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All comments and remarks are welcome.

1 Introduction

1.1 Motivation

Asymmetric information can be considered as the problem of an insider trading : some financial agent knows something about the future. Thus, a market model is built on a filtered probability space $(\Omega, (\mathcal{F}_t, t \in [0,T]), \mathbb{P})$, the prices of assets being solution of a SDE driven by W, a d-dimensional Brownian motion. From the beginning, $t = 0$, the investor (namely the "insider") knows a random variable $L \in L^1(\Omega, \mathcal{F}_T; \mathbb{R}^\kappa)$, $\kappa \in \mathbb{N}$, for instance, he knows that some trading will be done and when it will be done; for two assets of prices S^1 et S^2, the random variable could be their ratio at time T : $L = \ln S^1_T - \ln S^2_T$. The "natural" filtration known by the insider trader is $\mathcal{F}_t \vee \sigma(L)$. But on the filtered probability space $(\Omega, (\mathcal{Y}_t, t \in [0,T]), \mathbb{P})$, the process W is no longer a semi-martingale. This is the so-called initial enlargement of filtration problem. This one is widely studied in LN 1118. Karatzas and Pikovsky [34] studied similar problems on some examples of real or vectorial random variables : $L = W_1$, $L = (\lambda_i W^i_1 + (1-\lambda_i)\mathcal{E}_i)_{i=1,d}$ with a family of independent Gaussian variables $(\mathcal{E}_i)$, or $L = S^1$ the price at time 1, or $L = \mathbf{1}_{\{S^1 < p\}}$. The common point is the so called hypothesis $\mathbf{H_J}$ which will be defined below.

But if there is a set of investors with different information on the market, the trading manages only if a price is got between them, this problem is called an “equilibrium problem". Here we don't manage with other types of equilibrium, such as “Arrow-Debreu" or “Arrow-Radner” (cf. [25]), meaning a set of agents, receiving an endowment, using it to optimize their consumption, while balancing the market; the market is to be “clear".

Another point of view could be the so called filtering problem, meaning incomplete information, given a signal, how to estimate the prices given a signal for the best. In any cases, the agents have an optimization problem to solve. Such problem are solved with Bellman principle, for instance.

1.2 Plan

Actually, this course will offer seven chapters, including an introduction to the useful tools (Subsection 1.3).

- We first introduce some elements on enlargement of filtration, filtering, optimal control, and Bellman principle [13, 16, 31].
- Then in Chapter 2 we present Kyle's seminal paper about “insider trading and rational anticipation" [36]: the aim is to set the existence of an equilibrium price when there exists not only market maker and noise traders but also an insider trader; the model is a discrete time model.
- Always in a discrete time model, Chapter 3 concerns an extension of this insider trading whith nonlinear equilibria with risk neutrality, respectively with risk aversion, following El Karoui and Cho [8].
- Chapter 4 presents continuous case, first Back's point of view [3] and secondly Cho's extension [9].
- This work is then extended in [42] to **strategic** noise traders (Chapter 5).
- Chapter 6 presents another type of equilibrium (cf. [25]).
- Finally, in Chapter 7 we quickly present some other points of view (e.g. Campi and Cetin, Jouini and Napp, Schweizer, Lasserre.... cf. some lectures in AMaMef workshop-Toulouse January 2007).

1.3 Stochastic tools

1.3.1 Initial enlargement of filtrations

Let a filtered probability space $(\Omega, (\mathcal{F}_t, t \in [0,T]), \mathbb{P})$, with some processes given by the equation :

$$S_t^i = S_0^i + \int_0^t S_s^i b_s^i ds + \int_0^t S_s^i (\sigma_s^i, dW_s), 0 \leq t \leq T, S_0 \in \mathbb{R}^d, i = 1 \cdots, d.$$

where W is d-dimensional Brownian motion and $(.,.)$ denotes the scalar product in $\mathbb{R}^d$. We "enlarge" the filtration with an initial information, for instance a random variable $L \in L^1(\Omega, \mathcal{F}_T; \mathbb{R}^\kappa), \kappa \in \mathbb{N}$, or a σ-algebra $\mathcal{G}$. The "enlarged" filtration is $\mathcal{F}_t \vee \sigma(L)$. To apply the standard results, we use the associated right continuous filtration, denoted by $\mathcal{Y} : \mathcal{Y}_t = \cap_{s>t}(\mathcal{F}_s \vee \sigma(L)), t \in [0, T]$.

But on the filtered probability space $(\Omega, (\mathcal{Y}_t, t \in [0, T]), \mathbb{P})$, the process W is no longer a semi-martingale. Following Föllmer and Imkeller [15], an equivalent probability measure Q is built such that under Q, for all $t < T$, the σ-algebra $\mathcal{F}_t$ is independent of $\sigma(L)$. Thus W is a $(\mathcal{Y}, Q)$-Brownian motion. Another useful method is the initial enlargement of filtrations, it allows to find some conditions on L (or on $\mathcal{G}$) so that there exist a $\mathcal{Y}$-Brownian motion B and an increasing process A satisfying $W_t = B_t + A_t$. This was studied when L is a Gaussian random variable by Yor [46], Chaleyat-Maurel and Jeulin [7].

More generally, Jacod [29] did the same when the family of conditional laws $Q_t(\omega, .)$ of L given $\mathcal{F}_t$ is dominated almost surely by a non-random measure ; see also Song [45]. The Bouleau-Hirsch [6] results give some simple conditions on L so that these conditional laws are dominated by the Lebesgue measure. With some extra hypotheses, Imkeller [27], specifies the decomposition of the semi-martingale W using Malliavin calculus.

Below, we give the links between some hypotheses allowing enlargement of filtration.

1.3.2 Hypotheses, cf. [21]

Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with filtration $(\mathcal{F}_t, t \leq T, \mathcal{F}_T \subset \mathcal{A})$; an informed agent has an initial (at $t = 0$) private information described by a σ-field $\mathcal{G} \subset \mathcal{A}$, (a general example is $\mathcal{G} = \sigma(L)$ for a random variable $L \in L^1(\Omega, \mathcal{F}_T; \mathbb{R}^\kappa), \kappa \in \mathbb{N}$). The right continuous filtration $(\mathcal{Y}_t = \cap_{s>t}(\mathcal{F}_s \vee \mathcal{G}), t \in [0, T])$ will denote the information of the informed agent.

This section is devoted to show the link between different hypotheses which allow the enlargement of filtration which is necessary to study market with informed agent. Indeed the semi-martingales describing prices evolution in the market without an informed agent $(\mathcal{F})$ need to be also semi-martingales in the market with informed agent $(\mathcal{Y})$.

The first hypothesis is

$$(H') \ : \ \forall M \in \mathcal{M}_{loc}(\mathcal{F}, \mathbb{P}), \exists M' \in \mathcal{M}_{loc}(\mathcal{Y}, \mathbb{P}) \text{ such that } M'_t = M_t - A_t, \forall t \in [0, T[,$$

where A is a finite variations process. (H') is granted as soon as the conditional law of L given $\mathcal{F}_t$ is absolutely continuous with respect to a deterministic measure for all $t \in [0, T[)$ (Jacod [29]).

The second and third hypotheses are (H_3) and (H_J) which imply (H') (cf. Proposition 1.4 below) :

(H_3) : $\exists Q \sim \mathbb{P}$; such that under $Q, \forall t < T, \mathcal{F}_t$ is independent of $\mathcal{G}$.

(H_J) : The conditional law of L given $\mathcal{F}_t$ is equivalent to the law of $L, \forall t < T$.
Here $q(t, .)$ will denote the density between the two probability laws.

When $\mathcal{G} = \sigma(L)$, (H_J) implies (H') and under (H_J) the probability law defined by, for all $A < T$, $Q = \frac{1}{q(A,L)}\mathbb{P}$ on $\mathcal{Y}_A$ has the following properties:

under Q, $\mathcal{F}_t$ and $\mathcal{G}$ are independent and $Q_{|\mathcal{F}_t} = \mathbb{P}_{|\mathcal{F}_t}, \forall t \leq A$.

Remark 1.1 *J. Amendinger (his thesis and [1]) supposes (H_J), then shows (H_3), and actually always uses the property (H_3) in his proofs. So, we can apply his results. He establishes also that under (H_J), filtration $\mathcal{F}_. \vee \mathcal{G}$ is right continuous as soon as $\mathcal{F}_.$ is (Proposition 3.4, [1]).*

Simple examples for which Hypothesis (H_3) is verified are given in [19] ; examples, for which hypothesis (H') is verified but (H_3) is not, are given in [17].

Remark 1.2 *Let R a probability law which verifies (H_3). Then, there exists an equivalent probability Q, satisfying (H_3), and such that $\forall t < T, Q_{|\mathcal{F}_t} = \mathbb{P}_{|\mathcal{F}_t}$. Indeed, let $Z_t^* = E_{\mathbb{P}}[\frac{dR}{d\mathbb{P}}/\mathcal{F}_t]$ and $Q = (Z_t^*)^{-1}R$ on $\mathcal{Y}_t$; let $H \in L^\infty(\mathcal{F}_t)$ and $Y \in L^\infty(\mathcal{G})$, if $H' = (Z_t^*)^{-1}H$, this random variable is in $L^1(\mathcal{F}_t, R)$, then*

$$E_Q[H.Y] = E_R[(Z_t^*)^{-1}.H.Y] = E_R[H'.Y] = E_R(H')E_R(Y) = E_{\mathbb{P}}(H)E_R(Y).$$

So $\mathcal{G}$ and $\mathcal{F}_t$ are independent under Q and $Q_{|\mathcal{F}_t} = \mathbb{P}_{|\mathcal{F}_t}, \forall t < T$.

Hypothesis (H_J) is linked with Hypothesis (27) in Föllmer-Imkeller [15]. The next proposition is near to Remarks (28) and (29) in this paper.

Proposition 1.3 *If $\mathcal{G} = \sigma(L)$, then (H_3) and (H_J) are equivalent.*

Proof : The proof done in [19] never uses the continuity of processes in the story....so it can be used here. •

Proposition 1.4 *Hypothesis (H_3) implies hypothesis (H').*

Proof : Let $Q = H\mathbb{P}, H \in L^1(\Omega, \mathcal{A}, \mathbb{P})$, be the probability law under which $\mathcal{F}$ and $\mathcal{G}$ are independent and such that $Q|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$; let $Z_t = E_{\mathbb{P}}[H/\mathcal{F}_t]$, it is a $(\mathcal{F}, \mathbb{P})$ martingale. Let $M \in \mathcal{M}_{loc}(\mathcal{F}, \mathbb{P})$, then (Protter page 109 [43]) we have

$$M' = M - (Z)^{-1}.[Z, M] \in \mathcal{M}_{loc}(\mathcal{F}, Q). \tag{1}$$

But if $t < T$, $Q|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$ then $Z_t = 1$ and $M' = M$.

The independence of $\mathcal{G}$ and $\mathcal{F}_t, \forall t < T$, under Q shows that $M' \in \mathcal{M}_{loc}(\mathcal{Y}, Q)$. The process $Z_t^* = E_Q[H^{-1}/\mathcal{Y}_t]$ is a $(\mathcal{Y}, Q)$ martingale. Then again using Girsanov theorem,

$$M'' = M' - (Z^*)^{-1}.[Z^*, M'] = M - (Z^*)^{-1}.[Z^*, M] \in \mathcal{M}_{loc}(\mathcal{Y}, \mathbb{P}) \tag{2}$$

thus we get the $(\mathcal{Y}, \mathbb{P})$-semimartingale decomposition of M.

•

Remark 1.5 *If the martingale M, in the proof of the previous proposition, admits a representation property then $[Z^*, M]$ is absolutely continuous with respect to $[M, M]$. This is done for martingales driven by a Brownian motion and a point process as an example.*

In Grorud-Pontier 2001 [21] is given a sufficient condition including hypothesis (H') to satisfy (H_3) for models in which prices are driven by a Brownian motion and a point process (actually similar to Novikov condition in case of Girsanov formula).

