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INTRODUCTION

We can consider this problem as a particular case of a control problem : some agents trade financial assets on the market. They want to choose an optimal strategy, they know some information on the market, the political and economical situation, and so on. This world is a random world, and it is modeled with respect to some elements :

- alea (for instance underlying the prices) : Ω.

- daily information (observable filtration) : $(\mathcal{F}_t, t \geq 0)$ where $\mathcal{F}_t$ is the natural filtration associated to the observed prices at time t.

- goods consumption, exogeneous endowments.

- portfolio, i.e. decisions $\mathcal{D}$ taking their values in $\mathbb{R}$, with respect to some constraint to define admissible controls.

- agents' preference ordering so that $\mathcal{D}$ is ordered ; actually the wealth-consumption utility.

Vocabulary :

- exchange goods

- securities, stocks...

- traders

- endowments, for instance wages.

First, a simple model (chapter 1) :

. one time period, two dates : 0 and 1

. finite alea at time 1 : $\Omega = \{\omega_1, \cdots, \omega_K\}$, and firstly, $\Omega = \{\omega_1, \omega_2\}$.

. trading at times 0 and 1

We can generalize to n times $t_i, i = 1, \cdots, n$ with a finite Ω, endowed with a filtration, sigma-algebras increasing sequence $\mathcal{F}_{t_i}$, i.e. the information at time t_i, and trading at time t_i. Generally, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{P}(\Omega)$, chapter 2.

Finally, a continuous model is a larger generalization: $card(\Omega) = \infty$ and $t \in [0, T]$. It is out of our purpose.

1 Two periods market

Cf. R.A. Dana and M. Jeanblanc [1], chapter 1. Dothan [2], chapters 1 and 2.

1.1 Two times, two states, two assets

1.1.1 Model

At time 0, the asset price is S_0, at time 1, it is denoted as S_1 and its values are S_h, or S_b. S_h and S_b are the two states for this risky asset. At time 0, the bond price is 1, at time 1, it is $1+r$,r is the bond return.

Definition 1.1 *A "call option" is the following contract: at time 0, the buyer pays q so that he has the right at time 1 to buy the asset at price K even if $S_1 > K$. It is not an obligation, only a right.... When at time 1 $S_1 > K$ he buys, so that he wins $S_1 - K - q$. In the other case, and if he does nothing, he loses q. Globally, he wins $(S_1 - K)^+ - q$.*

A "put option " is the following contract: at time 0, the buyer pays q so he has the right at time 1 to sell the asset at price K even if $S_1 < K$. It is not an obligation, only a right.... When at time 1 $S_1 < K$ he buys, so he wins $K - S_1 - q$. In the other case, and if he does nothing, he loses q. Globally, he wins $(K - S_1)^+ - q$.

Then the problem is to fix a "fair price" q, between the buyer and the seller, of this contract.

1.1.2 Hedging portfolio, option pricing

Option pricing means what is the fair price q.

. Obviously, the only interesting case is $K \in]S_b, S_h[$ and $E(S_1) \geq S_0(1+r)$: if not, the best is to only buy bond ! We look for a portfolio $(\alpha, \beta), \alpha$ is the amount on the bond, β on the risky asset, which "hedges" the option, i.e. its value at time 1 is the same as the option value: $\alpha \to (1+r)\alpha; \beta S_0 \to \beta S_1$.

This portfolio terminal value is $(1+r)\alpha + \beta S_1$.

The couple (α, β) is solution to the system:

$$(1+r)\alpha + \beta S_h = S_h - K; (1+r)\alpha + \beta S_b = 0$$

with the initial condition $q = \alpha + S_0\beta$.
Exercise: solve the system and compute the price q

$$\alpha = -\frac{S_b(S_h - K)}{(S_h - S_b)(1+r)}; \beta = \frac{S_h - K}{S_h - S_b}.$$

The option price, (the fair price) is

$$q = \alpha + S_0\beta = \frac{S_h - K}{S_h - S_b}(S_0 - \frac{S_b}{1+r}).$$

Interpretation : with as least q, the seller can obtain the portfolio (α, β) which induces $(S_1 - K)^+ \ \forall\omega$ and so he can pay the buyer.
The buyer doesn't want to pay more than q, because in other case with such a portfolio, he could win more than $(S_1 - K)^+$.

. When K is not in $]S_b, S_h[$, the system is:

$(1+r)\alpha + \beta S_h = (S_h - K)^+$ denoted as C_h; $(1+r)\alpha + \beta S_b = (S_b - K)^+$ denoted as C_b

thus

$$\alpha = \frac{C_b S_h - S_b C_h}{(S_h - S_b)(1+r)}; \beta = \frac{C_h - C_b}{S_h - S_b},$$

and the option price is

$$q = \frac{S_h C_b - S_b C_h + (1+r)S_0(C_h - C_b)}{(S_h - S_b)(1+r)}.$$

This could be written as:

$$(1) \qquad (1+r)q = \pi C_h + (1-\pi)C_b, \text{ with } \pi = \frac{(1+r)S_0 - S_b}{S_h - S_b}.$$

Exercise: (i) prove that $\pi \in [0,1] \leftrightarrow S_b \leq (1+r)S_0 \leq S_h$

$$(1+r)p = \pi P_h + (1-\pi)P_b, \text{ avec } P_. = (K - S_.)^+,$$

(ii) compute the put option price.

1.1.3 Risk neutral probability

The π definition yields the relation $S_0(1+r) = \pi S_h + (1-\pi)S_b$. When π belongs to $[0,1]$, which is equivalent to $S_b \leq (1+r)S_0 \leq S_h$ (*exercise*), π can be looked as a new probability on space Ω and the price equation is now

$$(1+r)q = E_\pi[(S_1 - K)^+] \text{ respectively } (1+r)p = E_\pi[(K - S_1)^+]$$

Definition 1.2 *This probability is called "Risk neutral probability".*

Interpretation : under this probability, on an average it is equivalent to buy risk-less or risky asset.....

Proposition 1.3 *The option fair price is the discounted profit mean computed under the risk neutral probability:* $q = E_\pi[\frac{1}{1+r}(S_1 - K)^+]$.

Proof: the remark above the definition.

1.1.4 Put-call parity

Let us notice that $(S_1 - K)^+ - (K - S_1)^+ = S_1 - K$ and $(1 + r)S_0 = E_\pi(S_1)$.
So $E_\pi[(S_1 - K)^+] - E_\pi[(K - S_1)^+] = (1 + r)S_0 - K$, i.e. the call price is the put price plus $S_0 - K : (1 + r)$: $q = p + S_0 - K/(1 + r)$

1.1.5 Arbitrage opportunity

Definition 1.4 *An "***arbitrage opportunity***" is the opportunity to have a portfolio such that the initial wealth $X_0 < 0$ and the terminal wealth $X_1 \geq 0$, or $X_0 \leq 0, X_1 \geq 0$ with $X_1(\omega_1)$ or $X_1(\omega_2) > 0$.*

Exercise: $S_b \leq (1 + r)S_0 \leq S_h$ is equivalent to the absence of arbitrage opportunities.

. On one hand, if $(1 + r)S_0 < S_b$, obviously it is better to borrow S_0 on the bond and to buy risky asset $(\alpha = -S_0, \beta = 1)$. Thus $X_0 = 0$ and $X_1 = -(1 + r)S_0 + S_1$ which is strictly positive (cf. hypothesis): then $(\alpha = -S_0, \beta = 1)$ is an arbitrage opportunity.
Similarly if $S_h > (1 + r)S_0$, it is better to borrow risky asset S_1 and to buy some bond, once again, $(\alpha = S_0, \beta = -1)$ is an arbitrage opportunity.

.. Conversely if we assume that $S_b \leq (1 + r)S_0 \leq S_h$, we found $\pi = \frac{(1+r)S_0 - S_b}{S_h - S_b}$, risk neutral probability, which satisfies $E_\pi[X_1] = (1 + r)X_0$ for any portfolio since $S_1^0 = (1 + r)S_0^0, E_\pi[S_1^1] = (1 + r)S_0^1$. If there exists an arbitrage opportunity, a portfolio (α, β) could exist which would induce an initial wealth $X_0 = \alpha + \beta S_0 < 0$, and a terminal wealth $X_1 = \alpha(1 + r) + \beta S_1 \geq 0$.
We compute $E_\pi[X_1]$ under the probability measure π i.e. $E_\pi[X_1] = \alpha(1 + r) + \beta S_0(1 + r)$ which is nonnegative, so it is a contradiction.

1.1.6 Risk of an option

Let be p the probability of the event $\{S_1 = S_h\}$. The mean return of the asset is $\frac{pS_h + (1-p)S_b}{S_0}$ denoted as m_S. This return variance is

$$E[(\frac{S_1}{S_0} - m_S)^2] = p(1 - p)\frac{(S_h - S_b)^2}{S_0^2}.$$

Definition 1.5 *The* **volatility** *of an asset is its standard deviation: $\sqrt{p(1 - p)}\frac{S_h - S_b}{S_0}$, denoted as v_S. It is a measure of the mean risk.*

Concerning the option, let Δ be the quantity β to put on the risky asset to hedge the contract: $\Delta = \frac{C_h - C_b}{S_h - S_b}$. Recall that the part β in the portfolio is used to hedge the option. Economists call it the "sensibility" of C to S, roughly speaking the relative variation of the C range (option price) with respect to this of the asset price S.

Definition 1.6 *The option* **elasticity** *is* $\Omega = \frac{S_0}{q}\Delta$.

Proposition 1.7 *The option mean return is* $m_C = p\frac{C_h}{q} + (1-p)\frac{C_b}{q}$ *and its volatility is* $\sqrt{p(1-p)}\frac{C_h - C_b}{q}$*, denoted as* v_C*.*

Let us notice that $v_C = v_S \times \Omega$.

Here we assume that $m_S \geq 1 + r$, if not it could be better to only buy bond...

Proposition 1.8 *We assume that* $m_S \geq 1 + r$*. Then:*

(i) $v_C \geq v_S$*.*

(ii) The call excess return is more than the asset one i.e. :

$$m_C - (1 + r) \geq m_S - (1 + r).$$

Proof:
1. Actually we have to prove $\Omega \geq 1$, i.e. $\frac{S_0}{q}\frac{C_h - C_b}{S_h - S_b} \geq 1$.
But recall that $q = \frac{S_h C_b - S_b C_h + (1+r)S_0(C_h - C_b)}{(1+r)(S_h - S_b)}$, so we have to compare $(1 + r)S_0(C_h - C_b)$ and $S_h C_b - S_b C_h + (1 + r)S_0(C_h - C_b)$.
Otherwise, $S_h C_b - S_b C_h \leq 0$ is easy to verify so the first assertion is proved.
2. Let us recall $m_S = \frac{pS_h + (1-p)S_b}{S_0}$ and $m_C = \frac{pC_h + (1-p)C_b}{q}$; so yields

$$(m_S - (1 + r))S_0 = pS_h + (1 - p)S_b - \pi S_h - (1 - \pi)S_b = (p - \pi)(S_h - S_b)$$

and similarly

$$(m_C - (1 + r))q = pC_h + (1 - p)C_b - \pi C_h + (1 - \pi)C_b = (p - \pi)(C_h - C_b).$$

Thus yields the equality :

$$\frac{m_C - (1 + r)}{m_S - (1 + r)} = \frac{S_0(C_h - C_b)}{q(S_h - S_b)},$$

i.e. exactly Ω which we know to be more than 1.

The interpretation is that the option return is larger than the underlying asset one ; the risk (variance, volatility) is larger. My comment is: if you take more risks, don't be surprised to loose money.... the probability of such an event is non null....

