch integral, which obeys the usual rules of calculus such as $\int_0^t W_s \circ dW_s = \frac{1}{2}W_t^2$.
indication: explicit an approximation of Ito integral $\int_0^t W_s \circ dW_s$ and of W quadratic variation; then apply Ito formula to W_t^2.
Or: write $S_\varepsilon(\Pi)$ with a combination of $W_{t_{i+1}}^2 - W_{t_i}^2$ and $(W_{t_{i+1}} - W_{t_i})^2$.

5 Itô formula

1. The quadratic covariation of two continuoussquare integrable semi martingales X and Y is the limit in probability, when $\sup_i |t_{i+1} - t_i| \to 0$ of:

$$\langle X, Y\rangle_t = \lim_{proba} \sum_{i=1}^{n} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}).$$

Prove this covariation is null when X is a continuous semi-martingale and Y a finite variation process.
2. **Lévy Theorem** : Let be X a continuous (semi-)martingale, $X_0 = 0$ almost surely.
X is a real Brownian motion if and seulement if X is a continuous local martingale s.t. $\langle X\rangle_t = t$.
First step: compute the $\mathcal{F}_s$-conditional characteristic function of $X_t - X_s$ using Itô formula, $\forall$ $s \leq t$.
3. Prove that the unique solution in $\mathcal{C}_b^{1,2}(R^+, R^d)$ of the partial differential equation (heat equation)

$$\partial_t f = \frac{1}{2}\Delta f, f(0, x) = \varphi(x), \ \forall x \in R^d$$

where $\varphi \in \mathcal{C}_b^2(R^d)$ is $f(t, x) = E[\varphi(x + B_t)]$, B $d-$dimensional Brownian motion.
Peut-on se passer de l'hypothèse que les dérivées de f and ϕ sont bornées ?

4. *Long and tedious proof...* Let be M a d-dimensional vector of continuous martingales, A an adapted contious d-dimensional vector with with finite variation, X_0 a $\mathcal{F}_{\prime}$-measurable random variable; let be $f \in C^{1,2}(\mathbb{R}^+, \mathbb{R}^d)$ and $X_t := X_0 + M_t + A_t$. Prove that $\mathbb{P}$ almost surely:

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \partial_t f(s, X_s)ds + \int_0^t \sum_i \partial_i f(s, X_s)dM_s^i + \int_0^t \sum_i \partial_i f(s, X_s)dA_s^i \\ &+\frac{1}{2}\int_0^t \sum_{i,j} \partial_{ij}^2 f(s, X_s)d\langle M^i, M^j\rangle_s \end{aligned}$$

5.
a)Use exercise 4 with two semi-martingales $X = X_0 + M + A$ and $Y = Y_0 + N + C$. Prove that $\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \langle X, Y\rangle_t$.
This the **integral by part formula**.
b) Use Ito formula to get the stochastic differential of the processes

$$t \mapsto X_t^{-1} \ ; \ t \mapsto \exp(X_t) \ ; \ t \mapsto X_t.Y_t^{-1}.$$

6. Prove that

$$(\exp \int_0^t a_s ds)(x + \int_0^t b_s \exp(-\int_0^t a_u du)dB_s)$$

is solution to the SDE

$$dX_t = a(t)X_t dt + b(t)dB_t, \ t \in [0, T], \ X_0 = x,$$

after justification of any integral in the formula.
7. Stratonovitch integral is defined as:

$$\int_0^t Y_s \circ dX_s = \int_0^t Y_s \circ dX_s + \frac{1}{2}\langle Y, X\rangle_t.$$

Let be $\varepsilon = \frac{1}{2}$. Prove that:

$$\lim_{\|\pi\|\to 0} S_\varepsilon(\Pi) = \sum_{i=0}^{m-1}[(1-\varepsilon)W_{t_i} + \varepsilon W_{t_{i+1}}](W_{t_{i+1}} - W_{t_i}) = \int_0^t W_s \circ dW_s = \frac{1}{2}W_t^2$$

where $\|\pi\| = \sup_i(t_{i+1} - t_i)$.