1.3.3 Stochastic Filtering, cf. Bain and Crisans [4]

A signal process X is observed via another continuous process Y, namely the "observation process". This one generates its natural filtration, $\mathcal{Y}$. The aim is to estimate the signal for the best using the observations, more precisely, to get the $\mathcal{Y}_t$-conditional law of X_t for any time t, denoted as π_t. Process π is characterized as a solution of a SDE. This $\mathcal{Y}_t$-conditional law of X_t is a stochastic process $t \mapsto \pi_t$ taking its value in the space of probability measures. Generally, the observation process is

$$Y_t = Y_0 + \int_0^t h(X_s)ds + W_t, \ t \geq 0.$$

Under some topological or stochastic properties as separability, optional projection or weak topology are needed. Then, the construction of a regular conditional probability can be done, with some sufficient conditions on function h to get the *innovation process*,

$$I_t = Y_t - \int_0^t \pi_s(h)ds$$

as a Brownian motion. Now let the famous Fujisaki-Kallianpur-Kunita representation theorem:

Theorem 1.6 *If for all $t \geq 0$, $h(X_.) \in L^2(\Omega \times [0,t])$ and $\int_0^t \|\pi_s(h)\|^2 ds < \infty$, then $\forall \eta$ $\mathcal{Y}_\infty$-measurable there exists a $\mathcal{Y}$-progressively measurable process ν such that*

$$\eta = E(\eta) + \int_0^t \nu_s.dI_s.$$

An important point in this theory is the filtering equations, in the case of the signal X is a diffusion driven by a p-Brownian motion V:

$$X_t^i = X_0^i + \int_0^t f^i(X_s)ds + \sum_{j=1,p} \int_0^t \sigma^{ij}(X_s)dV_s^j, \ i = 1, \cdots, d.$$

Its initial law is denoted as π_0, and there exists an operator A on the bounded Borelian functions such that $\phi(X_t) - \phi(X_0) - \int_0^t A\phi(X_s)ds$ is a martingale. Thus yields Kushner-Stratonovitch equation:

$$\pi_t(\phi) = \pi_0(\phi) + \int_0^t \pi_s(A\phi)ds + \int_0^t [\pi_s(\phi.h) - \pi_s(\phi)\pi_s(h)]dI_s.$$

1.3.4 Optimal control, Bellman principle

This is useful starting from Chapter 4.

Let a value function to maximize with respect to a set of strategies $\mathcal{S}$, meaning a set of predictable processes:

$$\alpha \mapsto E[U(\alpha, X)].$$

Usually the optimal conditional value after t is defined:

$$\Phi_\alpha(t) = ess\sup\{E[U(\tilde{\alpha}, X)/\mathcal{F}_t],\ \tilde{\alpha}_{|[0,t]} = \alpha_{|[0,t]}\}.$$

Lemma 1.7 *cf. [13] (N. El Karoui's version of Bellman's principle).*
For all $\alpha \in \mathcal{S}$, Φ_α has to be a supermartingale.
Moreover, α is an optimal control is equivalent to the fact that Φ_α is a martingale.

As soon as X is a diffusion process with infinitesimal generator $\mathcal{L}$, this lemma and Ito formula imply that the value function is solution of a partial differential equation, named Hamilton-Jacobi-Bellman equation. Indeed, since X is a Markov process, actually there exists a function F^α (supposed for the moment to be smooth) such that

$$\Phi_\alpha(t) = F^\alpha(t, X_t) = F^\alpha(0, X_0) + \int_0^t \mathcal{L}F^\alpha(s, X_s)ds + M_t$$

where M is a martingale. So

$$\alpha \text{ is an optimal control } \Leftrightarrow \mathcal{L}F^\alpha(t, X_t) = 0\ dt \otimes d\mathbb{P}\ a.s.$$

If the support of process X is $[0,T] \times U$ we get a PDE:

$$\mathcal{L}F^\alpha(s, x) = 0,\ \forall (t, x) \in [0,T] \times U$$

with boundary conditions convenient to the problem to be solved.

Bellman's principal could be named Dynamic Programming Principle and leads to Hamilton-Jacobi-Bellman equation. If process X is controlled by α:
$dX_t = b(X_t, \alpha)dt + \sigma(X_t, \alpha)dW_t$, with the function to be maximized $E[\int_0^T f(t, X_t, \alpha)dt]$, we introduce the so-called *Hamiltonian*:

$$H(t, x, p, M) = \sup_{\alpha \in \mathcal{S}} \left[-b(x, \alpha).p - \frac{1}{2}\sigma^2(x, \alpha).M - f(t, x, \alpha)\right]$$

and the above value function F is solution to:

$$-\partial_t F(t, x) + H(t, x, \partial_x F, (t, x)\partial^2_{xx} F(t, x)) = 0,\ \ F(T, x) = D(x)$$

if $D(X_T)$ is the terminal cost.

2 Insider trading and rational anticipation

In the seminal paper [36], Kyle asks three questions:

- how quickly is new private information about the underlying value of a speculative commodity incorporated into market prices?

- how does noise trading affect the volatility of prices?

- what determines the liquidity of a speculative market?

Kyle considers a market with one risky assets and one riskless assets under Gaussian hypotheses. There are three agents:

- the insider trader has access to a private observation of the *expost* liquidation value of the risky assets,

- uninformed noise traders trade randomly (according to a centered Gaussian law),

- the market maker set prices *efficiently*, meaning conditionnally on the public information he has about the assets quantities tradered by all the traders.

Dicrete-time model can be considered, so trading times are each auction times, or continuous-time model with continuous trading. At each time, the traders choose the quantities they will trade, using their own information. After that, the market maker set a price and trades the quantity which will make the market clear.

Here is summarized this discrete time model (1985) [36].

2.1 Model and notations.

Let the expost liquidation value of the risky assets be denoted as V, a Gaussian random variable $N(p, \sigma^2)$.
The noise trader trades the quantity $\tilde{u} \sim N(0, \sigma_u^2)$, random variable independent of V.
The insider trader trades the quantity $\tilde{x}$. The insider trader knows V from the beginning. We can assume that there exists a function X such that $\tilde{x} = X(V)$, X is the so-called insider's "strategy".
The market maker has to set the researched price, denoted $\tilde{p}$. He observes the global demand of the risky asset: $\tilde{x} + \tilde{u}$ but no separately and so we can assume that there exists a function P such that $\tilde{p} = P(\tilde{x} + \tilde{u})$.
Then the **insider's profit** will be on time 1:

$$\tilde{\pi} = (V - \tilde{p})\tilde{x} = (V - P(X(V) + \tilde{u}))X(V). \tag{3}$$

Finally, there exists functions $\tilde{\pi}$ and $\tilde{p}$ such that

$$\tilde{\pi} = \tilde{\pi}(X, P) \ ; \ \tilde{p} = \tilde{p}(X, P).$$

2.2 Equilibrium

The insider is risk neutral and his aim is to optimize his profit. Otherwise, the market maker has to set a price so that the market will be clear. So we get the following definition:

Definition 2.1 *An* **equilibrium** *is a pair of functions* (X, P) *satisfying:*

- X *is optimal: for any strategy* X', $E[\tilde{\pi}(X, P)/V] \geq E[\tilde{\pi}(X', P)/V]$,
- $\tilde{p}(X, P) = E[V/\tilde{x} + \tilde{u}]$ *(the price is said to be "rational" or "efficient").*

It appears like a fix point problem. This problem is too general and complex. Here the functions X and P are constrained to be affine functions, so we get:

Theorem 2.2 *(Th. 1 page 1319) There exists a unique equilibrium in which the functions* X *and* P *are linear:*

$$X(v) = \sqrt{\frac{\sigma_u^2}{\sigma^2}}(v - p) \ ; \ P(y) = p + \frac{1}{2}\sqrt{\frac{\sigma^2}{\sigma_u^2}}y. \tag{4}$$

Proof : Following the linear hypothesis, put:

$$P(y) = \mu + \lambda y \ ; \ X(v) = \alpha + \beta v.$$

A natural constraint is $\lambda > 0$, since the prices usually increase with the demand. Thus the insider's profit is

$$\tilde{\pi} = (V - P(\tilde{x} + \tilde{u}))\tilde{x} = (V - \mu - \lambda(\tilde{x} + \tilde{u}))\tilde{x},$$

and conditionnally on $V = v$, given the independence between V and $\tilde{u}$ and the fact that $\tilde{u}$ is centered:

$$E[\tilde{\pi}/V = v] = (v - \mu - \lambda\tilde{x})\tilde{x}.$$

On one hand, since $\lambda > 0$, the optimal profit is reached with

$$x^* = \frac{v - \mu}{2\lambda}$$

and the expected optimal profit is $\tilde{\pi}^* = \frac{(\mu - v)^2}{2\lambda}$.

On the other hand, we look for (μ, λ) such that the equilibrium pricing rule could be efficient given $x = x^*$:

$$\mu + \lambda y = E[V/x^* + \tilde{u} = y] = E\left[V/\frac{V - \mu}{2\lambda} + \tilde{u} = y\right].$$

Once again, Gaussian and independence hypotheses yield the pair $(V, \frac{V-\mu}{2\lambda} + \tilde{u})$ is Gaussian with mean $(p, \frac{p-\mu}{2\lambda})$ and covariance matrix Γ:

$$\Gamma = \begin{pmatrix} \sigma^2 & \frac{\sigma^2}{2\lambda} \\ \frac{\sigma^2}{2\lambda} & \sigma_u^2 + \frac{\sigma^2}{4\lambda^2} \end{pmatrix}$$

Thus the conditional expectation is:

$$E[V/x^* + \tilde{u} = y] = p + \frac{\frac{\sigma^2}{2\lambda}}{\sigma_u^2 + \frac{\sigma^2}{4\lambda^2}}(y - \frac{p-\mu}{2\lambda}).$$

The identification to the function $\mu + \lambda y$ yields the result: $\lambda = \frac{\sigma}{2\sigma_u}$ and $\mu = p$. •
Remember, if $(X, Y) \sim \mathcal{N}$,

$$E[X/Y = y] = E(X) + \frac{cov(X,Y)}{Var(Y)}(y - E(Y)).$$

2.3 Sequential auction equilibrium: N auctions

Then Kyle extends his results to a sequence of auctions, meaning that auctions occur on times $t_0 = 0 < t_1 < \cdots < t_{N-1} < 1$. Once again, the expost liquidation value of the risky assets is denoted as V, Gaussian random variable $N(p, \sigma^2)$. The noise trader trades u_n on time t_n, a Gaussian random variable $N(0, (t_n - t_{n-1})\sigma_u^2)$, independent of V. The insider trader trades x_n on time t_n. Then the market maker set the price $\tilde{p}_n$ on this time, depending of the global demand $u_n + x_n$. The insider observes V and also knows the prices on each time auction, so the strategy is given by:

$$x_n = X_n(\tilde{p}_0, \cdots, \tilde{p}_n, V).$$

Similarly, $\forall n$, there exists a function P_n:

$$\tilde{p}_n = P_n(u_o + x_o, \cdots, u_n + x_n).$$

The insider's profit is

$$\pi_n = \sum_{k=n}^{N}(V - \tilde{p}_k)x_k.$$

Definition 2.3 *A* **sequential auction equilibrium** *is a pair of* $\mathbb{R}^N$ *valued functions* (X, P) *satisfying:*

- X *is optimal:* $\forall n = 1, \cdots, N$, *for any strategy* X' *such that* $X_i = X'_i, i = 1, \cdots, n-1$, $E[\pi_n(X, P)/\tilde{p}_1, \cdots, \tilde{p}_n, V] \geq E[\pi_n(X', P)/\tilde{p}_1, \cdots, \tilde{p}_n, V]$,

- $\forall n = 1, \cdots, N$, $\tilde{p}_n(X, P) = E[V/x_i + u_i, i = 1, ..., n]$.