1.2 Two times, N assets, K states, several agents.

We have two times 0 and 1 where exchanges occur and goods are consumed. But at time 1, $\Omega = \{\omega_1, \cdots, \omega_K\}$, the nature states set, endowed with the probability measure $\mathbb{P}$. There exists only a perishable good, we can't stock it.

Otherwise now there exist N assets to be traded at time 0. If one of them is risk-less, we index them from 0 to N; the risk-less asset is not necessarily with the initial value 1.

Let us assume:

- no transaction cost.

- the nth action at time 1 is a random variable on $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$.

- there are I traders; everything is deterministic at time 0, and at time 1, they know the precise ω to be observed. They are characterized by
. endowments : $e^i(0), e^i(1, \omega), i = 1, ..., I$,
. consumption processes : $c^i = (c^i(0), c^i(1, \omega))$
(random variables as the prices, taking their values in $X = \mathbb{R} \times \mathbb{R}^K$, **not necessarily positive** contradicting the intuition!!)

- X is endowed with a preference $\prec$ i.e. a **total** binar relation, and satisfying:
$c \in X$ implies $\{c' \prec c\}$ are $\{c \prec c'\}$ are closed subset in X.
if all the coordinates of c' are $\geq$ these of c, then $c \prec c'$.
if $c \prec c'$ and $c \prec c''$, then $\forall \alpha \in [0, 1]$, $c \prec \alpha c' + (1 - \alpha)c$"

1.2.1 Budget set

Let $S_0 = (S_0^1, ..., S_0^N)$ be the N assets prices at time 0 and denote as $\langle x, y \rangle$ the scalar product in $\mathbb{R}^N$, D the prices matrix at time 1: $D = [d_n(\omega_k)\ n = 1, ...N; k = 1, ...K]$ and finally S is the price system (S_0, D).

If there exists a risk-less asset: $S_0 = (S_0^0, S_0^1, ..., S_0^N)$, we get $S_1^0(\omega_k) = S_0^0(1 + r), \forall k$.

Definition 1.9 *The $i-th$ agent's* **budget set** *is the set*

$$B(e^i, S) = \{c \in X/\exists \theta_1, ..., \theta_N : c(0) = e^i(0) - < \theta, S_0 >; c(1) = e^i(1) + D\theta\}$$

i.e. there exists a buying strategy which can finance the terminal consumption.

There are several agents, so we say "trading strategies", and we say that c is generated by (e^i, θ).
Exercise: Let the prices system for $K = 2, N = 4$

S_0	50	4	22	44
$d_.(\omega_1)$	100	40	60	120
$d_.(\omega_2)$	100	0	40	80

Does a risk-less asset exists ? if yes, what is its rate ?
The agent's endowment being $e(0) = 9, e(1, \omega_1) = 10, e(1, \omega_2) = 20$, *describe the budget set.*
In $X = \mathbb{R}^3$, $B(e, S)$ is the set of all $c \in \mathbb{R}^3$ such that there exists a portfolio $\theta \in \mathbb{R}^4$ which is solution to the following system:

$$\begin{aligned} c(0) - 9 &= -50\theta_1 - 4\theta_2 - 22\theta_3 - 44\theta_4 \\ c(1, \omega_1) - 10 &= 100\theta_1 + 40\theta_2 + 60\theta_3 + 120\theta_4 \\ c(1, \omega_2) - 20 &= 100\theta_1 + 0\theta_2 + 40\theta_3 + 80\theta_4 \end{aligned} \tag{2}$$

(3)

To find c in $\mathbb{R}^3$ is obtained after cancelling θ; after some computations we get:

$$c(0) + \frac{1}{10}c(1, \omega_1) + \frac{4}{10}c(1, \omega_2) = 18$$

i.e. the budget set $B(e, S)$ is a plane in $\mathbb{R}^3$.

1.2.2 Equilibrium

The agents'aim is obviously to optimize their preferences! But the trading occur only if there exists an "equilibrium". More precisely, we need that the markets are "clear",i.e. for any asset $n, n = 1, ..., N$:

$$\sum_{i=1}^{I} \theta_n^i = 0$$

in other words..... "rien ne se perd, rien ne se crée"...

Definition 1.10 *There exists an* **equilibrium** *(Arrow-Debreu : cf [1]) with respect to the prices system* S *if* (S, θ) *satisfies* $\forall e, \forall i$, *the consumption process* c^i *in* $B(e^i, S)$ *generated by* (e^i, θ^i) *is optimal for each agent and if the market is "clear".*

Such a set of consumption processes $\{c^i, i = 1, ...I\}$ associated with the equilibrium trading is called the **equilibrium allocation** w.r.t. (e, S).

1.2.3 Pareto efficiency

It is a weak normative criterion for the social "desiderability" of an allocation of consumption w.r. t. endowments e in price system S.

Definition 1.11 *An allocation* $\{c^i, i = 1, ...I\}$ *is* **feasible** *in this system if*

$$\sum_{i=1}^{I} c^i(t) = \sum_{i=1}^{I} e^i(t), t = 0, 1.$$

Comparing the definitions, remark that an equilibrium allocation is always feasible.

Definition 1.12 *Let the system* (e, S)*: the feasible allocation* $\{c^i, i = 1, ...I\}$ *is* **Pareto efficient** *if there is no other feasible allocation strictly better, i.e. there is no other feasible allocation* $\{b^i, i = 1, ...I\}$ *such that* $\forall i,\ c^i \prec b^i$ *and at least one of* b^i *differs from* c^i*.*

Definition 1.13 *A consumption process* c *is* **accessible** *(or simulable) in the price system* (S_0, D) *if there is an endowment process* e *such that:*

$$e(0) > 0,\ \ e(1) = 0\ ;\ \ c \in B(e, (S_0, D)).$$

Let us denote M the set of accessible/simulable consumption processes.
The following are equivalent:

$$c \in M \Leftrightarrow \exists e\ :\ c \in B(e, S), e(1) = 0 \Leftrightarrow \exists \theta, e\ : c(0) = e(0) - < \theta, S_0 >\ ;\ c(1) = D\theta,$$

i.e. there is a portfolio which allows to attain this consumption process.
We now prove:

Theorem 1.14 *If every consumption process is accessible, then every equilibrium allocation is Pareto efficient.*

Proof:
(i) Assume that the equilibrium allocation $(c^i, i = 1, \cdots, I)$ is not Pareto efficient , i.e. there exists another feasible allocation $(b^i, i = 1, \cdots, I)$ which is strictly better; so

$$(*)\ \sum_i c^i = \sum_i e^i = \sum_i b^i,\ c^i \prec b^i,\ \forall i = 1, \cdots, I \text{ and there exists } i_0 \text{ such that } c^{i_0} \neq b^{i_0}.$$

Let us denote $a^i = b^i - c^i$: the hypothesis yields that this allocation is accessible and there exists (θ, α) such that $\forall i = 1, \cdots, I$:

$$b^i(0) - c^i(0) = a^i(0) = \alpha^i - \langle \theta^i, S_0 \rangle\ ;\ a^i(1) = D\theta^i.$$

(*) proves that $\sum_i a^i(t) = 0,\ t = 0, 1$, i.e. $\sum_i \sum_n \theta^i_n d_n = \sum_i D\theta^i = 0,\ \forall \omega$, and
$\sum_i \alpha^i = \sum_i \langle \theta^i, S_0 \rangle = \sum_i \sum_n \theta^i_n S^n_0$.

(ii) Assume that this sum $\sum_i \alpha^i$ is strictly positive. Suppose that one agent in the equilibrium changes his strategy θ in the strategy : $\theta' = \theta - \sum_i \theta^i$ so this is an equilibrium and his consumption becomes

$$c'(0) = c(0) + \langle \sum_i \theta^i, S_0 \rangle > c(0) \ ; \ c'(1) = D\theta' = D\theta - \sum_i D\theta_i = c(1)$$

which is a strictly better consumption than c, and which belongs to the budget set thanks the portfolio θ'. This is impossible because of the optimality hypothesis of allocation c at equilibrium: by definition, c is optimal in $B(e, S)$ so $\sum_i \alpha^i \leq 0$.
Similarly, using the portfolio $\theta' = \theta + \sum_i \theta^i$ when the sum is strictly negative yields a contradiction thus actually $\sum_i \alpha^i = 0$.

(iii) Now assume that there exists i such that $\alpha^i < 0$: denoting η^i the optimal portfolio associated to c^i, we get

$$b^i(0) - \alpha^i = c^i(0) - \langle \theta^i, S_0 \rangle,$$

$$b^i(1) = a^i(1) + c^i(1) = D(\eta^i + \theta^i).$$

Then the couple $(b^i(0) - \alpha^i, b^i(1))$ is a consumption belonging to the budget set $B(e^i, S)$ via the strategy $\theta^i + \eta^i$; moreover

$$b^i(0) - \alpha^i > b^i(0) \geq c^i(0),$$

so this consumption in $B(e^i, S)$ is better than b^i so better than c^i, and this is a contradiction to the optimality of c^i in $B(e^i, S)$: thus $\alpha_i \geq 0$, $\forall i$ and since their sum is null all α_i are null.
But then the previous equations show that $b^i \in B(e^i, S)$; the hypothesis shows that b^i is strictly better than c^i, this fact contradicts that c^i is the optimal consumption (cf.(*)) in the budget set $B(e^i, S)$ at the equilibrium: so the allocation (b^i) doesn't exist and the allocation $(c^i, i = 1, \cdots, I)$ is Pareto efficient. □

Theorem 1.15 *Assume the matrix D rank is equal to K, then every consumption process is accessible and every equilibrium allocation is Pareto efficient.*

Proof: The consumption c is accessible if and only if $\forall \omega$ the system $c(1, \omega) = \langle D(\omega), \theta \rangle$ admits a solution, this is equivalent to the fact that the matrix D rank is equal to K. In such a case $e(0) = c(0) + \langle \theta, S_0 \rangle$. □

The conclusion is the following : we need enough independent assets to hedge the alea, the hazard....

1.2.4 Accessible set

It is a similar notion, analogue to the budget set, but without endowments.
Let $D : \mathbb{R}^N \longrightarrow \mathbb{R}^K$ such that $D(\theta) = D.\theta$.

Theorem 1.16 *The accessible consumption processes set M is $\mathbb{R} \times Im(D)$.*

Actually it is the set generated by $c(0), c(1) = D.\theta$, so we get
$c(0) \in \mathbb{R}$; $c(1) \in$ Im D.

Theorem 1.17 *For every endowment process e and every price system $S = (S_0, D)$:*

$c \in B(e, (S_0, D)) \Leftrightarrow c - e$ *is accessible with initial endowment* 0, *i.e.* $c - e \in B(0, (S_0, D))$.

Proof: We use a sequence of equivalence:

(4) $c \in B(e, S) \quad \Leftrightarrow \quad \exists \theta / c(0) = e(0) - < \theta, S_0 >; c(1) = e(1) + D\theta$ on one hand,
$c - e \in B(0, S) \quad \Leftrightarrow \quad \exists \theta / c(0) - e(0) = - < \theta, S_0 >; c(1) - e(1) = D\theta$ on the other hand

1.2.5 Arbitrage strategies

As in the previous section, the key is as following: a trader who uses an arbitrage strategy is sure to obtain a return without any initial investment.

Definition 1.18 *An* **arbitrage strategy** *θ is a trading strategy which allows to get a strictly positive consumption with a null initial wealth; more precisely:*

- *either* $< \theta, S_0 > \leq 0$; $D(\omega)\theta \geq 0$ *and* $\exists \omega / D(\omega)\theta > 0$,
- *or* $< \theta, S_0 > < 0$ *and* $\forall \omega D(\omega)\theta \geq 0$

Such a strategy allows to win something without any initial wealth.