Let be X and Y two continuous semi-martingales,and π a partition [0,t]. Prove that

$$\lim_{\|\pi\|\to 0} \sum_{i=0}^{m-1} \frac{1}{2}(Y_{t_{i+1}} + Y_{t_i})(X_{t_{i+1}} - X_{t_i}) = \int_0^t Y_s \circ dX_s.$$

Let be X a d-dimensional vector of continuous semi-martingales, and f a C^2 function. Prove that:

$$f(X_t) - f(X_0) = \int_0^t \partial_i f(X_s) \circ dX_s^i.$$

6 Stochastic differential equations

1. Prove that the process $t \mapsto (\exp \int_0^t a_s ds)(x + \int_0^t b_s \exp(-\int_0^s a_u du) dB_s)$ is solution to the SDE $dX_t = a(t)X_t dt + b(t)dB_t,\ t \in [0,T],\ X_0 = x$, after justification of any integral in the formula. (meaning specify useful hypotheses on parameters a and b.

2. Let be B a real Brownian motion. Prove that $B_t^2 = 2\int_0^t B_s dB_s + t$.
If $\forall t\ X \in \mathcal{L}_t(B)$, then:

$$(X.B)_t^2 = 2\int_0^t (X.B)_s X_s dB_s + \int_0^t X_s^2 ds.$$

Let be $Z_t = \exp((X.B)_t - \frac{1}{2}\int_0^t X_s^2 ds)$. Prove that Z is solution to the SDE:

$$Z_t = 1 + \int_0^t Z_s X_s dB_s.$$

Prove that $Y = Z^{-1}$ is solution to the SDE:

$$dY_t = Y_t(X_t dt - X_t dB_t).$$

Prove that there exists a unique solution to the SDE $dX_t = X_t b_t dt + X_t \sigma_t dB_t,\ X_t = x \in \mathbb{R}$ when $b, \sigma^2 \in L^1(\mathbb{R}^+)$, computing the stochastic differential of two solutions ratio.
3. Let be Ornstein Uhlenbeck stochastic differential equation:

$$dX_t = -\alpha X_t dt + \sigma dB_t,\ X_0 = x,$$

where $x \in L^1(\mathcal{F}_0)$.
(i) Prove that the following is the solution of this SDE:

$$X_t = e^{-\alpha t}(x + \int_0^t \sigma e^{\alpha s} dB_s).$$

(ii) Prove that the expectation $m(t) = E[X_t]$ is solution of an ordinary differential equation which is obtained by integration of $X_t = x - \alpha \int_0^t X_s ds + \sigma B_t$. Deduce $m(t) = m(0)e^{-\alpha t}$.
(iii) Prove the covariance

$$V(t) = Var[X_t] = \frac{\sigma^2}{2\alpha} + (V(0) - \frac{\sigma^2}{2\alpha})e^{-2\alpha t}.$$

(iv) Let be x a $\mathcal{F}_0$-measurable variable, with law $\mathcal{N}(0, \frac{\sigma^2}{2\alpha})$, Prove that X is a Gaussian process with convariance function $\rho(s,t) = \frac{\sigma^2}{2\alpha} e^{-\alpha|t-s|}$.

7 Black-Scholes Model

1.Assume that a risky asset price process is solution to the SDE

$$dS_t = S_t b dt + S_t \sigma dW_t, S_o = s, \tag{1}$$

b is named "trend' and σ "volatility". Prove that (**??**) admits a unique solution, using Ito formula to compute the ratio $\frac{S^1}{S^2}$ with $S^i, i = 1, 2$ two solutions to the SDE.