Once again, he restricts the component functions X and P to be affine functions, and in this case, they are recursive. Thus he gets

Theorem 2.4 *th 2 page 1322.*
There exists a unique linear equilibrium and this equilibrium is a recursive one.

$$\begin{aligned} x_n &= \beta_n(V - \tilde{p}_{n-1})\Delta t_n, \\ \Delta\tilde{p}_n &= \lambda_n(x_n + u_n), \\ \sigma_n^2 &= Var(V/x_1 + u_1, \cdots, x_n + u_n) \\ E[\tilde{\pi}_n/p_1, \cdots, p_n, v] &= \alpha_{n-1}(v - p_{n-1})^2 + \delta_{n-1}. \end{aligned} \tag{5}$$

Here the constants are the unique solution to the difference equation system, $n = 1, ..., N$:

$$\begin{aligned} \alpha_{n-1} &= \frac{1}{4\lambda_n(1 - \alpha_n\lambda_n)}, \ \alpha_N = 0, \\ \delta_{n-1} &= \delta_n + \alpha_n\lambda_n^2\sigma_u^2\Delta t_n, \ \delta_N = 0, \\ \beta_n\Delta t_n &= \frac{1 - 2\alpha_n\lambda_n}{2\lambda_n(1 - \alpha_n\lambda_n)}, \\ \lambda_n &= \frac{\beta_n\sigma_n^2}{\sigma_u^2}, \\ \sigma_n^2 &= (1 - \beta_n\lambda_n\Delta t_n)\sigma_{n-1}^2, \end{aligned} \tag{6}$$

given the condition $\lambda_n(1 - \alpha_n\lambda_n) > 0$.

We here omit the proof.

How to interpret these parameters ? We quote Kyle (page 1323):

- "The parameters $\beta_n, n = 1, \cdots, N$, characterize the insider's strategy and measure the intensity with which he trades on the basis of his private information,
- the parameters $\lambda_n, n = 1, \cdots, N$, characterize the recursive pricing rule and measure the depth of the market (small λ_n correspond to a deep market),
- the parameters $\sigma_n, n = 1, \cdots, N$, give the error variance of prices after the n^{th} auction and measure how much of the insider's private information is not yet incorporated into prices (as estimated by market makers),
- the parameters α_{n-1}, δ_{n-1}, define a quadratic profit function which gives the value of trading opportunities at auction n."

2.4 Conclusion

The answers to Kyle's questions (2):
the informed trader trades in such a way that his information is incorporated into prices gradually.
The constant volatility reflects the fact that information is incorporated into prices at a

constant rate. Furthermore, all of the insider's private information is incorporated into prices by the end of trading in a continuous auction equilibrium.
An *ex ante* doubling of the quantities traded by noise traders induces the insider and market maker to double the quantities they trade, but has no effect on prices, and thus, doubles the insider's profit.

The meaning is given below, in a definition which will be useful later.

Definition 2.5 *A "doubling strategy" is to double the price up to winning.... to buy the assets with the hope that the traders will buy more and more and thus the prices would increase....*

This notion is more or less linked to arbitrage opportunities.

3 Insider trading and nonlinear equilibria, [8] 2000

Here the Gaussian hypothesis is relaxed for the risky assets law. Moreover, the authors exhibit necessary and sufficient conditions for the existence of an equilibrium. As Kyle does it, they only consider two times: $t = 0$ or $t = 1$. The insider's strategy only depends on the signal S and doesn't depend on the time, neither the market maker's observation, Y. Equilibrium price is characterized as a fixed point of a system, only depending on the assets law.

Two examples are studied where law of the risky assets are more accurate. Finally, the authors present what is the influence of the utility function on the equilibrium, depending of it is risk adverse or not.

3.1 The model

Once again, there are three agents

- the insider trader has access to a private observation: he knows a signal S on the *expost* liquidation value V of the risky assets, $S = V + \xi$, here ξ is a Gaussian random variable $N(0, \eta^2)$, independent of V, (the law of which is very general for the moment). Knowing S, he uses this knowledge to invest X on the risky assets and his strategy is $X = \alpha(S)$, his initial investment on the risky assets is X_0.

- the uninformed noise trader trades randomly and invests Z, Z law is a Gaussian law $N(0, \sigma_u^2)$,

- the market maker observes the sum $Y_1 = X + Z$, but not separately. Then he sets prices *efficiently*, meaning conditionnally on the public information he has about the assets quantities tradered by all the traders: $P_1 = E(V/Y_1) = H(Y_1)$.
On the one hand, H is a function depending on the strategy α, let us denote it H_α. On the other hand, the insider's aim is to optimize the expected utility function of his terminal wealth (profit), meaning $W_1(H, \alpha) = VX_0 + [V - H(\alpha(S) + Z)]\alpha(S)$, given his information $\mathcal{I} = \sigma(Y_0, S, X_0)$:

$$\alpha \mapsto E[u(VX_0 + [V - H(\alpha(S) + Z)]\alpha(S))/\mathcal{I}].$$

Remark that:

$H(\alpha(S) + Z)$ is the selling price,

$[V - H(\alpha(S) + Z)]\alpha(S)$ is the profit.

3.2 Nonlinear equilibrium, risk neutral utility

Definition 3.1 *An* **equilibrium** *is a pair of measurable functions (H, α) satisfying:*

- H is a rational price, meaning $H(Y_1) = E[V/Y_1]$.

- α is H-optimal: for any strategy α', $E[W_1(H, \alpha)/\mathcal{I}] \geq E[W_1(H, \alpha')/\mathcal{I}]$.

Firstly, the insider computes H_α given α, then he optimizes $\alpha \mapsto E[W_1(H_\alpha, \alpha)/\mathcal{I}]$.

In a first step, the authors assume $u = Id$ (meaning the insider is risk neutral). In such a case, X_0 is irrelevant, we assume $X_0 = 0$. They also suppose that $\xi = 0$, meaning that insider's information is not noisy.

Proposition 3.2 *Let f_Z the probability density function of the random variable Z. For any strategy α, below the expectation is w.r.t. V:*

$$H_\alpha(y) = E[V/\alpha(V) + Z = y] = \frac{E[V f_Z(y - \alpha(V))]}{E[f_Z(y - \alpha(V))]}.$$

Remark that anyway, $V \geq 0$ implies $H_\alpha(y) \geq 0$.
Proof : Using Bayes rule, the law of (V, Y_1) is given by the product of the conditionnal law of Y given V times the law of V, meaning $F_Z(y - \alpha(V)) \times F_V(v)$. By the way,

$$E[V/Y_1 = y] = \frac{\int v.f_Z(y - \alpha(v))F_V(dv)}{\int f_Z(y - \alpha(v))F_V(dv)}.$$

•

A corollary yields:

Corollary 3.3 *Since the law of Z is the Gaussian law $N(0, \sigma_u^2)$,*

$$H_\alpha(y) = \frac{E[V \exp(\sigma_u^{-2}(y\alpha(V) - \frac{1}{2}\alpha^2(V)))]}{E[\exp(\sigma_u^{-2}(y\alpha(V) - \frac{1}{2}\alpha^2(V)))]}.$$

The insider's second step is to optimize $\alpha \mapsto E[(V - H_\alpha(\alpha + Z))\alpha/\mathcal{I}]$ in the functions of V set.

Proposition 3.4 *The pair (H^*, α^*) is an equilibrium if (and only if?)*
1. $\forall v \in supp(V)$, $\alpha^*(v)$ solves $E[(1 + \sigma_u^{-2}xZ)H_{\alpha^*}(x + Z)] = v$.
2. $v \mapsto \alpha^*(v)$ is strictly increasing on $supp(V)$.
3. $H^* = H_{\alpha^*}$.

Proof ([8] page 26): in the case when $\xi = 0, X_0 = 0$, actually $\mathcal{I} = \sigma(V)$. So let

$$J: \ \alpha \mapsto E[W_1(H, \alpha)/\mathcal{I}] = E[(v - H(\alpha + Z))\alpha].$$

As a first step, assume that the function H is twice differentiable and that we can differentiate under the integral. In such a case, $\partial_\alpha J = E[v - H(\alpha + Z) - \alpha H'(\alpha + Z)]$ and $\partial^2_{\alpha^2} J = -E[2H'(\alpha + Z) + \alpha H"(\alpha + Z)]$. A sufficient condition for α^* to be optimal is to satisfy

$$E[H(\alpha + Z) + \alpha H'(\alpha + Z)] = v, \tag{7}$$
$$E[2H'(\alpha + Z) + \alpha H"(\alpha + Z)] \geq 0. \tag{8}$$

Since the law of Z is the Gaussian law $N(0,\sigma_u^2)$, we could get (notice here $y-\alpha(V)$ is Z)

$$\partial_\alpha E[H(\alpha+Z)] = \sigma_u^{-2} E[ZH(\alpha+Z)]$$

and the condition (7) could be written as following

$$E[(1+\alpha Z\sigma_u^{-2})H(\alpha+Z)] = v, \tag{9}$$

that is condition 1.

If we now consider the implicit equation which links α and v, yields:

$$\forall v \in Supp(V),\ \ E[H(\alpha(v)+Z)+\alpha(v)H'(\alpha(v)+Z)] = v$$

and we differentiate this equation with respect to v:

$$\alpha'(v)\left(E[2H'(\alpha+Z)+\alpha H"(\alpha+Z)]\right) = 1$$

meaning that condition (8) is equivalent to $\alpha'(v) > 0$, this is condition 2.

Thus conditions 1 and 2 are **sufficient** for α^* to be optimal, and so the function H^* defined in Proposition 3.2 with such α^*, the obtained pair (H^*,α^*) is an equilibrium. •

Conversely, I disagree the authors since these are not necessary conditions: indeed, a solution to (7) could be an optimum without the function J would be concave, id est Condition (8): keep it in mind as an exercise.

Anyway, the matter is still very difficult to solve because the implicit equation. Below two simple examples which can be solved.

3.3 Nonlinear equilibrium with Binomial law of V

Suppose that V is a Binomial random variable:

$$P(V=v_1) = p\ ;\ P(V=v_0) = q = 1-p, v_0 < v_1.$$

So let $\alpha(v_i) = \alpha_i,\ i = 0,1$. We obtain the closed formula for H_α:

$$H_\alpha(y) = \frac{pv_1 \exp(\sigma_u^{-2}(y\alpha_1 - \frac{1}{2}\alpha_1^2) + qv_0 \exp(\sigma_u^{-2}(y\alpha_0 - \frac{1}{2}\alpha_0^2)}{p\exp(\sigma_u^{-2}(y\alpha_1 - \frac{1}{2}\alpha_1^2) + q\exp(\sigma_u^{-2}(y\alpha_0 - \frac{1}{2}\alpha_0^2)}.$$

The sufficient optimality conditions are then:
when $v = v_i,\ v_i = E[(1+\sigma_u^{-2}\alpha_i Z)H(\alpha_i+Z)],\ \alpha_0 < \alpha_1$.

Proposition 3.5 *(prop 5 page 28)*
Under the hypothesis of a Binomial random variable V, $\alpha(V)$ is optimal yields $\alpha_1\alpha_0 = -\sigma_u^2$.

Then a unique equilibrium is defined by

$$H^*(y) = \frac{pv_1\exp(\sigma_u^{-2}(y\alpha_1 - \frac{1}{2}\alpha_1^2)) + qv_0\exp(\sigma_u^{-2}(y\alpha_0 - \frac{1}{2}\alpha_0^2))}{p\exp(\sigma_u^{-2}(y\alpha_1 - \frac{1}{2}\alpha_1^2)) + q\exp(\sigma_u^{-2}(y\alpha_0 - \frac{1}{2}\alpha_0^2))} \quad (10)$$

$$\alpha_1^* = \text{the unique positive solution of} \quad (11)$$

$$0 = E\left[\frac{1+\sigma_u^{-2}xZ}{p\exp[Z(\sigma_u^{-2}x + x^{-1}) + \frac{1}{2}\sigma_u^{-2}(x+\sigma_u^2 x^{-1})^2] + q}\right],$$

$$\alpha_0^* = -\sigma_u^2(\alpha_1^*)^{-1}. \quad (12)$$

Proof : (only an idea...) Actually the first condition is a degenerate linear system in (v_0, v_1) so we produce a necessary condition on (α_0, α_1). Nevertheless, the uniqueness is not so easy to prove, nor the last condition. Actually, in Appendix A, the authors prove condition (12) but only a numerical approach in Appendix B indicates the existence of a unique solution of (11). •

Naturally, higher the variance σ_u^2 of the noise trading is, deeper the market is, and thus larger the insider's optimal strategy. Note that the optimal strategy doesn't depend on explicit value v_i, but only depends on "good or bad news". The postannouncement value W_{1+} depends on $v_1 - v_0$.

3.4 Nonlinear equilibrium with continuous law of V

3.2. pages 29-30
Assumption A: We assume that law of V is such that there exists a strictly increasing function h satisfying $\Theta = h^{-1}(V)$ is a standard Gaussian random variable.
This means that if V admits a continuous distribution function F, $F^{-1}(x)$ defined as the minimum of y such that $F(y) = x$; let Φ be the standard Gaussian law distribution function, thus h could be $F^{-1}\circ\Phi$. Remark that actually this function h is almost surely differentiable.