Example : Let a system with three states and three assets. The initial price is $S_0 = (8, 10, 3)$; the matrix D is :

$$\begin{matrix} 6 & 11 & 3 \\ 5 & 11 & 3 \\ 12 & 9 & 3 \end{matrix}$$

We can prove that the portfolio $\theta = (1, 7/2, -87/6)$ is an arbitrage strategy:

$$< \theta, S_0 > = -\frac{1}{2}; < \theta, D(\omega_1) > = 1; < \theta, D(\omega_2) > = 0; < \theta, D(\omega_3) > = 0.$$

Usually we assume the **arbitrage free hypothesis** (denoted below as A.O.A.). We can prove the useful result (Farkas lemma):

Theorem 1.19 *The hypothesis A.O.A. is equivalent to the existence of* $\beta \in (\mathbb{R}_*^+)^K$*, called state price, such that* $S_0^i = \sum_{j=1}^K S_1^i(\omega_j)\beta_j, \forall i = 0, ...N.$

The key proof is the Minkowski separation theorem: if C_1 C_2 are non empty convex subset of $\mathbb{R}^k$, C_1 being closed and C_2 being compact, there exists $a \in \mathbb{R}^k$, non null, $b_1, b_2 \in \mathbb{R}$ such that $\langle a, x\rangle \leq b_1 < b_2 \leq \langle a, y\rangle, \forall x \in C_1, y \in C_2$.
Let us denote the simplex $\Delta_n = \{y \in (\mathbb{R}^+)^{n+1}, \sum_i y_i = 1\}$.

Proof: let $U = \{z \in \mathbb{R}^{K+1}, z_0 = -\langle S_0, x\rangle, ; (z_1, \cdots, z_K) = D.x, x \in \mathbb{R}^N\} = C_1$. Otherwise, $\Delta_K = C_2 \subset \mathbb{R}^{K+1}$.
(i) Assume AOA. Then $U \cap (\mathbb{R}^+)^{K+1} = \{0\}$ because if not, when $\exists z \neq 0$ belonging to this intersection, it could be $z_0 = -\langle S_0, x\rangle \geq 0$ i.e. $\langle S_0, x\rangle \leq 0$, and the other coordinates $D.x_j$ could be positive, and this is an arbitrage except if $z = 0$. Please note that $U \cap \Delta_K = \emptyset$, U is a closed convex subset, Δ_K is a convex compact subset. So we apply Minkowski theorem: $\exists \beta \in (\mathbb{R}^{K+1})_*, \langle \beta, z\rangle \leq b_1 < b_2 \leq \langle \beta, y\rangle, \forall z \in U, \forall y \in \Delta_K$.

Using $0 \in U$, then $b_1 \geq 0, b_2 > 0$ and $y = (0, ...0, 1, 0, ..., 0) \in \Delta_K, \beta_j > 0$. We can suppose $\beta_0 = 1$ and $(\beta_1, \cdots, \beta_K) \in (\mathbb{R}_*^+)^K$. Then $\forall z \in U, \langle \beta, z\rangle = -\langle S, x\rangle + \beta^t.D.x \leq b_1, \forall x \in \mathbb{R}^N$. But this is possible only if $S_0 = D^t.\beta : S_0^i = \sum_{j=1}^k S_1^i(\omega_j)\beta_j$.

(ii) Conversely, if there exists $\beta \in (\mathbb{R}_*^+)^K$ such that $S_0 = D^t\beta$, we compute the portfolio θ value:

$X_0 = \langle \theta, S_0\rangle, X_1 = D.\theta$. So $X_0 = \theta^t D^t \beta = \langle X_1, \beta\rangle$. Any portfolio can't to be an arbitrage strategy , since this last equation shows that one can't have $X_0 < 0$ and $X_1 \geq 0$ together. $\square$

The probabilistic interpretation is the following: $\forall i = 1, \cdots, N, S_0^i = \sum_j \beta_j S_1^i(\omega_j)$ and if there exists a risk-less asset (number 0) $j, d^0(\omega_j) = 1 + r$ then
$1 = \sum_j \beta_j(1 + r)$, and this defines a probability measure:

Π on $\Omega : \pi_j = \beta_j(1 + r), j = 1, \cdots, K$.

Please remark that $S_0^i = \sum_j \frac{\pi_j}{1+r} S_1^i(\omega_j) = \frac{1}{1+r} E_\pi[S_1^i]$. It yields
$(1+r)\langle \theta, S_0\rangle = (1+r)\sum_{i,j} \theta^i \beta_j S_1^i(\omega_j) = E_\pi[\langle \theta, S_1^i\rangle]$: the portfolio initial value, discounted at time 1, is equal to the portfolio mean value at time 1 under the probability Π.

Definition 1.20 Π *is called a* **risk neutral probability measure** *if it satisfies: under* Π*, it is equivalent to buy only the bond (risk-less) or to buy only risky assets.*

The asset i return under the state ω_j is $\frac{S_1^i(\omega_j)}{S_0^i}$ and its $\Pi-$mean is equal to
$\sum_j (1 + r)\beta_j \frac{S_1^i(\omega_j)}{S_0^i} = 1 + r$.

Proposition 1.21 *Assume AOA, then there exists a risk neutral probability measure under which the assets prices at time* 0 *are the mean of the assets discounted prices at time* 1*.*

Theorem 1.22 *When the situation is an equilibrium there exists no arbitrage strategy.*

Proof: $\forall i$, let $(\theta_1^i, ..., \theta_N^i)$ the strategy which finances the i-th agent equilibrium consumption process. We suppose that there exists an arbitrage strategy θ (first definition in Definition 1.18). Then, $\theta' = \theta^i + \theta$ allows the consumption $c'(1) = D\theta^i + D\theta$ and $c'(1) > c^i(1)$ for at least one ω whereas $c'(0) = c^i(0) - < \theta, S_0 > \geq c^i(0)$. But $c' \in B(e^i, S)$. This contradicts the fact that c^i could be the ith agent's optimal consumption in $B(e^i, S)$.

Please, do the proof for the second definition in Definition 1.18 as an *exercise*. □

1.2.6 Complete markets

Theorem 1.15 in Section 1.2.3 tells us that if the matrix D rank is equal to K, every equilibrium allocation is Pareto efficient.

Let us add an asset of price S_0^{N+1} at time 0 and S_1^{N+1} at time 1. In such a case we get an important consequence. Let us consider the equilibrium of the market with prices$((S_0^1, ..., S_0^{N+1}), D)$. Otherwise the matrix D_N, gathering the first N assets is of rank K. Then

$$D_N.\theta = S_1^{N+1} \text{ admits a solution } (\theta_1, ..., \theta_N)$$

Necessarily, $S_0^{N+1} = \sum_{n=1}^N \theta_n S_0^n$, because if $S_0^{N+1} > \sum_{n=1}^N \theta_n S_0^n$, then the strategy $\theta' = (\theta_1, ..., \theta_N, -1)$ could be an arbitrage strategy: indeed θ' initial value is $< \theta, S_0 > -S_0^{N+1} < 0$ and the terminal value $D\theta - S_1^{N+1} = 0$. But there exists no arbitrage when the situation is an equilibrium (cf 1.22).
If $S_0^{N+1} < \sum_{n=1}^N \theta_n S_0^n$, using symmetry, the strategy $\theta'' = (-\theta_1, ..., -\theta_N, 1)$ could be an arbitrage strategy.

In such a case , any more asset price is obtained as a linear combination of the previous assets prices.

Definition 1.23 *We say that a market is* **complete** *if every consumption process is accessible, i.e. if for every contingent claim $C \in \mathbb{R}^K$, there exists a portfolio θ which finances this claim:*

$$D.\theta = C, \ \sum_{i=1}^N \theta_i S_1^i(\omega_j) = C(\omega_j), j = 1, \cdots, K.$$

Proposition 1.24 *A market is complete if and only if the matrix D rank is equal to K, the cardinal of Ω .*

The proof is : the application $f : \theta \mapsto D.\theta$ is surjective onto $\mathbb{R}^K$.

Economical interpretation of the state price in a complete market :

. If the contingent claim C coordinates are null except the jth equal to 1, and if there exists a portfolio θ^j which finances C, compute the following scalar product with *beta* (cf. Theorem 1.19) : $\langle \beta, C \rangle = \beta_j = \beta^t D \theta^j = \langle S_o, \theta^j \rangle$: we get that β_j is the price to pay at time 0 to earn 1F at time 1 if the world state is then ω_j.
This is the reason of the word "state price".

Theorem 1.25 *On a complete market, every equilibrium allocation is Pareto efficient.*

The proof is the same as this of Theorem 1.14 in Section 1.2.3 □

1.2.7 Equilibrium measure, or risk neutral probability measure

Let us consider a complete market and the portfolio $\theta = (\theta_1, \cdots, \theta_N)$ such that $D\theta(\omega) = 1 \ \forall \omega$. Then the initial portfolio value is $\langle S_0, \theta \rangle = S_0^t D\theta = \sum_{j=1}^K S_0^j$. Without missing generality, we can suppose that the asset 1 is risk-less: with this asset we get $\sum_{j=1}^K S_0^j = \frac{1}{1+r}$, that is the amount obtained with this portfolio.

Let us assume that the coordinates of the state price are strictly positive and define $\pi_j = (1+r)\beta_j$: this is the risk neutral probability measure on Ω.
Hypothesis: the asset number 0 is risk-less with initial price 1, i.e. it is a "bond"

$$S_1^0(\omega) = (1+r), \forall \omega \in \Omega.$$

Definition 1.26 *Let S a price system. If the system $D^t.Q = (1+r)S_0$ admits a solution in $\mathbb{R}^K$ with all its coordinates > 0, Q is said to be an* **equilibrium price measure**.

The index bond is o: $S_1^o = (1+r)$, $(1+r) > 0$ and S_0 denotes the vector $(1, S_0^1, ..., S_0^N)$. So yields the system:

$$\sum_{k=1}^K S_1^j(\omega_k) Q(\omega_k) = (1+r) S_0^j, j = 0, ...N.$$

Thus, if $Q(\omega_k) > 0$, $\forall k$, a measure is defined on Ω and this measure is a probability measure : for $j = 0, S_1^0(\omega_k) = (1+r) \ \forall k$, so for $j = 0$ the system equation becomes:

$$\sum_{k=1}^K (1+r) Q(\omega_k) = (1+r),$$

i.e. $\sum_{k=1}^K Q(\omega_k) = 1$.

Theorem 1.27 *Q exists if and only if the price system S forbids any arbitrage.*

Proof: actually it is another proof of Theorem 1.19, to skip for a first lecture.

Corollary 1.28 *An equilibrium price measure exists if and only if the price system (S_0, D) admits an equilibrium with respect to a population of traders the preferences of who are convex increasing continuous.*

Recall :
. $\prec$ is continuous on $\mathbb{R}^{K+1}$ if $\{c' \prec c\}$ and $\{c \prec c'\}$ are closed.
. $\prec$ is increasing if we have the implication: every coordinates of $c' \leq$ coordinates of c implies $c' \prec c$.
. $\prec$ is convex if $c \prec c'$ and $\prec c''$ implies $\forall \alpha \in]0,1[, c \prec \alpha c' + (1-\alpha)c$"
Proof: Using Theorem 1.22, when there exists an equilibrium the situation is arbitrage free. Theorem 1.27 then says that an equilibrium price measure exists.

Conversely suppose that there exists an equilibrium price measure Q and let $(e_1, \cdots, e_I)$ an endowments set. We define a preference as following:

$$b \prec c : c(0) + \frac{1}{1+r} E_Q(c(1)) \geq b(0) + \frac{1}{1+r} E_Q(b(1)).$$

Exercise: this is a convex increasing continuous relation.