2. Assume that the portfolio θ value $V_t(\theta)$ is such that there exists a $C^{1,2}$ regular function C satisfying

$$V_t(\theta) = C(t, S_t). \tag{2}$$

Otherwise, θ is the pair (a, d) and

$$V_t(\theta) = a_t S_t^0 + d_t S_t = \langle \theta_0, p_0 \rangle + \int_0^t a_s dS_s^0 + \int_0^t d_s dS_s. \tag{3}$$

With this "self-financing" strategy θ the option seller (for instance option $(S_T - K)^+$) could "hedge" the option with the initial price $q = V_0$: $V_T(\theta) = C(T, S_T)$.
Use two different ways to compute the stochastic differential of $V_t(\theta)$ to get a PDE (partial differential equation) the solution of which will be the researched function C.

3. Actually this PDE is solved using the change of (variable,function) :

$$x = e^y, y \in \mathbb{R} \text{ ; } \mathrm{D(t,y) = C(t, e^y)}.$$

Thus, prove that we turn to the Dirichlet problem

$$\partial_t D(t, y) + r\partial_y D(t, y) + \frac{1}{2}\partial^2_{y^2} D(t, y)\sigma^2 = rD(t, y), y \in \mathbb{R},$$
$$D(T, y) = (e^y - K)^+, y \in \mathbb{R}.$$

Now let be the SDE:

$$dX_s = rds + \sigma dW_s, s \in [t, T], X_t = y.$$

Dedduce the solution

$$D(t, y) = E_y[e^{-r(T-t)}(e^{X_T} - K)^+],$$

and the explicit formula, "Black-Scholes" formula, which uses the fact that the law of X_T is a Gaussian law.

8 Change of probability measures, Girsanov theorem

1. Let be the probability measure Q equivalent to $\mathbb{P}$ defined as $Q = Z.P$, $Z \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$ meaning $Q_{|}\mathcal{F}_t = Z_t.P$, $Z_t = E_P[Z/\mathcal{F}_t]$.
Prove that $\forall t$ and $\forall Y \in L^\infty(\Omega, \mathcal{F}_t, P), E_P[YZ_t/\mathcal{F}_s] = Z_s E_Q[Y/\mathcal{F}_s]$.
Indication: compute $\forall A \in \mathcal{F}_s$, *the expectations* $E_P[1_A Y Z_t]$ and $E_P[1_A Z_s E_Q[Y/\mathcal{F}_s]]$.
2. Let be $T \geq 0$, $Z \in \mathcal{M}(\mathbb{P})$ and $Q = Z_T\mathbb{P}$, $0 \leq s \leq t \leq T$ and a $\mathcal{F}_t-$measurable random variable $Y \in L^1(Q)$. Prove (Bayes formula)

$$E_Q(Y/\mathcal{F}_s) = \frac{E_{\mathbb{P}}(YZ_t/\mathcal{F}_s)}{Z_s}.$$

3. Let be M a $\mathbb{P}$-martingale, $X \in \mathcal{L}(B)$ such that $Z = \mathcal{E}(X.B)$ is a $\mathbb{P}$-martingale (remember: $dZ_t = Z_t X_t dB_t$, $Z_0 = 1$). Let be $Q := Z_T\mathbb{P}$ an equivalent probability measure to $\mathbb{P}$ on σ-algebra $\mathcal{F}_T$.
(i) Prove that $d\langle M, Z\rangle = ZXd\langle M, B\rangle$.
(ii) Use Itô formula to developp $M_t Z_t - M_s Z_s$, calculer $E_{\mathbb{P}}[M_t Z_t/\mathcal{F}_s]$.
(iii) Use Itô formula between s and t to process $Z_. \int_0^. X_u d\langle M, B\rangle_u$.
(iv) Deduce $M_. - \int_0^. X_u d\langle M, B\rangle_u$ is a $Q-$martingale.
4. The following is a contra-example when Novikov condition is not satisfied: let be the stopping time $T = \inf\{1 \geq t \geq 0, t + B_t^2 = 1\}$ and

$$X_t = -\frac{2}{(1-t)^2} B_t 1_{\{t \leq T\}} \ ; \ 0 \leq t < 1, \ X_1 = 0.$$

(i) Prove that $T < 1$ almost surely, so $\int_0^1 X_t^2 dt < \infty$ almost surely.
(ii) Apply Itô formula to the processt $\rightarrow \frac{B_t^2}{(1-t)^2}$; $0 \leq t < 1$ to prove:

$$\int_0^1 X_t dB_t - \frac{1}{2}\int_0^1 X_t^2 dt = -1 - 2\int_0^T \frac{t}{(1-t)^4} B_t^2 dt < -1.$$