In this subsection, we restrain the insider's strategies to be linear w.r.t. Θ:

$$\mathcal{A} = \{\alpha : V \mapsto ah^{-1}(V) + b \ ; \ a > 0, \ b \in \mathbb{R}\}.$$

So yields a "quasi-linear equilibrium", and $Y = \alpha(V) + Z = ah^{-1}(V) + b + Z = a\Theta + b + Z$. In such a case, the pair (Y, Z) is a Gaussian one. Remember $Z \sim \mathcal{N}(0, \sigma_u^2)$.

Proposition 3.6 *(prop. 7 p.30)*
When the distribution function F of V is a continuous one, and the insider's strategies $\alpha \in \mathcal{A}$, $h = F^{-1}\circ\Phi$, there exists a unique quasi-linear equilibrium as soon as there exists $(a,b) \in \mathbb{R}_*^+ \times \mathbb{R}$ such that h satisfies

$$E\left[h[\frac{a}{a^2+\sigma_u^2}(ax+b) + X] + \frac{a}{a^2+\sigma_u^2}(ax+b)h'[\frac{a}{a^2+\sigma_u^2}(ax+b)+X]\right] = h(x), \ \forall x \in \mathbb{R},$$

where X is a Gaussian random variable $N(-b\frac{a}{a^2+\sigma_u^2}, 1-\frac{a^4}{(a^2+\sigma_u^2)^2})$. In such a case, the equilibrium is the pair (α, H):

$$\begin{aligned}\alpha(V) &= ah^{-1}(V)+b, && (13)\\ H(y) &= E[h(\frac{ay}{a^2+\sigma_u^2}+J)],\ J\sim N(-\frac{ab}{a^2+\sigma^2}, \frac{\sigma_u^2}{a^2+\sigma_u^2}). && (14)\end{aligned}$$

Proof : The first step is to compute the function H using Corollary 3.3:

$$H_\alpha(y) = \frac{E[h(\Theta)\exp(\sigma_u^{-2}(y(a\Theta+b)-\frac{1}{2}(a\Theta+b)^2))]}{E[\exp(\sigma_u^{-2}(y(a\Theta+b)-\frac{1}{2}(a\Theta+b)^2))]},$$

remark that $(\sigma_u^{-2}(y(at+b)-\frac{1}{2}(at+b)^2)) - \frac{1}{2}t^2 = -\frac{1}{2}\frac{a^2+\sigma_u^2}{\sigma^2}(t-\frac{a(y-b)}{a^2+\sigma_u^2})^2 + C(a,b,y,\sigma_u)$, then a cancellation yields:

$$H_\alpha(y) = \int h(t)\sqrt{\frac{\sigma_u^2}{2\pi(a^2+\sigma_u^2)}}\exp -\frac{1}{2}\frac{a^2+\sigma_u^2}{\sigma_u^2}\left(t-\frac{a(y-b)}{a^2+\sigma_u^2}\right)^2 dt$$

thus, putting $t = \frac{a(y-b)}{a^2+\sigma_u^2} + u\sqrt{\frac{\sigma_u^2}{(a^2+\sigma_u^2)}}$,

$$H_\alpha(y) = \int h\left(\frac{a(y-b)}{a^2+\sigma_u^2} + u\sqrt{\frac{\sigma_u^2}{(a^2+\sigma_u^2)}}\right)\sqrt{\frac{1}{2\pi}}\exp -\frac{1}{2}u^2 du$$

which can be summarized as

$$H_\alpha(y) = E\left[h\left(\frac{a(y-b)}{a^2+\sigma_u^2} + U\sqrt{\frac{\sigma_u^2}{(a^2+\sigma_u^2)}}\right)\right],\ \text{where } U\sim\mathcal{N}(0,1). \quad (15)$$

Now since $a>0$ condition (8) in Proposition 3.4 proof is satisfied.
So we look for $(a,b)\in\mathbb{R}_+^*\times\mathbb{R}$ such that condition (7) could be satisfied $\forall x\in\mathbb{R}$,:

$$h(x) = E_{U,Z}\left[h\left(\frac{a(ax+Z)}{a^2+\sigma_u^2} + U\sqrt{\frac{\sigma_u^2}{a^2+\sigma_u^2}}\right) + \frac{a(ax+b)}{a^2+\sigma_u^2}h'\left(\frac{a(ax+Z)}{a^2+\sigma_u^2} + U\sqrt{\frac{\sigma_u^2}{a^2+\sigma_u^2}}\right)\right]$$

In the particular case where F is a Gaussian distribution function, meaning

$$h: x\mapsto x\sigma_V + p,\ h' = \sigma_V,$$

we obtain Kyle's equilibrium as a particular case, cf. (4) above: following the proposition, $\forall x\in\mathbb{R},\ (a,b)$ has to satisfies

$$x = \frac{a(ax)}{a^2+\sigma_u^2} + \frac{a(ax+b)}{a^2+\sigma_u^2} \Rightarrow a^2 = \sigma_u^2,\ b=0.$$

recovering Kyle's result: $\alpha^*(V) = \sqrt{\frac{\sigma_u^2}{\sigma_V^2}}(V-p)$.

Corollary 3.7 *cor 8, page 30.*
Let V a Gaussian random variable $N(p,\sigma_V^2)$, then there exists a unique quasilinear equilibrium which is exactly Kyle's equilibrium:

$$H^*(y) = \frac{1}{2}\sqrt{\frac{\sigma_V^2}{\sigma_u^2}}y + p \ ; \ \alpha^*(s) = (s-p)\sqrt{\frac{\sigma_u^2}{\sigma_V^2}}.$$

3.5 Risk adverse versus risk neutral

4. pages 30-33
Finally, the authors study the influence of utility, but in a simpler case: V is a Gaussian random variable $N(m,\sigma_V^2)$ and $H(y) = \lambda y + \mu,\ \lambda > 0$. The signal $S = V + \xi,\ \xi$ being a Gaussian random variable $N(0,\eta^2),\ \eta^2 < \sigma_V^2$.

3.5.1 Risk neutral case

Proposition 3.8 *Under these hypotheses, there exists a unique linear equilibrium*

$$H^*(y) = \frac{1}{2}\sqrt{\frac{\sigma_V^2-\eta^2}{\sigma_u^2}}y + m \ ; \ \alpha^*(s) = (s-m)\sqrt{\frac{\sigma_u^2}{\sigma_V^2-\eta^2}}.$$

Proof : The optimal value is $J(s) = \sup_\alpha E[(s - H(\alpha+Z))\alpha]$. Once again, H is constrained to be $y \mapsto \lambda y + \mu,\ \lambda > 0$. Since $E(Z) = 0$, we go to maximise
$\alpha \mapsto \alpha(s - \lambda\alpha - \mu)$, meaning $\alpha^* = \frac{s-\mu}{2\lambda}$.

Now we turn to a rational price, meaning

$$H(y) = E[V/\alpha + Z = y],\ \alpha + Z = \frac{V+\xi-\mu}{2\lambda} + Z,$$

the pair $(V, \frac{V+\xi-\mu}{2\lambda} + Z)$ being a Gaussian vector with mean $(m, \frac{m-\mu}{2\lambda})$ and covariance matrix:

$$\Gamma = \begin{pmatrix} \sigma_V^2 & \frac{\sigma_V^2}{2\lambda} \\ \frac{\sigma_V^2}{2\lambda} & \sigma_u^2 + \frac{\sigma_V^2+\eta^2}{4\lambda^2} \end{pmatrix}$$

Thus the conditional expectation is:

$$E[V/\alpha + Z = y] = m + \frac{\frac{\sigma_V^2}{2\lambda}}{\sigma_u^2 + \frac{\sigma_V^2+\eta^2}{4\lambda^2}}(y - \frac{m-\mu}{2\lambda}).$$

The identification to the function $\mu + \lambda y$ yields the result:

$$\lambda(\sigma_u^2 + \frac{\sigma_V^2+\eta^2}{4\lambda^2}) = \frac{\sigma_V^2}{2\lambda} \ \Rightarrow \ \lambda = \frac{1}{2}\sqrt{\frac{\sigma_V^2-\eta^2}{\sigma_u^2}}, \tag{16}$$

$$\mu = m - \frac{1}{2}(m-\mu) \ \Rightarrow \ \mu = m. \tag{17}$$

•

Remark 3.9 *As η^2 increases to σ_V^2, the insider observes less and less the value V. So the price pressure λ^* goes to 0 and α goes to infinity. The reason is that it is unrealistic to suppose the insider to be risk neutral.... So the authors go to an insider who is risk adverse.*

3.5.2 Risk adverse case

For instance, let the utility function $u_\gamma : x \mapsto \gamma e^{\gamma x}$, $\gamma < 0$, $|\gamma|$ is the risk aversion rate. Despite the unrealistic aspect, let us assume that $X_0 = x_0$ is a public observation, and we admit the following result:

Proposition 3.10 *prop 10 page 32.*
Suppose that V a Gaussian random variable $N(m, \sigma_V^2)$, and the insider's utility function is $u_\gamma : x \mapsto \gamma e^{\gamma x}$, $\gamma < 0$. Then there exists a unique LINEAR equilibrium defined as following.

$$H^*(y) = \lambda^* y + m - \frac{\gamma \eta^2 x_0 \lambda^* \phi^*}{1 - \lambda^* \phi^*}, \quad (18)$$

$$\alpha^*(s) = \phi^*[s - m + \frac{\gamma \eta^2 x_0}{1 - \lambda^* \phi^*}], \quad (19)$$

where ϕ^* is a positive solution to $g(\phi) =$

$$\gamma\eta^2(\sigma_V^2 + \eta^2)^2\phi^5 - (\sigma_V^4 - \sigma_u^4)\phi^4 + \gamma\sigma_u^2[\sigma_V^4 + 2\eta^2(\sigma_V^2 + \eta^2)]\phi^3 + 2\sigma_u^2\eta^2\phi^2 + \gamma\sigma_u^4\eta^2\phi + \sigma_u^4 = 0$$

and

$$\lambda^* = \frac{\sigma_V^2 \phi^*}{\sigma_u^2 + (\sigma_V^2 + \eta^2)(\phi^*)^2}.$$

Obviously, a positive solution exists since $g(0) = \sigma_u^4 > 0$ and $\lim_{\phi \to \infty} g(\phi) = -\infty$. But the author claims the uniqueness of this positive solution ϕ^*, nevertheless the proof is not so obvious and needs to be completed.

Remark that this equilibrium goes to the risk neutral equilibrium (above) when γ goes to 0.

Look also after the behaviour when γ goes to $-\infty$: in such a case the equation goes to

$$\phi[\eta^2(\sigma_V^2 + \eta^2)^2\phi^4 + \sigma_u^2[\sigma_V^4 + 2\eta^2(\sigma_V^2 + \eta^2)]\phi^2 + \sigma_u^4\eta^2] = 0$$

meaning there exists no more solution when the risk aversion rate increases too much, except $\phi^* = \alpha^* = 0$...

Finally, by contrast to the risk neutral case, the trading strategy doesn't explode even though the price pressure is very small.

4 Continuous time (cf. [3] 1992 and [9] 2003.)

Below we look at Back's work, then Cho's extension, both in continuous time frame.

4.1 Back model

Once again, the insider trader is risk neutral. As is the case in many other models, the continuous-time version is more tractable than the discrete-time version: all the processes could be semi-martingales. This also allows the price law to be more general than a Gaussian law. Moreover, it concerns non necessary linear functions X and P.

4.1.1 The model

Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space. Here, t belongs to $[0,1]$. There exists a constant riskless assets. The insider knows the expost terminal price V, the signal, with law F: trading price *after* the release of information. Once again we assume **Assumption A** (see above Section 3.4, there exists strictly increasing h such that $\Theta = h^{-1}(V)$ is a standard Gaussian random variable), $V \in L^2$, the interior U of $supp(V)$ is supposed to be an interval (finite or not).