Then we verify that the situation is an equilibrium. Let for the agent i a consumption process $c^i \in B(e^i, S)$ financed by a strategy θ^i :

$$c^i(0) = e^i(0) - < \theta^i, S_0 >; c^i(1) = e^i(1) + D\theta^i.$$

One computes the preference criterium for this consumption process:

$$c^i(0) + \frac{E_Q(c^i(1))}{1+r} = e^i(0) + \frac{E_Q(e^i(1))}{1+r} + \frac{1}{1+r} \sum_{k,n} Q(\omega_k)\theta^i_n S^n_1(\omega_k) - \langle \theta^i, S_0 \rangle. \tag{5}$$

The fact that Q is an equilibrium price measure implies $\forall n$,
$\sum_k Q(\omega_k) S^n_1(\omega_k) = (1+r)S^n_0$, so for all trader i, yields:

$$c^i(0) + \frac{1}{1+r} E_Q(c^i(1)) = e^i(0) + \frac{1}{1+r} E_Q(e^i(1)).$$

But $e^i \in B(e^i, S)$ for the null strategy with which the market is clear! Thus e^i and c^i are equivalent, e^i is optimal in $B(e^i, S)$ and attained thanks to a strategy with which the market is clear: so we exhibited the equilibrium $(S_0, e, c = e)$. □

Theorem 1.29 *When there exists an equilibrium price measure, Q is unique if and only if the market is complete.*

Preuve : The hypothesis is that the system $D^t Q = (1+r)S_0$ admits a solution with all the coordinates > 0. Then, classically, the uniqueness of the solution is equivalent to the fact that the matrix D rank $= K$ and this is equivalent to the fact that the market is complete (cf. 1.2.6). $\square$

Exercise: let us the initial prices vector: $S_0 = (1,3,9)$ *and the following prices matrix*

$$D = \begin{matrix} 1 & 3 & 9 \\ 1 & 1 & 5 \\ 1 & 5 & 13 \end{matrix}$$

Show that the market is not complete, that there exists an equilibrium price measure, that all the accessible consumption processes verify $c_3 = 2c_1 - c_2$*. Finally, define the set of risk neutral probability measures.*

1.2.8 Arbitrage pricing in a complete market

Proposition 1.30 *Under the arbitrage free hypothesis and if the market is complete, to finance a contingent claim* $C \in \mathbb{R}^K$*, at time* 1 *we need to start with the initial wealth* $\frac{1}{1+r}\sum_{j=1}^K \pi_j C(\omega_j)$ *i.e. the average of discounted* C *under the risk neutral probability measure.*

Proof: assuming these two hypotheses (and using *Exercise* 1.2.8. Feuille 1), it is known that the application defined by the matrix D is surjective and that there exists a portfolio θ which allows to attain C and the initial value of this portfolio $\langle\theta, S_0\rangle$ doesn't depend on the chosen θ. (Such a portfolio is called a "hedging portfolio"). Finally we show that this initial value satisfies: $\langle\theta, S_0\rangle = \theta^t D^t \beta = \frac{1}{1+r} E_\Pi[C]$. $\square$

Such a process is called "pricing" since the initial value (price) of an asset is computed using the arbitrage free hypothesis.
Exercise: let a contingent claim equal to a "call" option value: $C = (S_1^i - K)^+$*. Compute this option price.*

1.2.9 Optimization problem

We now go to look for an optimal strategy which maximizes the utility of the consumption:

$$F : B(e,S) \to \mathbb{R}^+ \ ; \ c \mapsto U(c).$$

The function U is supposed to be positive, increasing w.r.t. each component, strictly concave.

We now define a particular budget set satisfying the constraint that the wealth has to stay positive:

$$B = \{c/X_0 = e(0) - c(0) - \langle \theta, S_0 \rangle \geq 0 \; ; X_T = e(1) - c(1) + D\theta \; \geq 0\}.$$

Proposition 1.31 *There exists an optimal consumption if and only if the market satisfies the free arbitrage hypothesis. In such a case, the optimal solution belongs to the set* $(\mathbb{R}_+^*)^{K+1}$.

Proof: we here deliver the proof only in the case of injective matrix D, $N+1 \leq K$ and D rank is $N+1$ (i.e. the number of assets on the market).
(i) Let us assume the existence of an optimal consumption c^* : there exists a portfolio θ such that the initial wealth $X_0 = e(0) - c^*(0) - \langle \theta, S_0 \rangle$ and the terminal wealth $X_1 = e(1) - c^*(1) + D\theta$. Let us suppose that there exists an arbitrage strategy: a portfolio θ_1, i.e. $\langle \theta_1, S_0 \rangle = 0$ and $D\theta_1 \geq 0$ and non null. Then using the portfolio $\theta + \theta_1$, the trader's consumption is always in the the budget set, the initial wealth X_0 is the same and the terminal wealth is more than X_1 so he can obtain a consumption $c_1(1) = c^*(1) + D\theta_1 \geq c^*(1)$ better than the optimal c^*: this is a contradiction and we conclude to arbitrage free.
(ii) Conversely, under the arbitrage free hypothesis, the hypotheses concerning U imply that an optimum exists in the budget set B as soon as B is bounded. Suppose B unbounded: there exists a sequence (c_n, θ_n) satisfying the constraint and $\|\theta_n\| \to \infty$.

Let θ^* be a " valeur d'adhérence" of the sequence $(\frac{\theta_n}{\|\theta_n\|})$; the constraint implies $\forall n$ the following relations:

$$\frac{c_n(0)}{\|\theta_n\|} + \langle \frac{\theta_n}{\|\theta_n\|}, S_0 \rangle \leq \frac{e(0)}{\|\theta_n\|} \; ; \frac{c_n(1)}{\|\theta_n\|} \leq \frac{e(1)}{\|\theta_n\|} + D\frac{\theta_n}{\|\theta_n\|}$$

finally let n goes to infinity:

$$\langle \theta^*, S_0 \rangle \leq 0 \; ; \; D\theta^* \geq 0$$

i.e. an arbitrage unless $D\theta^* = 0$. But this is not possible since by definition the norm of θ^* is 1 and by hypothesis D is injective: the budget set is bounded and c^* exists in $B(e, S)$.

(iii) Finally we have to show that all the coordinates of this optimal consumption are strictly positive. This is a convex optimization problem under linear constraint: at the optimum, the constraint are equalized, i.e.

$$c^*(0) + \langle \theta, S_0 \rangle = e(0) \; ; c^*(1) - D\theta = e(1). \tag{6}$$

This optimization problem is solved using the Lagrange multipliers method:

$$\mathcal{L}(c, \theta, \lambda) = U(c) - \lambda_0[c(0) + \langle \theta, S_0 \rangle - e(0)] - \sum_{j=1}^{K} \lambda_j [c_1(\omega_j) D_j \theta - e_1(\omega_j)].$$

So we get the following differential system:

$$
\begin{aligned}
\nabla_c \mathcal{L} = \nabla U - \lambda &= 0 \in \mathbb{R}^{K+1} \\
\nabla_\theta \mathcal{L} = \lambda_0 S_0 - \sum_{j=1}^{K} \lambda_j S_1(\omega_j) &= 0 \in \mathbb{R}^{N+1},
\end{aligned} \tag{7}
$$

plus the two last derivations which are the Lagrangian derivatives w.r.t. to λ, i.e. the constraint saturation (6). The utility function U is strictly increasing so all the optimal Lagrange multipliers satisfy $\lambda_j^* > 0$.
Let us define $\beta_j = \frac{\lambda_j}{\lambda_0} = \frac{\nabla_j U}{\nabla_0 U}$: then (7) yields

$$S_0^i = \sum_{j=1}^{K} \beta_j S_1^i(\omega_j), i = 0, \cdots, N$$

which is a pricing formula.

The saturation equations (6) prove that the optimal consumption c^* is strictly positive as following. There exists $\varepsilon \in]0,1[$ such that

$$c(0) = c^*(0) + \varepsilon\langle\theta, S_0\rangle > 0; c(1) = c^*(T) - \varepsilon D\theta > 0,$$

indeed, if one of c^* coordinates is strictly positive, there exists ε small enough so that the c correspondent component stays strictly positive; if one c^* component is null, using Equation (6), either (if $c*(0) = 0$) $\langle\theta, S_0\rangle = e(0) > 0$ or if there exists ω such that $c*(1) - D(\omega)\theta = e(1,\omega) > 0$ so ε exists.

This consumption c belongs to the budget set via the portfolio $(1-\varepsilon)\theta$. Otherwise, since the utility function U is concave:

$$U(c) - U(c^*) \geq \sum_j (c_j - c_j^*)\nabla_j U(c) = \varepsilon[\langle\theta, S_0\rangle\nabla_0 U(c) - \sum_j (D\theta)_j \nabla_j U(c)]$$

which, when ε goes to zero, goes to $\langle\theta, S_0\rangle\nabla_0 U(c^*) - \sum_j (D\theta)_j \nabla_j U(c^*)$ which is null (7); so c could be better than c^* and this contradicts the optimality of c^*: $c = c^*$ which so is strictly positive. □

In the case of a complete market, we get more results since then the vector λ^* is unique and we get the relation :

$$c^*(0) + \langle\beta, c^*(T)\rangle = e(0) + \langle\beta, e(T)\rangle.$$

2 A discrete multi-period financial model

We extend the model of the previous chapter to T periods: $(0, \cdots, T)$ instead of $(0, 1)$. The modeling principle is the same, but now there are random processes on $\Omega \times (0, \cdots, T)$ instead of random variables for the consumption processes, the endowments and the prices. The strategies are depending on the time and the concerned asset.

2.1 The model

As for any control models, the economical agents take in account the available information on the prices and so the strategies (i.e. the controls) have to be adapted processes with respect to the observations. Thus the model is defined with:

- time $(0, \cdots, T)$,

- alea (usually underlying the prices): $\Omega = \{\omega_1, \cdots, \omega_K\}$, endowed with the "natural" probability $\mathbb{P}$,

- the daily informations (observable filtration): $\mathcal{F}$ generated by the observed prices and let us suppose that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{P}(\Omega)$: finally, everything is known! This sequence of increasing knowledge can be represented by an arborescence. This is well modeled with the **filtration**, i.e. the increasing sequence of sigma-algebras on Ω which model the daily information:

$$\begin{aligned}\mathcal{F}_0 &= \{\Omega, \emptyset\},\\ \mathcal{F}_T &= \mathcal{P}(\Omega),\\ \mathcal{F}_t &= \sigma(\{A_j(t), j \in \Omega_t\}\end{aligned}$$

where $A_j(t)$ are the atoms of $\mathcal{F}_t$, union of atoms $A_l(t+1)$ in $\mathcal{F}_{t+1}$.

- only one, perishable, consumption good

- finite number of endogeneous securities): one bond and N risky assets the prices of which are random variables

$$S^n(t); n = 1, \cdots, N; t = 1, \cdots; T, S^n(t) > 0 \;;$$

let be S_0 the initial prices vector and D the process of prices matrix which is a $\mathcal{F}$-adapted process (i.e. $\forall t$, the random vector D_t, sometimes denoted as S_t, is $\mathcal{F}_t$-measurable):

$$S_0 = (S^1(0), \cdots, S^n(0)) \;;\; \forall t = 1, \cdots, T,\; D_t = S_t = [S^n(\omega_k, t)\; n = 1, ...N; k = 1, ...K].$$

Let us denote $S = (S_0, D)$ the prices system at the $T + 1$ times.

- the I agents (traders) who receive endowments (for instance salaries) $e_i(t)$ and consume $c_i(t)$, e, c real adapted processes ($\forall t, e_t, c_t$ are $\mathcal{F}_t$-measurable).

- no transaction cost.