(iii) The local martingale $\mathcal{E}(X.B)$ is not a martingale: we deduce from (ii) that $E[\mathcal{E}_t(X.B)] \leq \exp(-1) < 1$ and this fact contradicts that for any martingale $E(M_t) = M_0$, here it could be 1.... Anyway, prove that $\forall n \geq 1$ and $\sigma_n = 1-(1/\sqrt{n})$, the stopped process $E(X.B)^{\sigma_n}$ is a martingale.

5(*). Let be B the standard Brownian motion on the filtered probability space $(\Omega, (\mathcal{F}_t; t \in \mathbb{R}^+), \mathrm{P})$ and $H \in L^2(\Omega \times [0, t])$. $\forall t$, let be $W_t := B_t + \int_0^t H_s ds$. Prove that the law of W is equivalent to the Wiener measure according to the density on $\mathcal{F}_T$ $\frac{d\mu_B}{d\mu_W} = \exp[\int_0^T -H_s dB_s - \frac{1}{2}\int_0^T -H_s^2 ds$ where (on the canonical space) $\mu_B(A) := \mathbb{P}\{\omega : \mathrm{t} \mapsto \mathrm{B_t}(\omega) \in \mathrm{A}\}$.

9 Representation theorems, martingale problem

Recall:
$\mathcal{H}_0^2 = \{M \in \mathcal{M}^{2,c}, M_0 = 0, \langle M\rangle_\infty \in L^1\}$,
M and N are said to be orthogonal if $E[M_\infty N_\infty] = 0$, noted $M \perp N$,
and strongly orthogonal if MN is a martingale, noted as $M \dagger N$.
Let be $\mathcal{A} \subset \mathcal{H}_0^2$: denote $S(\mathcal{A})$ the smallest stable closed vectorial subspace which contains $\mathcal{A}$.

1. Let be $M \in \mathcal{H}_0^2$ and Y a centered Bernoulli random variable independent on M. Let be $N := YM$. Prove $M \perp N$ but no $M \dagger N$.
2. Let be M and $N \in \mathcal{H}_0^2$. Prove the equivalencies:

$$(i) M \dagger N, \qquad (ii) S(M) \dagger N$$
$$(iii) S(M) \dagger S(N) \qquad (iv) S(M) \perp N$$
$$(v) S(M) \perp S(N)$$

3. Let be $\mathcal{M}(\mathcal{A})$ the set of probability measures Q on $\mathcal{F}_\infty$, $Q << \mathbb{P}$, $\mathbb{P}_{|\mathcal{F}_0} = \mathbb{Q}_{|\mathcal{F}_0}$, and such that $\mathcal{A} \subset \mathcal{H}_0^2(Q)$. Prove that $\mathcal{M}(\mathcal{A})$ is convex.
Study carefully the difference between $\mathcal{M}(\mathcal{A})$ and $M(\mathcal{A})$ (cf. Def 6.1 and 6.17 in Lecture Notes).
4. Let be B a $n-$dimensional Brownian motion on $(\Omega, \mathcal{F}_t, \mathbb{P})$. Prove that $\forall M \in \mathcal{M}_{loc}^{c,2}$, $\exists H^i \in \mathcal{P}(B^i), i = 1, \cdots, n$, such that:

$$M_t = M_0 + \sum_{i=1}^{n} (H^i.B^i)_t.$$

Indication: apply extremal probability measure theorem (th 6.14) to the set $M(\mathcal{B})$ (actually the singleton $\{\mathbb{P}\}$) when $\mathcal{B}$ is the set of Brownian motion, then localize.
5. Prove that the above vector process H is unique, meaning $\forall H'$ satisfying $M_t = M_0 + \sum_{i=1}^{n} (H'^i.B^i)_t$ is such that :

$$\int_0^t \sum_{i=1}^{n} |H_s'^i - H_s^i|^2 ds = 0 \text{ almost surely.}$$

6. Let be M a vector martingale, the components of which are not strongly orthogonal two by two. Prove the inclusion

$$\{H, \ \forall i H^i \in \mathcal{L}(M^i)\} \subset \mathcal{L}(M)$$

but the equality is false.

10 Example: optimal strategy for a small investor

Let be a set of price processes: $S_t^n = \mathcal{E}_t(X^n), t \in [0,T]$, with:

$$dX_t^n = \sum_{j=1}^{d} \sigma_j^n(t) dW_t^j + b^n(t)dt, n = 1, \cdots, N; dX_t^0 = r_t dt.$$

Suppose the matrix σ satisfies $dt \otimes d\mathbb{P}$ almost surely : $\sigma\sigma^* \geq \alpha I$, σ^* is the tranpose matrix of σ and $\alpha > 0$. The coefficients b, σ, r are$\mathcal{F}-$adapted bounded $[0,T] \times \Omega$ processes.
1. Look for a condition so that the market is viable, meaning a condition such that there is no arbitrage opportunity.
(i) Prove that a market is viable as soon as there exists a risk neutral probability measure Q.
(ii) Propose some hypotheses on the above model, sufficient for the existence of Q.
2. Propose some hypotheses on the above model, sufficient for the market be complete, meaning any contingent claim is "atteignable" (hedgeable).
Start with case $N = d = 1$, then $N = d > 1$.
Remark: If $d < N$ and σ surjective, there is no uniqueness of vector u so that $\sigma dW + (b-r)dt = \sigma d\tilde{W}$. In this case, the market is not complete and the set $\mathcal{Q}_S$ is bijective with $\sigma^{-1}(r-b)$.

Recall: let be a set of price processes S, a **risk neutral probability measure** on$(\Omega, (\mathcal{F}_t))$ is a probability measure Q equivalent to $\mathbb{P}$ such that the discounted prices $e^{-rt}S^n$, denoted as $\tilde{S}^n$, are uniformly integrable Q-martingales; denote their set $\mathcal{Q}_p$.

3. Let be θ an admissible strategy. Prove it is self-financing if and only if the discounted portfolio value $\tilde{V}_t(p) = e^{-rt}V_t(p)$ satisfies:

$$\tilde{V}_t(p) = V_0(p) + \int_0^t < \theta_s, d\tilde{p}_s > .$$

(use Ito formula)
4. Let be the relation defined as

$$c_1 \prec c_2 \text{ si } \psi(c_1) \leq \psi(c_2)$$

where the application ψ is defined on the consumption set X by:

$$\psi(a, Y) = a + E_Q[Y].$$

Prove that it is a convex increasing continuous complete preference relation.
5. A sufficient and necessary condition for a strategy (π, c) to be admissible: let be fixed the discounted "objective" consumption $\int_0^T e^{-rs}c_s ds$. Prove that

$$(*) \qquad E_Q[\int_0^T e^{-rs}c_s ds] \leq x$$

is equivalent to the existence of an admissible stategy π such that $X_T = x + \int_0^T \pi_s . d\tilde{S}_s$.
6. Optimal strategies.
Prove that actually the problem is as following: the small investor evaluates the quality of his investment with an "utility function" (U_i is positive, concave, strictly increasing, C^1 class); he look for the maximisation:

$$(c, X_T) \to E_{\mathbb{P}}[\int_0^T U_1(c_s)ds + U_2(X_T)]$$

under the above constraint 5 (*). Solve this constrained optimisation problem using Lagrange method and Kuhn and Tucker Theorem.