Noise traders order according to a process Z, Brownian motion independant of V. The insider trader orders according to a process X, càdlàg semi-martingale $A+M$. The market maker observes the sum $Y = X + Z$. On time t, he set the price $P_t = H(Y_t, t)$, $H \in C^{2,1}(\mathbb{R}, [0,1])$, $H(Z_1, 1) \in L^2$, $\forall t,\ y \mapsto H(y,t)$ is strictly monotoneous. Thus H^{-1} exists and the insider observes P_t, he also knows Z_t.

Different filtrations are defined on $(\Omega, \mathcal{A}, \mathbb{P})$ concerning the different information: $\mathcal{F}^Z$, $\mathcal{F} := \mathcal{F}^Z \vee \sigma(V)$, $\mathcal{F}^Y$ for instance.

Let (B, X) the insider's portfolio, so his wealth is the semi-martingale $W_t = B_t + P_t X_t$. We suppose that this portfolio is self financing, so $dW_t = X_{t^-} dP_t$, since the riskless asset is constant, equal to 1.

Notice that the price could have a jump on terminal time: $P_1 = H(Y_1, 1) \neq V$.

Thus, assuming $W_0 = 0$, the insider's terminal wealth is:

$$W_1 = (V - P_1)X_1 + \int_0^1 X_{t^-} dP_t.$$

Integration by part formula (Ito formula) yields:

$$X_1 P_1 = \int_0^1 X_{t^-} dP_t + \int_0^1 P_{t^-} dX_t + [X, P]_t,$$

so

$$W_1 = \int_0^1 (V - P_{t-}) dX_t - [X, P]_t.$$

This can be interpreted as the value of the terminal position $V.X_1$, less the cost of acquiring it: $\int_0^1 P_{t^-} dX_t$. This formula can be linked to Kyle's formula concerning insider's profit (3)

$$\tilde{\pi} = (V - \tilde{p})\tilde{x} = (V - P(X(V) + \tilde{u}))X(V)$$

4.1.2 Notations and definitions

We denote $\mathcal{H}$ the set of functions H satisfying
$H \in C^{2,1}(\mathbb{R}, [0,1]),\ H(Z_1, 1) \in L^2,\ \forall t,\ y \mapsto H(y,t)$ is strictly monotoneous, so $\partial_y H > 0$.
$\mathcal{H}$ is the set of pricing rules.

Recall that the filtration generated by the process Z and the random variable V above is denoted as $\mathcal{F}$.

We denote $\mathcal{X}$ the set of $\mathcal{F}$-semi-martingales X such that when $H \in \mathcal{H}$,

$$E[\int_0^1 H^2(X_{t-} + Z_t, t)dt] < \infty,$$

(this hypothesis avoids the price to be too great, to rule out “doubling strategies"', page 394 l -12 (7), meaning to double the bet up to win), $\mathcal{X}$ is the set of the insider's trading strategies. Obviously X_t is a measurable function of V.

Definition 4.1 *A pricing rule H is said to be* **rational** *if $dt \otimes d\mathbb{P}$ almost surely*

$$H(Y_t, t) = E[V/\mathcal{F}_t^Y].$$

A strategy X is said to be **optimal** *if it maximizes the conditional terminal wealth, meaning the application on $\mathcal{X}$*

$$X \mapsto E[\int_0^1 (V - P_{t-})dX_t - [X, P]_1/\sigma(V)]. \tag{20}$$

4.1.3 Equilibrium

Definition 4.2 *A pair $(H, X) \in \mathcal{H} \times \mathcal{X}$ is an* **equilibrium** *if H is a rational price and X an optimal trading strategy.*

Let Φ the Gaussian distribution function of Z_1 and F this one of V, supposed to be continuous strictly increasing, so F^{-1} exists and is too continuous strictly increasing. The main results (Theorems 1, 2, 3 pages 396-397) are summarized below.

Theorem 4.3 *There exists an equilibrium (H, X) defined as following:*

$$H(y,t) = E[F^{-1} \circ \Phi(y + Z_1 - Z_t)], \tag{21}$$

$$X_t(V) = (1-t)\int_0^t \frac{\Phi^{-1} \circ F(V) - Z_s}{(1-s)^2} ds. \tag{22}$$

Proof: This “verification theorem" is a consequence of the following technical lemmas.

Lemma 4.4 *Let h be a differentiable strictly monotoneous function s.t. $h(Z_t) \in L^1$. Suppose the pricing rule on $\mathbb{R} \times [0,1]$ is*

$$H : (y,t) \mapsto E[h(y + Z_1 - Z_t)].$$

Let on $U \times \mathbb{R}$, $j(v,y) = \int_y^{h^{-1}(v)} (v - h(x))dx$, and on $U \times \mathbb{R} \times [0,1]$,

$$J(v,y,t) = E[j(v, y + Z_1 - Z_t)],$$

where v is considered as a constant. If $J(v,0,0) < \infty$, then J is a smooth solution of the Bellman equation on $U \times \mathbb{R} \times]0,1[$

$$\max_{\alpha \in \mathbb{R}} \left[\partial_t J + \partial_y J \alpha + \frac{1}{2}\sigma^2 \partial^2_{yy} J + (V - H)\alpha \right] = 0, \tag{23}$$

and boundary condition:

$$J(v,y,1) > J(v, h^{-1}(v), 1) = 0 \quad \forall v \in U, \forall y < h^{-1}(v). \tag{24}$$

Be cautious below differentiating the expectation, but formally, with such a definition, J satisfies $\partial_y J = H - V$, and using Ito formula for $j(v, y + Z_1 - Z_s)$ from t to 1 and its expectation, we check $\partial_t J + \frac{1}{2}\sigma^2 \partial^2_{yy} J = 0$, so yields (23).

Boundary condition (24) is a consequence of h strict monotonicity: $y < x < h^{-1}(v) \Rightarrow h(y) < h(x) < v \Rightarrow J(v,y,1) = j(v,y) > 0$. •

Lemma 4.5 *Let H an arbitrary pricing rule. Suppose a nonnegative, smooth solution of (23), (24), J, exists. Then for any trading strategy X, the expected profit (20) is no larger than $E[J(V,0,0)]$.*
Any trading strategy $X := A + M$ which has continuous paths, for which $M = 0$, and which implies $H(Y_1, 1) = V$ almost surely, yields an expected profit equal to $E[J(V,0,0)]$ and is therefore an optimal strategy.

If X is any trading strategy that includes discrete orders, or has a nonzero local martingale part, or does not imply $H(Y_1,1) = V$ almost surely, then the expected profit from X is strictly less than $E[J(V,0,0)]$.

- Since J, H satisfy (23), $\partial_y J = V - H$, $\partial_t J + \frac{1}{2}\sigma^2 \partial^2_{yy} J \leq 0$. Using Itô developpement we get

$$J(V, Y_1, 1) - J(V,0,0) = \int_0^1 \partial_y J(V, Y_s, s) dY_s + \int_0^1 \partial_t J(V, Y_s, s) ds$$

$$+ \int_0^1 \frac{1}{2} \partial^2_{yy} J(V, Y_s, s) d[Y^c, Y^c]_s + \sum_{0 \leq s \leq 1} [J(V, Y_s, s) - J(V, Y_{s-}, s) - \Delta_s X \partial_y J(V, Y_{s-}, s)].$$

Technical computations and independence between Z and V yield the result concerning the expected profit:

$$E[\int_0^1 (V - P_{t-}) dX_t - [X,P]_1 / \sigma(V)] \leq J(V,0,0).$$

- The equality is reached when X is a continuous finite variation process and yields to $H(Y_1, 1) = V$).

- If not (meaning X doens't satisfy one of the previous conditions), then

$$E[\int_0^1 (V - P_{t-})dX_t - [X, P]_1/\sigma(V)] < J(V, 0, 0).$$

$\bullet$

Lemma 4.6 *If the insider trader follows the strategy (22)*

$$X_t(V) = (1-t)\int_0^t \frac{\Phi^{-1} \circ F(V) - Z_s}{(1-s)^2} ds,$$

then the process Y is a $\mathcal{F}$-Brownian bridge with instantaneous variance σ^2, terminating at $\Phi^{-1} \circ F(V)$.
Moreover Y is a $\mathcal{F}^Y-$Brownian motion with zero drift and instantaneous variance σ^2.

Actually, with this strategy

$$dY_t = \frac{\Phi^{-1} \circ F(V) - Y_t}{1-t} dt + dZ_t,$$

which is (6.23) in Karatzas Schreve 1987, page 358, so by (6.26)

$$Y_t = t\Phi^{-1} \circ F(V) + (1-t)\int_0^t \frac{dZ_s}{1-s}$$

so that Y is a $\mathcal{F}-$Brownian bridge: actually it can be proved identifying dY_t as $dX_t + dZ_t$ with X_t defined in the lemma, and $dY_t = d[t\Phi^{-1} \circ F(V) + (1-t)\int_0^t \frac{dZ_s}{1-s}]$.

More precisely, by a developpement and an integration by part:

$$X_t(V) + Z_t = (1-t)\int_0^t \frac{\Phi^{-1} \circ F(V) - Z_s}{(1-s)^2} ds + Z_t = Y_t.$$

Moreover, with respect to $\mathcal{F}^Y$, $\frac{1}{\sigma}Y$ is a $\mathcal{F}^Y-$Brownian motion (think of innovation process in filtering theory) $\bullet$

Proof Theorem 4.3 needs to check two points

- $H(y, t)$ is a rational price, meaning $H(y, t) = E[H(Y_1, 1)/Y_t = y]$,
- the proposed strategy is optimal.

The first point is a consequence of Lemma 4.6: equality in law between Z on $\mathcal{F}$ and Y on $\mathcal{F}^Y$, so

$$E[H(Z_1,1)/Z_t = y] = E[F^{-1}\circ\Phi(Z_1)/Z_t = y] = E[F^{-1}\circ\Phi(y+Z_1-Z_t)/Z_t = y],$$

$$H(y,t) = E[F^{-1}\circ\Phi(y+Z_1-Z_t)]$$

For the second point, $H(Y_1,1) = F^{-1}\circ\Phi(Y_1) = V$ using Lemma 4.6 and independence between V and Z. Since the proposed strategy satisfies this property, and is continuous with finite variation, Lemma 4.5 concludes the proof.

Before proving converse and uniqueness results, we need the following two lemmas.

Lemma 4.7 *Let H an arbitrary pricing rule. If there exists a smooth solution to (23), (24), then the process $t \mapsto H(Z_t,t)$ is a martingale on the filtration $\mathcal{F}$.*
If the trading strategy $X := A + M$ has continuous paths, for which $M = 0$,

$$\forall t,\ H(Y_t,t) = H(0,0) + \int_0^t \partial_y H(Y_s,s)dY_s.$$

Actually author's proof doesn't use (23), (24) but the following consequences:

$$\begin{aligned}
\partial_y J(V,y,t) &= H(y,t) - V,\\
(\partial_t J &+ \frac{1}{2}\sigma^2\partial^2_{yy}J)(V,y,t) = 0,\\
J(V,y,t) &= E[J(V,y+Z_1-Z_t)/V].
\end{aligned}$$

So, $H(y,t) = \partial_y J(V,y,t) + V$.
Differentiating the second relation above w.r.t. y, $(\partial_t H + \frac{1}{2}\sigma^2\partial^2_{yy}H)(y,t) = 0$ so

$$dH(Z_t,t) = \int_0^t \partial_y H(Z_s,s)dZ_s$$

is a $\mathcal{F}-$martingale.

In case of continuous finite variation process X, Itô formula applied to $H(Y_t,t)$ concludes the proof.

•

Lemma 4.8 *Let (H,X) be an equilibrium and J a smooth solution to (23), (24). Then, on the filtration $\mathcal{F}^Y$, the process Y is a Brownian motion with zero drift and instantaneous variance σ^2.*

Proof Lemma 4.5 and X optimal shows that X is a continuous finite variation process. Then, since (H,X) is an equilibrium and J a smooth solution to (23,24), Lemma 4.7 yields:

$$dH(Y_t,t) = \partial_y H(Y_t,t)dY_t.$$

Moreover, H being a pricing rule, it is a $\mathcal{F}^Y-$martingale and $\partial_y H > 0$, thus

$$dY_t = (\partial_y H(Y_t,t))^{-1} dH(Y_t,t)$$

is a $\mathcal{F}^Y-$ local martingale.
Finally, remember $Y_t = Z_t + X_t$ with X a finite variation process, so the brackets $\langle Y\rangle_t = \langle Z\rangle_t = \sigma^2 t$. Lévy Theorem concludes the proof. •

This first theorem concerns uniqueness of equilibrium.