- let us denote X the consumption set: $\mathbb{R} \times \{$adapted processes $\}$, endowed with a preference $\prec$ complete, continuous, increasing and convex, i.e.:
. $(X, \prec)$ is totally ordered
. if $c \in X$, $\{c' \in X \ : \ c' \prec c\}$ and $\{c' \in X \ : \ c \prec c'\}$ are closed sets.
. if any c coordinates are more than these of c', then $c \prec c'$.
. if $c \prec c'$ and $c \prec c''$ then $\forall \alpha \in [0,1]$, $c \prec \alpha c' + (1-\alpha)c$".

2.2 Strategies and budget set

Moreover we suppose that there exists a risk-less asset, number 0, with initial price 1 and deterministic price S_t^0, for instance $(1+r)^t$.
The strategies (cf. Lamberton and Lapeyre p.13) are processes taking their values in $\mathbb{R}^{N+1}$:

$$\theta = (\theta_t^0, ..., \theta_t^N) \ ; \ t = 1, ..., T.$$

The process θ has to be **predictable**, i.e. θ_t is $\mathcal{F}_{t-1}$ measurable: the previous information is used to change the portfolio, before knowing the next prices.

Using the strategy θ_t at time t, the agent's wealth

$$V_t(\theta) = D_t \theta_t \text{ denoted as } \sum_{i=0}^{N} \theta_t^i . S_t^i \text{ or } \langle \theta_t, S_t \rangle$$

is a random variable $\mathcal{F}_t$-measurable. Let us denote $\tilde{V}_t(\theta) = (S_t^0)^{-1} V_t(\theta)$ the portfolio discounted value, $\tilde{S}_t = (S_t^0)^{-1} S_t$ the discounted prices, the bond is used as a reference price. So we get $\tilde{S}_t^0 = 1$ at any time t. Let us denote $R_t = (S_t^0)^{-1}$ the discount coefficient on the market.

Definition 2.1 *θ is a self-financing strategy if for all* $t = 1, ..., T-1$ *:*

$$\langle \theta_t, S_t \rangle = \langle \theta_{t+1}, S_t \rangle$$

(cf. Dothan page 69 and Lamberton and Lapeyre p.14: no transaction costs).

Interpretation : the new portfolio is only done thanks to an internal redistribution between the assets; the consumption is only financed by the endowments.

Remark 2.2 *θ is a* **self-financing strategy** *if and only if*

$$V_{t+1}(\theta) - V_t(\theta) = \langle \theta_{t+1}, S_{t+1} - S_t \rangle,$$

i.e. the portfolio value variation is only a consequence of the prices variation.

Proof: *exercise.*

Definition 2.3 *Let us define a* **budget set** *with respect to an endowment process* e_i *and a prices* S *system:*

$$B(e^i, S) = \{c \in X / \exists \theta \text{ predictable } : c(t) = e^i(t) + \langle \theta_t - \theta_{t+1}, S_t \rangle ; \forall t \in \{0, ..., T\} \ ; \ \theta_o = \ \theta_{T+1} = 0\}$$

This means that there exists a buying strategy which allows to finance the terminal consumption. Let us remark that this definition doesn't include constraint on the sign of the consumption process.
Moreover, if θ is self-financing,

$$B(e^i, S) = \{c : c(0) = e^i(0) - \langle \theta_1, S_0 \rangle, c(t) = e^i(t), t = 1, \cdots, T-1, c(T) = e^i(T) + \langle \theta_T^i, S_T \rangle\}.$$

In this situation with several agents, we call them trading strategies, c is generated (or **simulated**, or **attainable**) by (e^i, θ).

Proposition 2.4 *The following are equivalent:*

(i) θ *is self-financing,*

(ii) $\forall t \in \{1, ..., T\}, V_t(\theta) = V_0(\theta) + \sum_{s=1}^t \langle \theta_s, S_s - S_{s-1} \rangle$,

(iii) $\forall t \in \{1, ..., T\}, \tilde{V}_t(\theta) = V_0(\theta) + \sum_{s=1}^t \langle \theta_s, \tilde{S}_s - \tilde{S}_{s-1} \rangle$.

Proof: we show the first equivalence using the remark above:

$$V_{s+1}(\theta) - V_s(\theta) = \langle \theta_{s+1}, S_{s+1} - S_s \rangle$$

and we compute the sum.
To show (i) $\iff$ (iii), θ is self-financing $\iff \langle \theta_t, \tilde{S}_t \rangle = \langle \theta_{t+1}, \tilde{S}_t \rangle \iff$
$\tilde{V}_{s+1}(\theta) - \tilde{V}_s(\theta) = \langle \theta_{s+1}, \tilde{S}_{s+1} - \tilde{S}_s \rangle \iff$ (iii). □

Proposition 2.5 *(cf. [3] p.15) For any predictable process* θ *taking its values in* $\mathbb{R}^N$ *and and for any* $V_0 \in \mathbb{R}$*, there exists a unique real predictable process* θ^0 *such that the process taking its values in* $\mathbb{R}^{N+1}$ $\bar{\theta} = (\theta^0, \theta)$ *is a self-financing strategy with initial value* V_0*.*

Proof(*do it as an exercise*) : the identity

$$\tilde{V}_t(\bar{\theta}) = \theta_t^o + \sum_{n=1}^N \theta_t^n . \tilde{S}_t^n = V_o + \sum_{s=1}^t \langle \bar{\theta}_s, (\tilde{S}_s - \tilde{S}_{s-1}) \rangle$$

and the fact that $\forall s$, $\tilde{p}_s^o = 1$, after some cancellations, yields:

$$(8) \qquad \theta_t^o = V_0 + \sum_{s=1}^{t-1} \sum_{n=1}^N \theta_s^n . (\tilde{S}_s^n - \tilde{S}_{s-1}^n) - \sum_{n=1}^N \theta_t^n . \tilde{S}_{t-1}^n, t > 1 \text{ and } \theta_1^0 = V_0 - \sum_{n=1}^N \theta_1^n . \tilde{S}_0^n,$$

and this is a predictable expression.

2.3 Equilibrium, Pareto efficiency

The agents' aim is to optimize their preferences. But the trading can only be done in a context of an "equilibrium". To be more precise, the market needs to be "clear", i.e. for any asset $n, n = 0, ..., N$:

$$\sum_{i=1}^{I} \theta_n^i = 0$$

meaning *rien ne se perd, rien ne se crée...* and in the same time anybody can optimize his strategy in his budget set.

Definition 2.6 *Let be an endowment set $\{e_i,\ i = 1, \cdots, I\}$, a prices system S and a set of strategies $\{\theta_i,\ i = 1, \cdots, I\}$ with which we can attain the consumption processes $\{c_i,\ i = 1, \cdots, I\}$; we say that the situation is an* **equilibrium** *if*
(i) $\forall i$, the consumption process c^i is optimal in $B(e^i, S)$ endowed with the preference $\prec$ for the agent i,
(ii) the market is "clear" at any time t and for all ω, i.e. for any asset $n, n = 0, ..., N$:

$$\sum_{i=1}^{I} \theta_n^i(t, \omega) = 0, \forall n, \forall \omega, \forall t.$$

Such a consumption processes set $\{c^i, i = 1, ...I\}$ is called the **equilibrium allocation** with respect to (e, S). (Recall: $S = (S_0, D)$).

Definition 2.7 *An allocation $\{c^i, i = 1, ...I\}$ is* **feasible** *if*

$$\forall i, c^i \in X \text{ and } \sum_{i=1}^{I} c^i(t) = \sum_{i=1}^{I} e^i(t),\ \forall t = 0, \cdots, T.$$

Definition 2.8 *A feasible allocation $\{c^i, i = 1, ...I\}$ is* **Pareto efficient** *if there doesn't exist another feasible allocation $\{b^i, i = 1, ...I\}$ which could be better (w.r.t. the preference relation) for each agent.*

Definition 2.9 *A consumption process is* **accessible** *in the prices system S if there exists an endowment process e such that:*

$$e(0) > 0\ ;\ e(t) = 0, t = 1, ..., T; c \in B(e, S).$$

Let be S fixed, and denote $M(S)$ the feasible consumption processes set. Then yields the equivalence:

$$c \in M(S) \Leftrightarrow \exists \theta \text{ which finances } c.$$

Then we get the following theorem.

Theorem 2.10 *(cf. [2] p.57) If any consumption process is accessible, then any equilibrium allocation is Pareto efficient.*

Exercise: The proof is the same as this in two periods case, cf. theorem 1.14), do it as an exercise.

2.4 Arbitrage and admissible strategies

This section comes from [3] page 15 and [2] 3.7, pages 69-71.

Definition 2.11 *A strategy θ is said to be* **admissible** *if it is self-financing and if the portfolio value $V_t(\theta) \geq 0\ \forall t$; meaning that the investor can at any time repay his borrowing, for instance selling his portfolio.*

Another version is to allow a given deficit: $V_t(\theta) \geq a$ where a is a non positive real number. This could be considered as a "limit position".

As in the previous chapter, the arbitrage is the possibility for an economical agent to get a return **without** any initial investment

Definition 2.12 *An* **arbitrage strategy** *is an admissible trading strategy which allows an agent with a null initial wealth to obtain a strictly positive consumption; more precisely:*

- $\langle \theta, S_0 \rangle(0) = 0$ *;* $D_T\theta(\omega) \geq 0$ *and* $\exists \omega / D_T\theta(\omega) > 0$.
With the admissibility extended definition ($V_t(\theta) \geq a$ where a is non positive real number), yields

- either $\langle \theta, S_0 \rangle(0) \leq 0$ *;* $D_T\theta(\omega) \geq 0$ *and* $\exists \omega / D_T\theta(\omega) > 0$.

- or $\langle \theta, S_0 \rangle(0) < 0$ *and* $\forall \omega,\ D\theta(\omega) \geq 0$

It is easy to prove that if there exists a risk-less asset, the existence of such strategies is equivalent to the existence of strategies satisfying:

- $\langle \theta, S_0 \rangle(0) = 0$; $D_T\theta(\omega) \geq 0$ and $\exists \omega / D_T\theta(\omega) > 0$.

The absence of arbitrage is characterized by martingales properties (cf. J. Neveu [4]), so we need some "recalls" . Let a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F}_t)$ and an adapted process M: M is a **martingale** if $\forall t, M_t \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ and if the conditional expectation satisfies $E_{\mathbb{P}}[M_{t+1}/\mathcal{F}_t] = M_t$. The main properties to be known are (cf. for instance [4]):

- M is a martingale $\Leftrightarrow \forall s \leq t\ E_{\mathbb{P}}[M_t/\mathcal{F}_s] = M_s$.

- $E_{\mathbb{P}}[M_t] = E_{\mathbb{P}}[M_0]$.

- the sum of two martingales is a martingale.

Definition 2.13 *A market is said to be* **viable** *if there doesn't exist any arbitrage strategy.*

Theorem 2.14 *A market is viable if and only if one of the two equivalent following conditions:*

(i) $\exists Q$ *probability equivalent to* $\mathbb{P}$ *such that the discounted prices are* $Q-$*martingales,*

(ii) $\forall t = 1, \cdots, T$ *for all atom* A_j^{t-1} *in the sigma-algebra* $\mathcal{F}_{t-1}$*, the linear system*

$$\sum_{A_k^t \in A_j^{t-1}} D_t^i(A_k^t) Q_t(A_k^t) = (1+r) D_{t-1}^i(A_j^{t-1}), i = 1, \cdots, N,$$

where A_k^t *are the atoms of* $\mathcal{F}_t$ *the union of which is* A_j^{t-1}*, admits a solution with strictly positive components.*

Definition 2.15 *In such a case* Q *is said to be a* **equilibrium price measure** *(or : risk neutral probability measure).*

Actually the systems above have to be understood as following: the number of systems is equal to the number of atoms of sigma-algebra $\mathcal{F}_{t-1}$, denoted as A_j^{t-1} :

$$\text{(9)} \qquad \forall A_j^{t-1} \text{ atom of } \mathcal{F}_{t-1}, \sum_i {}^t D(t, B_i) Q_t(B_i) = (1+r) S_{t-1}(\omega), \forall \omega \in A_j^{t-1},$$

where B_i are the atoms of $\mathcal{F}_t$ included in A_j^{t-1}, and this is exactly the equation which defines the equilibrium price measures in the two periods case, between $t-1$ and t. So the assertion (ii) is equivalent to no arbitrage between $t-1$ and t.

Proof: be cautious, this proof is not a definitive one, and a first lecture can skip it.

(i) $\Rightarrow$ **the market is viable** : let θ an admissible (thus self-financing) portfolio; then (cf. Proposition 2.4) the discounted value of the portfolio satisfies:

$$\tilde{V}_t(\theta) = V_o(\theta) + \sum_{s=1}^{t} \langle \theta_s, \tilde{S}_s - \tilde{S}_{s-1} \rangle.$$

This is a finite sum of integrable random variables; thus $\tilde{V}_t(\theta)$ is integrable ; moreover θ_t is $\mathcal{F}_{t-1}$-measurable; we then compute the conditional expectation:

$$\begin{aligned} E_Q[\tilde{V}_t(\theta)/\mathcal{F}_{t-1}] &= \tilde{V}_{t-1}(\theta) + E_Q[\langle \theta_t, \tilde{S}_t - \tilde{S}_{t-1} \rangle / \mathcal{F}_{t-1}] \\ &= \tilde{V}_{t-1}(\theta) + \langle \theta_t, E_Q[\tilde{S}_t - \tilde{S}_{t-1}/\mathcal{F}_{t-1}] \rangle = \tilde{V}_{t-1}(\theta) \end{aligned}$$

since the hypothesis yields that $\tilde{S}$ is $Q-$martingale. So $\tilde{V}_t(\theta)$ is $Q-$martingale and $E_Q[\tilde{V}_t(\theta)] = V_o(\theta)$. If θ would be an arbitrage strategy ($\langle \theta_o, S_o \rangle = V_o(\theta) = 0$ and $V_T(\theta) \geq 0$), it would be $E_Q[\tilde{V}_T(\theta)] = 0$, i.e. $\tilde{V}_T(\theta) = \langle \theta_T, S_t \rangle = 0$ i.e. arbitrage is impossible and the market is viable.

To prove the reciprocal is difficult directly (cf. pages 18-19 in Lamberton and Lapeyre [3]) but the proof is easier using (ii) and denoting that this reciprocal is equivalent to no arbitrage between $t-1$ and t.

the market is viable yields (ii).

Let us suppose that (ii) fails: there exists t and an admissible arbitrage strategy θ between times t and $t+1$, i.e.

$$\langle \theta_{t+1}, S_t \rangle = \langle \theta_t, S_t \rangle = 0 \ ; \ \langle \theta_{t+1}, S_{t+1} \rangle \ > 0.$$

We complete this portfolio to get a self-financing arbitrage strategy admissible between 0 and T. Let us propose:

$$\begin{aligned}
&\theta_1 = \cdots = \theta_{t-2} = 0\\
&\theta_{t-1} \text{ such that } \langle \theta_{t-1}, S_{t-1}\rangle = \langle \theta_t, S_{t-1}\rangle \text{ and } \langle \theta_{t-1}, S_{t-2}\rangle = 0\\
&\theta_j^o = \frac{\langle \theta_{t+1}, S_{t+1}\rangle}{S_{t+1}^o} \forall j \geq t+2\\
&\theta_j^n = 0 \ \forall n = 0, \cdots, N, \ \forall j \geq t+2
\end{aligned}$$

This strategy is admissible:
it is clearly predictable; it is moreover self-financing :

- $\forall j \geq t+2, \langle \theta_j, S_j\rangle = \langle \theta_{j+1}, S_j\rangle$ since θ_j is constant,
- $\langle \theta_{t+2}, S_{t+1}\rangle = \theta_{t+2}^o.S_{t+1}^o = \langle \theta_{t+1}, S_{t+1}\rangle$ by construction,
- $\langle \theta_{t+1}, S_t\rangle = \langle \theta_t, S_t\rangle$ by hypothesis,
- $\langle \theta_t, S_{t-1}\rangle = \langle \theta_{t-1}, S_{t-1}\rangle$ by construction,
- finally $\langle \theta_j, S_{j-1}\rangle = \langle \theta_{j-1}, S_{j-1}\rangle = 0, \forall j \geq t-1$ by construction.

In particular this last point shows that $\langle \theta_1, S_0\rangle = 0$, and otherwise we can easily verify that $\langle \theta_T, S_t\rangle = \langle \theta_{t+1}, S_{t+1}\rangle \frac{S_T^o}{S_{t+1}^o} > 0$. This says that θ is an arbitrage strategy and this contradicts the hypothesis.

(ii) implies (i)

The hypothesis allows to built a probability Q equivalent to $\mathbb{P}$ under which the discounted prices are Q-martingales. Indeed following (cf. Dothan [2], page 75 and sq.) we set:

$$Q(\omega) = \prod_{t=1}^{T} Q_t(f_t(\omega)),$$

where $f_t(\omega)$ is the atom in $\mathcal{F}_t$ such that $\omega \in f_t(\omega)$ and Q_t is solution of system (ii):

$${}^tD(t,\omega)Q_t = (1+r)S_{t-1}(\omega),$$

thus Q is actually a probability measure and moreover the discounted prices are $Q-$martingales.
More precisely, the hypothesis is the following:
let t fixed between 1 and T and let the atoms $A_1, \cdots, A_{n_{t-1}}$ in $\mathcal{F}_{t-1}$; $\forall j = 1, \cdots, n_{t-1}, A_j = \cup B_j^k$ where B_j^k are atoms in $\mathcal{F}_t$; the random variable D_t is constant on any B_j^k and the random variable S_{t-1} is constant on A_j and the system $\sum_k D_t(B_j^k)Q_t(B_j^k) = (1+r)S_{t-1}(A_j)$ admits a solution $(Q_t(B_j^k) > 0, j = 1, \cdots, n_{t-1})$.

Now let $\omega \in \Omega, \mathcal{F}_T = \mathcal{P}(\Omega)$: this singleton is included in a sequence of atoms for the different filtrations as following: $\{\omega\} \subset f_{T-1}(\omega) \subset \cdots \subset f_1(\omega)$. The definition is:

$$Q\{\omega\} = Q_T\{\omega\}Q_{T-1}(f_{T-1}(\omega)) \cdots Q_1(f_1(\omega)).$$

Property: let the system which gives Q_t and write its 0-th coordinate: $D_t^0(B_j^k) = (1+r)^t$ and $S_{t-1}^0 = (1+r)^t$, thus

$$\forall j = 1, \cdots, n_{t-1}, \ \sum_k Q_t(B_j^k) = 1.$$

a) Q is probability measure: each $Q\{\omega\}$ is positive and we sum them recursively:

$$\sum_\omega Q\{\omega\} = \sum_{\text{atoms of } \mathcal{F}_{T-1}} (\sum_{\omega \in f_{T-1}(\omega)} Q_T(\omega) Q_{T-1}(f_{T-1}(\omega)) \cdots Q_1(f_1(\omega)).$$

But remark that the atoms $f_i(\omega)$ are fixed as soon as $f_{T-1}(\omega)$ is fixed and that the product $Q_{T-1}(f_{T-1}(\omega)) \cdots Q_1(f_1(\omega))$ is one factor in the sum $\sum_{\omega \in f_{T-1}(\omega)} Q_T(\omega)$ which is equal to 1. Then yields:

$$\sum_\omega Q\{\omega\} = \sum_{\text{atoms of } \mathcal{F}_{T-1}} Q_{T-1}(A_{T-1}^j) \cdots Q_1(f_1(A_{T-1}^j)).$$

It is then easy to get recursively $\forall t$:

$$\sum_\omega Q\{\omega\} = \sum_{\text{atoms of } \mathcal{F}_t} Q_t(A_t^j) Q_{t-1}(f_{t-1}(A_t^j)) \cdots Q_1(f_1(A_t^j)),$$

because when j is fixed, the atoms of the sigma-algebrae with an index less than t $f_{t-1}(A_t^j), \cdots, f_1(A_t^j)$ are constant and we get the result with $t = 1$ since at the beginning $\cup_j A_1^j = \Omega$.

b) The discounted prices are martingales, i.e. we have to proof:

$$E_Q[\tilde{S}_{t+1}^n / \mathcal{F}_t] = \tilde{S}_t^n.$$

Let A be an atom of $\mathcal{F}_t$ and A_j atoms of $\mathcal{F}_{t+1}$ such that $A = \cup_j A_j$:

$$\int_A \tilde{S}_{t+1}^n dQ = \sum_j \int_{A_j} \tilde{S}_{t+1}^n dQ = \sum_j \tilde{S}_{t+1}^n(A_j) Q(A_j)$$

since by definition S_{t+1}^n is $\mathcal{F}_{t+1}$-measurable, so it is constant on the $\mathcal{F}_{t+1}$-atoms. The computation a) of $Q(\Omega)$ is done similarly to this of $Q(A_j)$ so we get recursively:

$$Q(A_j) = Q_{t+1}(A_j) Q_t(A) Q_{t-1}(f_{t-1}(A)) \cdots Q_1(f_1(A)),$$

and:

$$Q(A) = Q_t(A) Q_{t-1}(f_{t-1}(A)) \cdots Q_1(f_1(A)),$$

a substitution get:

$$\int_A \tilde{S}_{t+1}^n dQ = \sum_j \tilde{S}_{t+1}^n(A_j) Q_{t+1}(A_j) Q_t(A) \cdots Q_1(f_1(A)).$$

Using Q_t definition and the starting hypothesis, $\sum_j \tilde{S}_{t+1}^n(A_j) Q_{t+1}(A_j) = \tilde{S}_t^n \mathbf{1}_A$, thus we can conclude. $\square$

2.5 Complete markets

(cf. Lamberton and Lapeyre [3], pages 19-21 ; Dothan [2], pages 57 and sq.)
This notion is defined with respect to the prices system, as is done the notion of "simulability".

Definition 2.16 *A market with prices system S is said to be* **complete** *if any consumption process is accessible with a self-financing strategy, meaning that*

$$\exists e_0 > 0,\ \exists \theta \text{ self-financing } : c_0 = E_0 - \langle \theta_1, S_0 \rangle, c_T = \langle \theta_T, S_T \rangle.$$

Theorem 2.17 *When a market is complete, any equilibrium allocation is Pareto efficient.*

This is only Theorem 2.10 corollary. □

Remark 2.18 *When a market is viable, under a equilibrium prices measure Q, the discounted prices, so the discounted self-financing portfolio value, are $Q-$martingales. Thus, $\tilde{V}_t(\theta) = E_Q[\tilde{V}_T(\theta)/\mathcal{F}_t]$ is only defined via its terminal value.*

Theorem 2.19 *A viable market is complete if and only if there exists a unique equilibrium prices measure.*

Proof: Thanks to Theorem 2.14, there exists a risk neutral probability measure Q.
(i) If the market is complete, any random variable X $\mathcal{F}_T-$measurable and integrable is a "terminal" consumption and there exists a self-financing portfolio θ such that $X = V_T(\theta)$. If moreover the market is viable, let us suppose that there exist two equilibrium prices measures Q_1, Q_2 so:

$$\tilde{V}_t(\theta) = \langle \theta_t, \tilde{S}_t \rangle = E_{Q_i}[\tilde{V}_T(\theta)/\mathcal{F}_t],\ i = 1, 2$$

and for any event A of $\mathcal{F}_T$ and for $t = 0$ and $X = \mathbf{1}_A \times S_T^0$ yields:

$$V_0(\theta) = Q_1(A) = Q_2(A),$$

meaning the uniqueness of the equilibrium prices measure.