Theorem 4.9 *The pricing rule H (21) is the unique equilibrium price for which there exist a nonnegative constant σ^2 and a nonnegative smooth function J on $U \times \mathbb{R} \times [0,1]$ satisfying the Bellman equation on $U \times \mathbb{R} \times]0,1[$:*

$$\max_{\alpha\in\mathbb{R}} \left[\partial_t J + \partial_y J\alpha + \frac{1}{2}\sigma^2 \partial^2_{yy} J + (V-H)\alpha\right] = 0,$$

and boundary condition:

$$J(v,y,1) > J(v,h^{-1}(v),1) = 0 \quad \forall v \in U, \forall y < h^{-1}(v)$$

where $h : x \mapsto H(x,1)$.

Proof Let H be any equilibrium pricing rule such that there exists J a smooth solution to (23), (24).
Once again $\partial_t H + \frac{1}{2}\sigma^2\partial^2_{yy} H = 0$, so Itô formula applied to the process $s \mapsto H(y + Z_s - Z_t, s)$ from t to 1 get

$$H(y + Z_1 - Z_t, 1) - H(y,t) = \int_t^1 \partial_y H(y + Z_s - Z_t, s) dZ_s$$

is a $\mathcal{F}-$martingale and necessarily

$$H(y,t) = E[H(y + Z_1 - Z_t, 1)].$$

Lemma 4.5 says that an optimal strategy X is necessarily a continuous finite variation process and leads to Y_1 such that $H(Y_1,1) = V$. Let $h : y \mapsto H(y,1)$ which is strictly increasing, so $Y_1 = h^{-1}(V)$,

$$\forall a \in \mathbb{R},\ \mathbb{P}\{Y_1 \le a\} = \mathbb{P}\{V \le h(a)\} = F \circ h(a).$$

But Lemma 4.8 gives Y_1 law $\mathcal{N}(0,\sigma^2)$, thus $\Phi = F \circ h$ concludes the proof of Point 1. •

The following characterizes equilibrium

Theorem 4.10 *Let (H,X) an equilibrium. If there exists a function J such that H, J satisfies (23,24). Then*

$$dP_t = \partial_y H(Y_t,t) dY_t$$

and the process Y is distributed as a $\mathcal{F}^Y$-Brownian motion with variance σ^2.
The process $H(Z_t,t)$ is a $\mathcal{F}$-martingale.
If $F(v) = \int_0^v f(u)du$, and $E[\partial_y H(Z_1,1)] < \infty$, then $\partial_y H(Z_.,.)$ is a $\mathcal{F}$-martingale and $\partial_y H(Y_.,.)$ is a $\mathcal{F}^Y$-martingale.

Comment: the boundary condition (24) is a little unusual, one might expect J to be null at final time. The interpretation could be that J is defined by continuity at final time, and the remaining value $J(V, Y_t, t)$ at time t close to final time is near to zero if and only if Y_t is close to $h^{-1}(V)$, meaning P_t close to V.
Proof

- $P_t = H(Y_t, t)$ so the first assertion is included in Lemma 4.8.
- The second assertion is included in Lemma 4.8.
- The third assertion concerns $X = 0$, so actually

$$dH(Y_t, t) = \partial_y H(Z_t, t) dZ_t$$

  is a $\mathcal{F}-$martingale.
- Finally assuming that V law admits a density of probability f, if $h = F^{-1} \circ \Phi$, then $h' = \frac{\Phi'}{f \circ h}$.
  Actually, the hypothesis $E[\partial_y H(Z_1, 1)] < \infty$ allows us to commute differentiation and integration, thus $G = \partial_y H$ satisfies the same PDE as $H : \partial_t G + \frac{1}{2}\sigma^2 \partial^2_{yy} G = 0$. This proves that $\partial_y H(Z_., .)$ is a $\mathcal{F}-$martingale and $\partial_y H(Y_., .)$ is a $\mathcal{F}^Y-$martingale.

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Exercices Suppose V has the Gaussian law $N(p, \sigma^2)$, or $\log V$ has a Gaussian law. Then solve the problem in these two cases.

4.1.4 Conclusion

cf. page 404 [3]:"The key aspect of the continuous-time model is that the informed trader can move continuously up or down the residual supply curve. This flexibility on the part of the insider, combined with risk neutrality, helps to pin down the equilibrium beliefs of market makers. In equilibrium, the insider has many optima, because there is no expected cost in moving up and then back down the supply curve, or vice-versa, or simply delaying trading....In a competitive equilibrium with a risk-neutral agent and a fixed risk-free rate, expected returns on all assets are uniquely determined, but any portfolio is optimal for the risk-neutral agent.

The situation is very different when agents are risk adverse. It is important to determine to what extent the results of this articleare robust to risk adversion.

The model was solved in this article without recourse to the filtering technology used by Kyle. This permitted the analysis of general assets value distributions."

4.2 Insider trading: uniqueness and risk aversion

Here is a continuous time model (cf. [9] 2003) as in Back's paper: it is an extension of the previous models. K.H. Cho considers continuous time: $t \in [0,1]$, and here the equilibrium is nonlinear; two cases are studied: risk neutral and risk adverse cases. The main tool is optimal control theory, for instance think of Bellman's principle.

All the random variables are defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we further define some filtrations included in $\mathcal{A}$.

4.2.1 The model, notations and definitions

As previously there are three agents:

- the insider trader has access to a private observation of the *expost* liquidation value of the risky assets, the random value V, and invests X_t given V,

- the uninformed noise trader trades randomly: Z_t, according to a centered Gaussian law,

- the market maker set the price P *efficiently*, meaning conditionnally on the public information he has about the assets quantities tradered by all the traders ($Y_t = X_t + Z_t$, $t \in [0,1]$, which is the market maker's information). Thus

$$P_t = E[V/Y_s, s \leq t],\ t < 1.$$

The author here supposes $Y_0 = 0$. As it is usual, let us denote the complete càd filtration generated by a process D as $\mathcal{F}^D$, e.g. $\mathcal{F}^Y$, $\mathcal{F}^P$ and so on. Using this notation yields

$$P_t = E[V/\mathcal{F}_t^Y].$$

This introduces a functional, so called "pricing rule", P such that:

$$P_t = P(t,\ Y_s, s \leq t).$$

The set of such functionals is denoted as $\mathcal{P}$. As it was stressed above, P_1 is not necessarily V. The following hypotheses are assumed:

($\mathbf{H_1}$) The distribution function F of V is continuous strictly increasing.
Let us denote Φ the distribution function of $\mathcal{N}(0,1)$, then the law of $\Theta := \Phi^{-1} \circ F(V)$ is $\mathcal{N}(0,1)$.
Remark that (H_1) is equivalent to the existence of a function h (namely $F^{-1} \circ \Phi$) continuous strictly increasing such that the law of $h^{-1}(V)$ (denoted below as Θ) is $\mathcal{N}(0,1)$.

($\mathbf{H_2}$) The process $Z_.$ is $\sigma B_.$ with $\sigma \neq 0$ and B is a Brownian motion with respect to $(\mathbb{P}, \mathcal{F}^B)$, independent of V.

The insider's information is modellized by the filtration $(\mathcal{F}_t = \mathcal{F}_t^P \vee \sigma(V), t \geq 0)$. Actually, $\mathcal{F}_\mathbf{t}^\mathbf{X} = \mathcal{F}_\mathbf{t}$ (to be proved below). Denote $\mathcal{S}$ the set of the insider's strategies, $(X_t,\ t \in [0,1[)$, u his utility function, W_1 his terminal wealth, after the information V is revealed, depending on the pair $(P, X) \in \mathcal{P} \times \mathcal{S}$.

Definition 4.11 *For a given pricing rule $P \in \mathcal{P}$, a trading strategy X^* is said to be* <u>*P-optimal*</u> *if*

$$\forall X \in \mathcal{S},\ E[u(W_1(P,X))/V] \leq E[u(W_1(P,X^*))/V].$$

Definition 4.12 *For a given trading strategy $X \in \mathcal{S}$, a pricing rule $P^* \in \mathcal{P}$ is said to be* <u>*X-rational*</u> *if*

$$P^*(t,\ Y_s, s \leq t) = E[V/\mathcal{F}_t^Y]\ \ dt \otimes d\mathbb{P} a.s.$$

Definition 4.13 *An equilibrium is a pair $(P,X) \in \mathcal{P} \times \mathcal{S}$ such that P is X-rational and X is P-optimal.*

4.2.2 Survey of different $\mathcal{P}$ and $\mathcal{S}$ spaces

- Kyle 1985 (only Section 4, previous ones concern discrete time):
  The processes $P \in \mathcal{P}_k$ and $X \in \mathcal{S}_k$ are as following

  $$dP_t = \lambda_t dY_t\ ;\ dX_t = \phi_t(V - P_t)dt,$$

  λ and ϕ are real deterministic functions on $[0,1]$, $\lambda_t > 0$.
  In this model V is a Gaussian random variable, $W_1 = (V - P(X(V) + \tilde{u}))X(V)$.
  The disadvantage is that P_t could be non positive, and the set $\mathcal{S}_k$ is too small.

- Cho and El Karoui 2000.
  The time is discrete, $t = 0,1$. So these spaces don't concern processes spaces. This paper is to be linked to Kyle sections 1-3 and Chapter 2. Anyway,

  $$W_1 = VX_0 + (V - H(X+Z))X.$$

- Back 1992.
  The spaces here are

  $$\mathcal{P}_b = \{P_t = H(t,Y_t),\ H \in C^{1,2}\}\ ;\ \mathcal{S}_b = \{\mathcal{F} - \text{càd-làg semi-martingales}\}.$$

  He obtains uniqueness of equilibrium in Gaussian case.
  $W_1 = \int_0^1 (V - P_{t-})dX_t - [X,P]_t = (V - P_1)X_1 + \int_0^1 X_{t-} dP_t.$

- Here the author sets

  $$\mathcal{P} = \{P : t \mapsto H(t, \int_0^t \lambda_s dY_s)\ ;\ H \in C^{1,2}(]0,1[\times\mathbb{R}),\ \partial_y H(t,y) > 0\ \forall t\ ;\ \lambda \in C^1(]0,1[,\mathbb{R}_*^+)\}.$$

Thus, $P \in \mathcal{P}$ is denoted as (H,λ). He also assumes:

$$(H_3) \qquad H(.,\int_0^{.} \lambda_s dY_s) \in L^2([0,1] \times \Omega), \qquad (25)$$

to avoid "**doubling strategies**" (cf. Definition 2.5).

$$\mathcal{S} = \{X : t \mapsto \int_0^t \alpha_s ds,\ \alpha\ \mathcal{F}-adapted\}.$$

In this model

$$W_1(P, \alpha) = \int_0^1 (V - H(t, \int_0^t \lambda_s dY_s^\alpha))\alpha_t dt.$$

Remark the inclusions:

$$\mathcal{P}_k \text{ and } \mathcal{P}_b \subset \mathcal{P}\ ;\ \mathcal{S}_k \subset \mathcal{S} \subset \mathcal{S}_b.$$

But notice that, anyway, the optimal strategy in Back actually belongs to $\mathcal{S}$.

4.2.3 Links between different filtrations

Since $\lambda > 0$, setting $\xi_t = \int_0^t \lambda_s dY_s$, $\mathcal{F}^\xi = \mathcal{F}^Y$. By definition of a rational pricing rule, P is $\mathcal{F}^Y$-adapted so $\mathcal{F}^P \subset \mathcal{F}^Y$. Conversely, since $\forall t,\ y \mapsto H(t, y)$ is invertible, ξ is $\mathcal{F}^P$-adapted, so $\mathcal{F}^\xi \subset \mathcal{F}^P$ thus finally

$$\mathcal{F}^P = \mathcal{F}^Y = \mathcal{F}^\xi.$$

Otherwise, $Z = Y - X$ is $\mathcal{F}$-adapted (since by definition $\mathcal{F}^Y = \mathcal{F}^Y \vee \sigma(V)$), so B is $\mathcal{F}-$adapted, moreover by definition $\sigma(V) \subset \mathcal{F}_t$ thus: $\mathcal{F}_t^B \vee \sigma(V) \subset \mathcal{F}_t$.
Conversely, $Y_t = \sigma B_t + X_t$ is $\mathcal{F}_t^B \vee \mathcal{F}_t^X$-measurable, so $\mathcal{F}_t^Y \subset \mathcal{F}_t^B \vee \sigma(V)$ and finally

$$\forall t \geq 0,\ \mathcal{F}_t^B \vee \sigma(V) = \mathcal{F}_t.$$

Thus using the independance of B and V, B is also a $(\mathcal{F}, \mathbb{P})$ Brownian motion.