(ii) Reciprocally, since the market is viable, (let us denote Q a risk neutral probability measure) but no complete, there exists a random variable X $\mathcal{F}_T-$measurable and integrable and positive non null and no accessible. Let us then define the set $\mathcal{V}$ of accessible wealth:

$$\mathcal{V} = \{u_o + \sum_{t=1}^{T} \langle \theta_t, \tilde{S}_t - \tilde{S}_{t-1} \rangle; u_o \in \mathbb{R};\ \theta \text{ predictable self-financing}\}.$$

Since X is not accessible, $\frac{X}{S^o_T}$ doesn't belong to $\mathcal{V}$ following the characterization (iii) of the self-financing strategies. Let Q be an equilibrium prices measure; $\mathcal{V}$ is a strict closed vector subspace of $L^2(Q)$ and so there exists $Y \in L^2(Q)$, orthogonal to $\mathcal{V}$. Thus $E_Q[Yh] = 0, \forall h \in \mathcal{V}$ and $E_Q[Y] = 0$ since all the constants belong to $\mathcal{V}$ (think of $\theta = 0$). Let us then define the measure:

$$Q' = (1 + \frac{Y}{2 \parallel Y \parallel_\infty})Q$$

which is equivalent to Q, so to $\mathbb{P}$, and which is a probability measure since $E_Q[Y] = 0$.

Then let $\forall A \in \mathcal{F}_t$ the strategy θ defined by

$$\theta^k_s = 1_A \delta_{n,k}, s = t+1 \; ; \; \theta_s = 0 \; \forall s \neq t+1, \; k = 0, \cdots, N$$

and $u_o = 0$: (u_o, θ) allows to finance the value $1_A(\tilde{S}^n_{t+1} - \tilde{S}^n_t)$ which so belongs to $\mathcal{V}$. Yields:

$$E_Q[Y 1_A(\tilde{S}^n_{t+1} - \tilde{S}^n_t)] = 0,$$

and:

$$E_{Q'}[1_A(\tilde{S}^n_{t+1} - \tilde{S}^n_t)] = E_Q[(1 + \frac{Y}{2 \parallel Y \parallel_\infty})1_A(\tilde{S}^n_{t+1} - \tilde{S}^n_t)] = E_Q[1_A(\tilde{S}^n_{t+1} - \tilde{S}^n_t)]$$

which is null, since the discounted prices are $Q-$martingales and also $Q'-$martingales, which contradicts the uniqueness hypothesis: $Q = Q'$ which implies $Y = 0$ and so $\mathcal{V} = L^2(Q)$. $\square$

2.6 Valuation and hedging

On a viable and complete market, let Q be the unique equilibrium price measure. At time T an aim is fixed, for instance the random variable h $\mathcal{F}_T$-measurable and integrable, and let θ a self-financing strategy which allows to finance h: $V_T(\theta) = h$. Let us recall that for all t, $\tilde{V}_t(\theta) = E_Q[\frac{h}{S^o_T}/\mathcal{F}_t]$ or $V_t(\theta) = S^o_t E_Q[\frac{h}{S^o_T}/\mathcal{F}_t]$, i.e. the portfolio θ value at time t, the initial wealth being $V_o(\theta) = E_Q[\frac{h}{S^o_T}]$. Starting with this value, and using the strategy θ, the existence of which is in force thanks to the completeness hypothesis, we are sure to obtain h at time T: the "hedging" is sure: θ "hedges" h, $V_0(\theta)$ is the "fair price".

This technique allows us to know what initial prices have to be fixed on the market for financial assets, defined as function of several actions. The most known are "options":

- a call option with terminal value $(S_T - K)^+$,

- a put option with terminal value $(K - S_T)^+$.

These options are called "European", there exist other options called "American options": i.e. we buy the right to exercise the sell or buy option before the maturity time T, at a random time and the aim is to optimize this random time in the stopping times set (stopping times are integer random variables with special properties):

$$h = \sup\{E[(S_\tau - K)^+] \; ; \tau \text{ stopping times } \leq T\}.$$

This is a special optimal control problem: optimal stopping time.

In the following section, we look for the strategy θ to be moreover admissible.

2.7 Optimization in a viable and complete market

Let $x = X_0$ be the agent's initial wealth. He looks for a strategy θ which has to be optimal with respect to a utility function U of the portfolio terminal value. As previously, U is a strictly concave increasing function on $\mathbb{R}^+$. Firstly we characterize the admissible strategies, i.e. θ has to be self-financing, at any time $V_t(\theta) \geq 0$, and $(V_t(\theta), t = 0, \cdots, T)$ is a Q- martingale, Q being the risk neutral probability measure.

Proposition 2.20 *Let be $V \in L^1_+(\Omega, Q)$ and $R_T = (S^0_T)^{-1}$. Then V is the terminal value of an admissible strategy with initial value x if and only if $E_Q[R_T V] = x$.*

Proof:

Let θ be the portfolio which allows to finance the aim V : $R_T V = R_T V_T(\theta)$. The martingale property shows that $E_Q[R_T V] = E_Q[\tilde{V}_T(\theta)] = V_0(\theta)$, i.e. x.

Reciprocally, since the market is complete, there exists a self-financing strategy θ such that $V = V_T(\theta)$ and in such a case $R_T V = V_0(\theta) + \sum_{s=1}^T \langle \theta_s, (\tilde{S}_{s+1} - \tilde{S}_s)\rangle$ (cf. (iii) in Proposition 2.4), and since Q is risk neutral, $M_t = V_0(\theta) + \sum_{s=1}^t \langle \theta_s, (\tilde{S}_{s+1} - \tilde{S}_s)\rangle$ is a martingale and $M_t = \tilde{V}_t(\theta) = E_Q[R_T V / \mathcal{F}_t]$ is positive since V is so; thus θ is admissible. Finally, by hypothesis $E_Q[R_T V] = x$, so $V_0(\theta) = x$, initial value of the admissible strategy θ. □

Now the aim is to find V^* which realizes the maximum of the application $V \mapsto E_{\mathbb{P}}[U(V)]$ under the constraint that V is the terminal value of an admissible strategy with initial value x. So, as in the two periods case, we have to solve an optimization problem under constraint. Let us define the Lagrangian function:

$$\mathcal{L}(V, \lambda) = E_{\mathbb{P}}[U(V)] - \lambda(E_Q[R_T V] - x)$$

which can be written as

$$\mathcal{L}(V, \lambda) = E_{\mathbb{P}}[U(V) - \lambda(\frac{dQ}{d\mathbb{P}} R_T V - x)].$$

The concavity hypotheses allow us to obtain the optimum cancelling the Lagrangian gradient:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial V} &= E_{\mathbb{P}}[U'(V) - \lambda \frac{dQ}{d\mathbb{P}} R_T] \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= E_Q[R_T V] - x. \end{aligned} \tag{10}$$

The strict concavity of U shows the existence of the function $I = (U')^{-1}$ and a solution to the system is for instance:

$$V^* = I\Big(\lambda^* \frac{dQ}{d\mathbb{P}} R_T\Big)$$

with λ^* such that $\mathcal{X}(\lambda^*) = x$ where

$$\begin{aligned} \mathcal{X} : \mathbb{R} &\rightarrow \mathbb{R} \\ \lambda &\mapsto E_{\mathbb{P}}[\frac{dQ}{d\mathbb{P}} R_T I(\lambda \frac{dQ}{d\mathbb{P}} R_T)]. \end{aligned} \tag{11}$$

Since this application is monotone (Lebesgue Theorem) and surjective on $\mathbb{R}^+$, there exists a unique $\lambda^* = \mathcal{X}^{-1}(x)$:

$$V^* = I(\mathcal{X}^{-1}(x) \frac{dQ}{d\mathbb{P}} R_T).$$

The optimal strategy θ^* is deduced from V^* as in Proposition 2.20.

FEUILLE 1

1. Soit une fonction U croissante positive et concave et $g(c) = E[U(c_T)]$. Montrer que la relation $c_1 \prec c_2$, définie par $g(c_1) \leq g(c_2)$, a les propriétés d'une préférence sur l'ensemble des consommations.

Les exercices qui suivent sont extraits du DANA-JEANBLANC

1.2.3. Soit $f : \mathbb{R}^{d+1} \to \mathbb{R}^k$ de matrice D. Montrer que D de rang k équivaut à f surjective ou f^* injective.

1.2.5. Certains auteurs définissent une opportunité d'arbitrage θ par

$$\sum_{i=0}^{d} \theta_i p^i = 0, \sum_{i=0}^{d} \theta_i d^i(\omega_j) \geq 0, j = 1, \cdots, k, \text{ et il existe } j_0 : \sum_{i=0}^{d} \theta_i d^i(\omega_{j_0}) > 0.$$

Montrer que si l'actif indexé 0 est sans risque, cette définition est équivalente à celle donnée dans le cours.

1.2.8. Soit un système de prix $S = (S_0, D), v_j^i = d^i(\omega_j)$, et $D\theta$ le vecteur de composantes $(D\theta)_j = \sum_{i=0}^{d} \theta_i v_j^i$ et $S.\theta$ le produit scalaire $\sum_{i=0}^{d} \theta_i p^i$. On fait l'hpothèse que l'actif 0 est sans risque.

a) montrer que l'hypothèse AOA équivaut à

$$(i) D\theta = 0 \Rightarrow S.\theta \geq 0.$$

$$(ii) D\theta \in \mathbb{R}_+^k, D\theta \neq 0 \Rightarrow S.\theta > 0.$$

b) Soit $z \in Im(D)$. On peut écrire $z = D\theta$. Montrer que l'application $\pi : z \to S.\theta$ ne dépend pas du choix de θ sous l'hypothèse AOA et définit une forme linéaire positive sur $Im\ D$.

c) Montrer que π se prolonge en une forme linéaire positive $\bar{\pi}$ sur $\mathbb{R}^k$.

d) Montrer en utilisant le théorème de Riesz que $\bar{\pi}(z) = \beta.z$ avec $\beta \in (\mathbb{R}_+^*)^k$.

e) En déduire le théorème : l'hypothèse AOA équivaut à l'existence d'une suite $(\beta_j)j = 1, \cdots, k$, de nombres strictement positifs, appelés prix d'états, tels que

$$S^i = \sum_{j=1}^{k} v_j^i \beta_j, i = 0, \cdots, d.$$

1.2.9. Supposons qu'il y ait des contraintes sur portefeuilles modélisées par un cône convexe fermé C, par exemple, $C = \{\theta \in \mathbb{R}^{d+1}, \theta_i \geq 0, i = r+1, \cdots, r+p\ ;\ \theta_i \leq 0, i = r+p+1, \cdots, d\}$. On modifie la définition d'AOA en se restreignant à C.

(i) On note pour $\bar{\theta} \in C, N_C(\bar{\theta}) = \{p \in \mathbb{R}^{d+1} \langle p, \theta - \bar{\theta} \rangle \leq 0, \forall \theta \in C\}$.Montrer que l'hypothèse d'AOA est équivalente à l'existence de $\beta \in (\mathbb{R}_+^*)^k$ tel que $-S + \tilde{D}\beta \in N_C(0)$.

(ii) On suppose qu'il y a $k \geq 4$ états de la nature et 4 actifs et l'hypothèse A.O.A. L'actif 0 est sans risque et le taux d'intérêt est r. Les autres sont risqués de matrice de rendement D. On suppose que les contraintes sont $\theta_2 \geq 0$ et $\theta_3 \leq 0$. Montrer qu'il existe une probabilité π telle que

$$S^1 = \frac{1}{1+r}\sum_{j=1}^{k} v_j^1 \pi_j, S^2 \geq \frac{1}{1+r}\sum_{j=1}^{k} v_j^2 \pi_j, S^3 \leq \frac{1}{1+r}\sum_{j=1}^{k} v_j^3 \pi_j.$$

II.

Un petit épargnant place chaque mois une proportion de sa fortune initiale dans un placement de taux aléatoire et épargne le reste à un taux nul. On suppose que les variables aléatoires qui modélisent les différents taux de chaque mois sont indépendantes et de même moyenne notée $m, m > 0$. Sa richesse initiale est $x, x > 0$ et il recherche une politique optimale sur N mois au sens où il veut maximiser en moyenne la somme de ses épargnes successives et de sa richesse terminale, qui est la quantité sur laquelle se fonde le banquier pour calculer le montant d'un prêt.

1. Définir le modèle de contrôle défini ci-dessus, en précisant ses différents éléments : espaces concernés, dynamique du système, valeur à optimiser...

2. Résoudre dans le cas d'une période de cinq mois avec $m = 0.5$.

3. Même question lorsque N est quelconque et $m > 1$.
Indication : montrer par récurrence que $V(k, X_k)$ est proportionnelle à X_k et que $\pi_k^*(X_k) = 1$ en indiquant le coefficient de proportionnalité.

4. Même question lorsque N est quelconque et $m \in [\frac{1}{N}, 1[$.
Indication : on pose $k_0 = [\frac{1}{m}]$; montrer par récurrence que $V(k, X_k)$ est proportionnelle à X_k et que $\pi_k^*(X_k) = 1, \forall k < N - k_0$; $\pi_k^*(X_k) = 0, \forall k \geq N - k_0$. de proportionnalité.

FEUILLE 2

1. On considère un marché comportant 3 actifs dont les prix d'achats sont

$$p_1 = 20 \ ; \ p_2 = 60 \ ; \ p_3 = 30$$

et les prix à l'instant T sont des variables aléatoires définies sur un espace de probabilité $\Omega = (\omega_1, \omega_2, \omega_3)$ et données par :

$$\begin{array}{lll} d_1(\omega_1) = 60 & d_2(\omega_1) = 100 & d_3(\omega_1) = 30 \\ d_1(\omega_2) = 50 & d_2(\omega_2) = 120 & d_3(\omega_2) = 30 \\ d_1(\omega_3) = 60 & d_2(\omega_3) = 120 & d_3(\omega_3) = 30 \end{array}$$

Déterminer l'ensemble budgétaire d'un agent disposant des ressources suivantes :

$$e(0) = 400, e(T, \omega_1) = 500, e(T, \omega_2) = 1300, e(T, \omega_3) = 300.$$

2. Montrer que dans un marché où tout processus de consommation est atteignable (marché complet), toute allocation d'équilibre a l'efficacité de Pareto : pour ceci, notant c^i le processus de consommation de l'agent i et e^i ses ressources à l'équilibre, b^i un autre processus de consommation préférable, et $a^i = b^i - c^i$, on suit les étapes suivantes (on note (θ_i, α_i) ce qui permet d'atteindre a_i) :

.i. traduire que c^i est une allocation d'équilibre ; que a^i est atteignable ; que b^i est faisable.

.ii. Montrer (par l'absurde) que $\sum_i \langle \theta^i, p \rangle = 0$ et en déduire que $\sum_i \alpha^i = 0$, où α_i est la ressource initiale associée à a_i.

.iii. Montrer que $\alpha_i = 0$ pour tout i, puis conclure.

3. On considère un marché comportant 3 actifs dont les prix d'achats sont

$$p_1 = 35 \ ; \ p_2 = 40 \ ; \ p_3 = 12$$

et les prix à l'instant T sont des variables aléatoires définies sur un espace de probabilité $\Omega = (\omega_1, \omega_2, \omega_3)$ et données par :

$$\begin{array}{lll} d_1(\omega_1) = 24 & d_2(\omega_1) = 44 & d_3(\omega_1) = 12 \\ d_1(\omega_2) = 20 & d_2(\omega_2) = 44 & d_3(\omega_2) = 12 \\ d_1(\omega_3) = 48 & d_2(\omega_3) = 36 & d_3(\omega_3) = 12 \end{array}$$

a) Quel est l'ensemble des processus de consommation atteignables ?
b) Le processus de consommation suivant :

$$c(0) = 0, c(T, \omega_1) = 6, c(T, \omega_2) = 5, c(T, \omega_3) = 12$$

est-il atteignable ? Donner une ressource initiale et une stratégie permettant de l'atteindre.
c) même question avec le processus de consommation :

$$c(0) = 0, c(T, \omega_1) = 9, c(T, \omega_2) = 1, c(T, \omega_3) = 17.$$

d) Ce système de prix permet-il une stratégie d'arbitrage ?

4. There are $K = 2$ states and $N = 3$ securities with payout

$$D = \frac{20\ 44\ 12}{48\ 36\ 12}$$

and prices $p_1 = 35, p_2 = 40, p_3 = 12$.
a. Is the market complete ? Find the set M of all attainable consumption processes.
b. Find an initial endowment and a trading strategy that attain the consumption process : $c(0 = 0, c(T, \omega_1) = 9, c(T, \omega_2) = 1$.
c. Does the given price system permit arbitrage strategies ?
d. Do equilibrium price measures exist ?

5. On considère un marché comportant 3 actifs dont les prix d'achats sont

$$p_1 = 8\ ;\ p_2 = 10\ ;\ p_3 = 3$$

et les prix à l'instant T sont des variables aléatoires définies sur un espace de probabilité $\Omega = (\omega_1, \omega_2, \omega_3)$ et données par :

$$\begin{array}{ccc} d_1(\omega_1) = 6 & d_2(\omega_1) = 11 & d_3(\omega_1) = 6 \\ d_1(\omega_2) = 5 & d_2(\omega_2) = 11 & d_3(\omega_2) = 6 \\ d_1(\omega_3) = 12 & d_2(\omega_3) = 9 & d_3(\omega_3) = 6 \end{array}$$

a. Le marché est-il complet ?
b. Existe-t-il une mesure de prix d'équilibre ? des stratégies d'arbitrage ?

FEUILLE 3

1. On considère un marché comportant 2 actifs dont l'évolution des prix au cours de deux périodes est la suivante :

$p^1(t,\omega)$	ω_1	ω_2	ω_3	$p^2(t,\omega)$	ω_1	ω_2	ω_3
$t=0$	1	1	1	$t=0$	5	5	5
$t=1$	2	3	3	$t=1$	2	6	6
$t=2$	3	3	8	$t=2$	3	4	8

a) Quelle est la filtration engendrée par les processus de prix ?
b) Donner des exemples de stratégies autofinancées.
c) Déterminer l'ensemble budgétaire d'un agent disposant du processus de ressource suivant :

$e(t,\omega)$	ω_1	ω_2	ω_3
$t=0$	10	10	10
$t=1$	20	30	30
$t=2$	0	0	0.

d) Quel est l'ensemble des processus de consommation accessibles ? (utiliser la question précédente avec $e(0)$ libre et $e(1)=e(2)=0$.)

2. On donne un système de prix de 2 actifs, évoluant sur 3 périodes, dans un espace d'aléas de cardinal $K=6$:

$p^1(t,\omega)$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	$p^2(t,\omega)$	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
$t=0$	8	8	8	8	8	8	$t=0$	20	20	20	20	20	20
$t=1$	7	7	7	7	7	7	$t=1$	10	10	30	30	30	30
$t=2$	9	9	9	9	10	10	$t=2$	15	15	40	40	40	40
$t=3$	10	10	10	10	10	10	$t=3$	10	20	30	40	50	60.

a) Quelle est la filtration engendrée par les processus de prix ?
b) Soit $(\theta(t); 1 \leq t \leq 3)$ un processus à valeurs dans R^2. A quelles conditions ce processus constitue-t-il une stratégie autofinancée ?
c) On introduit un "bond" sur le marché, de prix constant égal à 1 et on décide d'investir sur les actifs risqués 1 et 2 les quantités suivantes :

$$\theta(1) = (10;1)\ ;\ \theta(2,\omega_k) = (5;0), k=1,2; (5;3) \text{ sinon } \ ;$$

$$\theta(3,\omega_k) = (0;2), k=1,2; (1;5), k=3,4; (3;5) \text{ sinon } .$$

Quelle quantité $\theta^0(t)$ doit-on investir dans le bond à chaque instant t pour que (θ^0,θ) soit une stratégie autofinancée de richesse initiale $V_0 = 100$?

d) Ce marché est-il viable ? complet ?
e) Proposer un système de prix, engendrant la même filtration que le précédent, et tel que le marché induit ne soit pas complet .

FEUILLE 4

1. On considère un marché comportant 3 actifs dont les prix d'achats sont $p_1 = 1$; $p_2 = 2$; $p_3 = 7$ et les prix à l'instant T sont des variables aléatoires définies sur un espace de probabilité $\Omega = (\omega_1, \omega_2, \omega_3)$ et données par :

$$\begin{array}{lll} d_1(\omega_1) = 1 & d_2(\omega_1) = 3 & d_3(\omega_1) = 9 \\ d_1(\omega_2) = 1 & d_2(\omega_2) = 1 & d_3(\omega_2) = 5 \\ d_1(\omega_3) = 1 & d_2(\omega_3) = 5 & d_3(\omega_3) = 13. \end{array}$$

a) Existe-t-il des mesures de prix d'équilibre ? des stratégies d'arbitrage ?
b) Quel est l'ensemble M des processus de consommation atteignables ? Calculer $E_Q[c(T)]$ pour $c \in M$ et Q mesure de prix d'équilibre.
c) On considère un quatrième actif : une option call sur l'actif 2 de prix d'exercice 2. Calculer $E_Q(d_4)$ pour différentes mesures de prix d'équilibre.

2. On se place dans un marché, pas nécessairement complet, à plusieurs dates $0, 1, \cdots, T$, pour lequel on suppose qu'il existe au moins une mesure de prix d'équilibre. Montrer que pour toute consommation accessible par une stratégie autofinancée $c = (c(0), c(T))$, la valeur $E_Q[c(T)]$ ne dépend pas de la mesure de prix d'équilibre Q.

3. Un marché comportant 3 actifs est décrit par le système de prix suivant :

$$p_1 = 1 \ ; \ p_2 = 3 \ ; \ p_3 = 9$$

$$\begin{array}{lll} d_1(\omega_1) = 1 & d_2(\omega_1) = 3 & d_3(\omega_1) = 9 \\ d_1(\omega_2) = 1 & d_2(\omega_2) = 1 & d_3(\omega_2) = 5 \\ d_1(\omega_3) = 1 & d_2(\omega_3) = 4 & d_3(\omega_3) = 11 \\ d_1(\omega_4) = 1 & d_2(\omega_4) = 2 & d_3(\omega_4) = 7. \end{array}$$

a) Déterminer les mesures de prix d'équilibre.
b) Quel est l'ensemble M des processus de consommation accessibles ?

4. Come back to the market described in exercise 3, "feuille 2".
a) Do equilibrium price measures exist ? Find them all.
b) Find the equilibrium price of a fourth security : $d_4^t = (40, 20, 30)$.
c) Find the equilibrium prices of the following securities :

(i) A call option on security 1 with the exercise price of 25.

(ii) A put option on security 2 with the exercise price of 40.

(iii) A security whose terminal payout in each state is the maximum payout of all securities less the average payout of all securities in that state.

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