4.2.4 Bellman's optimality principle

cf. above Section 1.3.4
We now set a model of optimal control: the control process is the strategy X^α; let the conditional value function:

$$\Phi_\alpha(t) = ess\sup\{E[u(W_1(P, \tilde{\alpha}))/\mathcal{F}_t^\alpha],\ \tilde{\alpha} \in \mathcal{S}, \tilde{\alpha}_{|[0,t]} = \alpha_{|[0,t]}\},$$

where $\mathcal{F}_t^\alpha = \mathcal{F}^{Z+X^\alpha} \vee \sigma(V)$.
Remember that

$$W_t(P, \alpha) = \int_0^t [V - H(s, \int_0^s \lambda_u(\alpha_u du + dZ_u))]\alpha_s ds.$$

Remark that the system is a Markovian one, so actually there exists a measurable function such that (cf. Lemma 1.7, prop 2.3. in [42] or Proposition 5.3 in next Chapter):

$$\Phi_\alpha(t) = \Phi_\alpha(t, Y_t, V).$$

The following lemma will be usefull, using Bellman principle, cf. above Lemma 1.7.

Lemma 4.14 *Let $(H,\lambda) \in \mathcal{P}$ and $\alpha \in \mathcal{S}$. Then the three properties are equivalent:*

$$dt \otimes d\mathbb{P} \quad a.s. \partial_t H(t,\xi_t^\alpha) + \frac{1}{2}\sigma^2\lambda_t^2 \partial_{y^2}^2 H(t,\xi_t^\alpha) = 0, \tag{26}$$

$$\forall t \in [0,1], \;\; \hat{\alpha}_t = E[\alpha_t/\mathcal{F}_t^Y] = 0, \tag{27}$$

$$\xi_.^\alpha \text{ is a } \mathcal{F}^Y- \text{ martingale and } \langle \xi^\alpha, \xi^\alpha \rangle_t = \sigma^2 \int_0^t \lambda_s^2 ds. \tag{28}$$

Proof Let (H,λ) be a pricing rule in the class $\mathcal{P}$. Itô formula implies:

$$dH(t,\xi_t) = (\partial_t H(t,\xi_t) + \frac{1}{2}\sigma^2\lambda_t \partial_{x^2}^2 H(t,\xi_t))dt + \partial_x H(t,\xi_t)d\xi_t.$$

By definition, $t \mapsto H(t,\xi_t)$ is a $\mathcal{F}^Y$-martingale (rational price). On the other hand, $d\xi_t = \lambda_t dY_t = \lambda_t(\sigma dB_t + \alpha_t dt)$. Let the innovation process

$$dI_t = \lambda_t(\sigma dB_t + (\alpha_t - \hat{\alpha}_t)dt) = \lambda_t(dY_t - \hat{\alpha}_t dt)$$

which is also a $\mathcal{F}^Y$-martingale (cf. above Section 1.3.3). Thus,

$$dH(t,\xi_t) = (\partial_t H(t,\xi_t) + \frac{1}{2}\sigma^2\lambda_t \partial_{x^2}^2 H(t,\xi_t))dt + \partial_x H(t,\xi_t)dI_t + \lambda_t\hat{\alpha}_t dt.$$

This proves the equivalence between (26) and (27).

Finally, $d\xi_t = dI_t + \hat{\alpha}_t dt$ is a $\mathcal{F}^Y$-martingale is equivalent to $\hat{\alpha}_t = 0$ and the brackets of ξ and I are both equal to $\sigma^2\int_0^t \lambda_s^2 ds$.

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4.2.5 Risk neutral insider

Here $u = I_d$, recall $W_t(P,\alpha) = \int_0^t (V - H(s,\xi_s^\alpha))\alpha_s ds$, and $\forall t < 1$

$$\Phi_\alpha(t,\xi_t^\alpha) = W_t(P,\alpha) + ess\sup\{E[\int_t^1 (V - H(s,\xi_s^{\tilde{\alpha}}))\tilde{\alpha}_s ds/\mathcal{F}_t^\alpha], \;\; \tilde{\alpha}_{|[0,t]} = \alpha_{|[0,t]}\}.$$

The second term traditionnally is denoted as $J(t,\xi_t^\alpha)$. To solve the optimal problem, we need to get the conditional value Φ_α as a $\mathcal{F}^\alpha$-martingale, but, first, it has to be a supermartingale. We use Ito formula, assuming Φ^α smooth enough:

$$d\Phi_\alpha(t,\xi_t^\alpha) = (V - H(t,\xi_t^\alpha))\alpha_t dt + [\partial_t J(t,\xi_t) + \lambda_t\alpha_t\partial_y J(t,\xi_t) + \frac{1}{2}\sigma^2\lambda_t^2\partial_{y^2}J(t,\xi_t)]dt + \sigma\lambda_t\partial_y J(t,\xi_t)dB_t. \tag{29}$$

To necessarily be a supermartingale for all α, the finite variation part above has to be non positive, so yields:

$$\forall \alpha \in \mathcal{S}, \;\; (V - H(t,\xi_t^\alpha))\alpha_t + \partial_t J(t,\xi_t) + \lambda_t\alpha_t\partial_y J(t,\xi_t) + \frac{1}{2}\sigma^2\lambda_t^2\partial_{y^2}J(t,\xi_t) \leq 0 \;\; dt \otimes d\mathbb{P} a.s.$$

This is a linear function of α, so we get null the α coefficient, the remaining being non positive, $dt \otimes d\mathbb{P}$ a.s. and since the support of process ξ could be $\mathbb{R}$ we get $\forall y \in \mathbb{R}$:

$$V - H(t,y) + \lambda_t \partial_y J(t,y) = 0, \tag{30}$$

$$\partial_t J(t,y) + \frac{1}{2}\sigma^2 \lambda_t^2 \partial_{y^2} J(t,y) \leq 0. \tag{31}$$

The optimality is obtained when (31) is null.

By definition,

$$W_1(P,\alpha) = \int_0^1 (V - H(s,\xi_s^\alpha))\alpha_s ds.$$

It could be $W_1(P,\alpha) \neq \Phi_\alpha(1,\xi_1^\alpha)$. But actually α optimal makes the process $\Phi_\alpha(.,\xi_.^\alpha)$ a *U.I.* $\mathcal{F}^Y$-martingale, thus continuous. So

$$\lim_{t\mapsto 1} \Phi_\alpha(t,\xi_t^\alpha) = \Phi_\alpha(1^-,\xi_1^\alpha) = \int_0^1 (V - H(s,\xi_s^\alpha))\alpha_s ds.$$

Actually, if $\xi_\alpha^1 = [H(1,.)]^{-1}(V)$, i.e. $H(1,\xi_\alpha^1) = V$, then, $J(1,\xi_1) = 0$ as soon as $\xi_1 = h(V)$, this yields a boundary condition.

Proposition 4.15 *sufficient condition.*
Let a pricing rule $(H,\lambda) \in \mathcal{P}$ satisfying the system (30), (31) associated to $\alpha \in \mathcal{S}$ with (31)=0. Then necessarily λ is a constant, $E[\alpha_t/\mathcal{F}_t^Y] = 0$ a.s., $V = H(1,\xi_1^\alpha)$ a.s. and the pair $((H,\lambda),\alpha)$ is an equilibrium.

Proof

First step is to differentiate (30) w.r.t. y: $\partial_y H(t,y) = \lambda_t \partial_{y^2}^2 J(t,y)$.
This equation combined with (31) yields $\partial_t J(t,y) + \frac{1}{2}\sigma^2 \lambda_t \partial_y H(t,y) = 0$. This one is differentiated once again w.r.t. y:

$$\partial_{ty}^2 J(t,y) + \frac{1}{2}\sigma^2 \lambda_t \partial_{y2}^2 H(t,y) = 0. \tag{32}$$

Now let us differentiate (30) w.r.t. t: $\lambda_t' \partial_y J(t,y) + \lambda_t \partial_{ty}^2 J(t,y) - \partial_t H(t,y) = 0$.
But (30) says that $\partial_y J(t,y) = -\lambda_t^{-1}(V - H)$, so

$$\lambda_t' \lambda_t^{-1}(V - H) - \lambda_t \partial_{ty}^2 J(t,y) + \partial_t H(t,y) = 0 \tag{33}$$

and cancelling $\partial_{ty}^2 J(t,y)$ between (32) and (33) yield the following PDE for the function H:

$$\partial_t H(t,y) + \frac{1}{2}\sigma^2 \lambda_t \partial_{y2}^2 H(t,y) + \lambda_t' \lambda_t^{-2}(V - H) = 0.$$

But, H can't depend on the random variable V. So, necessarily, $\lambda' = 0$ and actually H is solution of the following PDE:

$$\partial_t H(t,y) + \frac{1}{2}\sigma^2 \lambda_t \partial_{y2}^2 H(t,y) = 0, \quad H(1,y) = E[V/Y_1 = y]. \tag{34}$$

Second step uses equivalences in Lemma 4.14, so $\hat{\alpha}_t = E[\alpha_t/\mathcal{F}_t^Y] = 0$ and $\xi^\alpha_. = \lambda Y_.$ is a $\mathcal{F}^Y-$martingale.

Finally, (31)$= 0$ implies that Φ^α is a martingale so Lemma 1.7 proves that α is optimal. A $\mathcal{F}^Y$-martingale being continuous, we get $\lim_{t \mapsto 1-} \Phi^\alpha(t) = \int_0^1 (V - H(s, \xi_s^\alpha)\alpha_s ds$ and the price $P_1 = H(1, \xi_1^\alpha)$ has to be exactly V (on final time, the information is revealed). So we get the terminal condition $V = H(1, \xi_1^\alpha)$.

With the solution H of (34) and λ a constant, $((H, \lambda), \alpha)$ is an equilibrium. •

Proposition 4.16 *Necessary condition*
Let V satisfying Hypothesis (H_1) such that $V = h(\Theta)$, (meaning h strictly increasing continuous and $\Theta \sim \mathcal{N}(0,1)$). Then an equilibrium pricing rule $(H, \lambda) \in \mathcal{P}$ necessarily satisfies:

$$\forall y \in \mathbb{R},\ H(1, y) = h(y/\lambda\sigma)\ ;\ H(t, y) = E[h(\frac{y}{\lambda\sigma} + \Theta_t)]$$

where $\Theta_t \sim N(0, 1-t)$.
Moreover the optimal ξ_1^α is the unique solution to $V = H(1, x)$ a.s.

Proof : Since $(H, \lambda) \in \mathcal{P}$ is an equilibrium pricing rule, there exists a strategy α such that $((H, \lambda), \alpha)$ is an equilibrium. As in the previous proof, necessarily there exists a regular function J such that the equalities (30) and (31) are satisfied. Thus the previous proposition can be applied: once again, λ is a constant and $\forall t \in [0,1],\ \hat{\alpha}_t = E[\alpha_t/\mathcal{F}_t^Y] = 0$; now using Lemma 4.14, the process ξ^α is a $\mathcal{F}^Y$-martingale with bracket $\sigma^2\lambda^2 t$, meaning that ξ_t^α law is $\mathcal{N}(0, \sigma^2\lambda^2 t)$.

On the other hand, Itô formula and (31) yield:

$$J(1, \xi_1^\alpha) - J(t, \xi_t^\alpha) = \int_t^1 \partial_y J(s, \xi_s^\alpha) d\xi_s^\alpha$$

and using (30)

$$J(1, \xi_1^\alpha) - J(t, \xi_t^\alpha) = \lambda \int_t^1 (H(s, \xi_s^\alpha) - V)(\alpha_s ds + dZ_s).$$

The right second term is a local $\mathcal{F}$-martingale null at time $t = 1$, but hypothesis (H_3) (25) makes it a true martingale:

$$E[J(t, \xi_t^\alpha) - J(1, \xi_1^\alpha)/\mathcal{F}_t] = E[\lambda \int_t^1 (V - H(s, \xi_s^\alpha))\alpha_s ds/\mathcal{F}_t].$$

By definition of process J, the right hand is maximum when α is optimal and the maximum actually is $J(t, \xi_t^\alpha)$, so $\forall t,\ E[J(1, \xi_1^\alpha)/\mathcal{F}_t] = 0$, thus $J(1, \xi_1^\alpha) = 0$. Following Back recalled in (24), this means that

$$V = H(1, \xi_1^\alpha).$$

Thus, $h(\Theta) = H(1,\xi_1^\alpha)$ almost surely, laws of $\sigma\lambda\Theta$ and ξ_1^α are the same, so we identify $\forall y \in \mathbb{R},\ \ h(y) = H(1,\sigma\lambda y)$ which concludes the proof.

Finally, H is solution of the so-called heat equation (34) with terminal condition as above and the last assertion is straightforward. •

The following proposition provides an explicit optimal solution.

Proposition 4.17 *Let V satisfying Hypothesis (H_1) such that $V = h(\Theta)$. Let a pricing rule $(H,\lambda) \in \mathcal{P}$ defined as following:*

$$\forall y \in \mathbb{R},\ H(t,y) = E[h(y+\Theta_t)]\ ;\ \lambda = 1/\sigma$$

where $\Theta_t \sim N(0,1-t)$, and a strategy $\alpha \in \mathcal{S}$:

$$\alpha_t = \sigma\frac{h^{-1}(V)-\xi_t}{1-t}.$$

Then this pair $(H,\lambda) \in \mathcal{P}$, $\alpha \in \mathcal{S}$ is an equilibrium.

Remark 4.18 *$\lambda = 1/\sigma$ means that $P_t = H(t,\frac{1}{\sigma}Y_t)$.*

Proof : *prop 2, Cho's proof page 68.... but actually I am not convinced...*
It is enough to check (30)(31) and $\hat{\alpha}_t = 0$. Then we apply Proposition 4.15.
The tools are: ξ^α is a $\mathcal{F}^Y$-martingale, $\langle\xi^\alpha\rangle_t = t$, and $\mathcal{F}^Y = \mathcal{F}^\xi$. •

Example with V as a Gaussian random variable $\mathcal{N}(m,\Sigma)$. This means $h: x \mapsto x\sqrt{\Sigma}+m$, and proves that $\alpha_t^* = \frac{V-P_t}{\lambda(1-t)\sqrt{\Sigma}} = \frac{\sigma(V-P_t)}{(1-t)\sqrt{\Sigma}}$. Here we see the influence of σ on insider's behaviour.

4.2.6 Risk adverse insider

The utility function is $u(x) = \gamma e^{\gamma x},\ \gamma < 0$. We only quote Cho's main results (Proposition 3 and 4):

Theorem 4.19 *1. There is no equilibrium on the space $\mathcal{P}\times\mathcal{S}$ unless V law is Gaussian.*

2. If V law is $\mathcal{N}(m,\Sigma)$, there exists a unique equilibrium $((H^*,\lambda)\alpha^*)$:

$$H^*(t,y) = m+y, \tag{35}$$

$$\lambda_t = \frac{\lambda_1}{\gamma\sigma^2\lambda_1(1-t)+1},\quad \lambda_1 = \frac{\Sigma}{\sigma}\left(\frac{1}{2}\gamma\sigma+\sqrt{1+(\frac{1}{2}\gamma\sigma)^2}\right), \tag{36}$$

$$\alpha_t^* = \frac{V-P_t}{\lambda_1(1-t)}. \tag{37}$$

5 Extension to strategic noise trader [42]

We consider an extension of the Kyle and Back's model [36, 3], meaning a model for the market with a continuous time risky asset and asymmetrical information. There are three financial agents : the market maker, an insider trader (who knows a random variable V which will be revealed at final time) and a non informed agent. Here we assume that the non informed agent is strategic, namely he/she uses a utility function to optimize his/her strategy. Optimal control theory is applied to obtain a pricing rule and to prove the existence of an equilibrium price when the insider trader and the non informed agent are risk-neutral. We will show that if such an equilibrium exists, then the non informed agent's optimal strategy is to do nothing, in other words to be non strategic.

5.1 Introduction

The purpose of this paper is to extend A. Kyle and K. Back's model [36, 3]. Firstly in 1985 Kyle [36] defined an equilibrium problem. On a Gaussian financial market in discrete time, there are three agents: a market maker, an insider trader, a non informed agent (noise trader). The market maker has to define a pricing rule in such way that an equilibrium does exist between the traders. Back [3] extended this model to continuous time. Then N. El Karoui and K. Cho [8] relaxed the Gaussian hypothesis in Kyle's model using fine tools in stochastic control [13, 16]. Finally, K. Cho [9] delivered a new version of Back's model, also relaxing the Gaussian hypothesis. In these four papers, the non informed agent is supposed to be non strategic and so he/she is called "noise trader". As in Cho [9], we like to ask the question: what happens if the non informed agent tries to be strategic instead of being only "a noise trader"?

On such a model, let us mention Guillaume Lasserre's thesis [37] and [38] which extended this problem to multivariate case in continuous time, the agents using a non specified utility function.

Among previous models of insider trading, let us mention [34, 1, 2, 18, 19, 20, 21, 44]. But these models are quite different: the main tools are enlargement of filtration [7, 29, 33, 46, 1, 2] and change of probability measure [15].

Finally, Kyle and Back's equilibrium model has to be distinguished from other models such that Arrow-Debreu or Arrow-Radner ones. These equilibrium were studied in an asymmetric information context by Pikovski and Karatzas (no published preprint) and Hillairet in her thesis (cf. also [25]).

5.2 The model

At time $t \in [0,1]$, the insider trader holds X_t units on risky asset, the non informed agent receives a random endowment E_t and holds Z_t. Let $Y_t = X_t + Z_t$, which is observed by the market maker. In order to discover price P_t of the risky asset, the following hypotheses are done :

There exists a $C^{1,2}$ function H on $]0,1[\times\mathbb{R}$ such that price P_t satisfies :

$$P_t = H(t, Y_t),\ \forall t,\ x \mapsto H(t,x) \text{ is non decreasing, } \partial_x H > 0. \tag{38}$$

On the filtered probability space $(\Omega, (\mathcal{F}_t^B, t \in [0,T]), \mathbb{P})$ associated to B, a standard Brownian motion, we get a random variable V independent of B and we suppose that the insider knows V (which could be the price at time 1, more precisely $V = P_{1+}$) so:

$$dX_t \;=\; \alpha(t, Y_t, V)dt, X_o \in L^1(\sigma(V)), \tag{39}$$

where α is a measurable function such that

$$\forall x, \alpha(., x, V) \text{ is càdlàg and } x \mapsto \alpha(s, x, V) \text{ is uniformly Lipschitz, on } [0,1]\times\mathbb{R}. \tag{40}$$

We will call such function $f(t,x,v)$ (or $f(t,x)$ resp.) with 3 (or 2) variables (t,x,v) (or (t,x)) **regular** when it satisfies the same condition (40) in (t,x) for each fixed v.
The non informed agent buys Z_t and consumes the remainder of her/his endowment denoted as E_t:

$$dE_t = e(t, Y_t)dt + \sigma dB_t,\ \text{where } \sigma > 0,$$

where e is regular in the above sense (40), and $dZ_t = -\beta(t, Y_t)dt + dE_t, Z_o \in \mathbb{R}$, where β is regular. Actually

$$dZ_t \;=\; (-\beta(t, Y_t) + e(t, Y_t))dt + \sigma dB_t, Z_o \in \mathbb{R}. \tag{41}$$

Notice that β represents the non informed agent's consumption speed. He/she invests his/her endowment minus his/her consumption.

So, we can introduce the following filtration:

$$\mathcal{F} = (\sigma\{V, (B_s, s \le t)\}, t \in [0,1]), \tag{42}$$

obviously, the filtration $(\mathcal{F}_t,\ t \in [0,1])$ is completed and get right continuous (cf. [1]): it satisfies the "usual" properties" (see for instance [43]). We can consider that $\mathcal{F}_t$ is the insider's information at time t.

More generally, $\mathcal{F}^M$ denotes the complete right continuous filtration generated by the process M. For instance, $\mathcal{F}^Y$ is the market maker's information at time t, the public information. Under the hypothesis that $\partial_y H > 0$, the knowledge on Y and P are the same, hence the filtrations $\mathcal{F}^Y$ and $\mathcal{F}^P$ are identical.

Proposition 5.1 *Under hypotheses (38) to (41), the following stochastic differential equation admits a unique strong solution :*

$$dY_t = [\alpha(t, Y_t, V) + e(t, Y_t) - \beta(t, Y_t)]dt + \sigma dB_t,\ t \in [0,T],\ Y_0 \neq 0. \tag{43}$$

This solution is a $\mathcal{F}$-Markov process.

Proof : the hypotheses are such that (43) satisfies the existence and uniqueness hypotheses (cf. [43] th. 6 page 194 for instance).
This equation is a diffusion equation with an initial $\sigma-$algebra non trivial: $\mathcal{F}_0 = \cap_{t>0}(\sigma(V) \vee \mathcal{F}_t^B)$. •

Corollary 5.2 *The filtration generated by the process Y and V is the same as the filtration $\mathcal{F}$:*

$$\mathcal{F}_t = \cap_{s>t}(\sigma(V) \vee \mathcal{F}_s^Y).$$

Proof : By construction, the unique solution to (43) is $\mathcal{F}$-adapted, and so, $\forall t, \mathcal{F}_t^Y \subset \mathcal{F}_t$, by definition, $\sigma(V) \subset \mathcal{F}_t$
$\Rightarrow \cap_{s>t}(\mathcal{F}_s^Y \vee \sigma(V)) \subset \cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$.

Reciprocally, since $\sigma > 0$ and

$$\sigma B_t = Y_t - \int_0^t [\alpha(s, Y_s, V) + e(s, Y_s) - \beta(s, Y_s)]ds,$$

$\forall t, B_t$ is $\mathcal{F}_t^Y \vee \sigma(V)$-measurable, and we conclude the proof. •

Let us remark that the independence between V and B implies that B is also a $(\mathcal{F}, \mathbb{P})$ Brownian motion (cf. [7] or [29] for instance). So we get the following :

Proposition 5.3 *Conditionally in V, Y is a $\mathcal{F}^Y$-Markov process, i.e. $\forall$ bounded borelian function f there exists a measurable function h on $[0,1] \times \mathbb{R} \times \mathbb{R}$ such that almost surely :*

$$E[\int_t^1 f(s, Y_s, V)ds/\mathcal{F}_t] = h(t, Y_t, V)$$

Proof : it is a consequence of Proposition 5.1 and Corollary 5.2. We obtain that Y is a $\mathcal{F}$-Markov process, so we get the conclusion. •

Remark 5.4 *The $\mathcal{F}$-Markov process Y is associated with the infinitesimal generator*

$$\mathcal{A}^{\alpha,\beta} = [\alpha + e - \beta]\partial_x + \frac{1}{2}\sigma^2\partial_{xx}^2.$$

5.3 The pricing rule

The market maker observes the filtration $\mathcal{F}^Y$, so that he/she can make price $H(t, Y_t)$.

Definition 5.5 *The function H, mentioned in the hypothesis (38), is called the* **pricing rule**.

The insider trader has to choose a strategy α, and the non informed agent has to choose a strategy β. The "admissible" strategies α, β, satisfy Hypotheses (39), (41) and then the stochastic differential equation (43) admits a unique strong solution. So we define

Definition 5.6 *The set of the regular functions* $\alpha(t,x,v)$ *is denoted by* $\mathcal{S}$*, whose element is called* **admissible strategy** *for the insider trader.*
On the other hand, $\mathcal{S}'$ *denotes the set of the regular functions* $\beta(t,x)$*, set of* **admissible strategies** *for the non insider trader.*

Definition 5.7 *(1) A strategy* $\beta^* \in \mathcal{S}'$ *is* **optimal** *if*

$$\beta^* \in argmax\{\beta \mapsto E[\int_0^1 -P_s[(e(s,Y_s) - \beta(s,Y_s))ds] + \int_0^1 \delta_s V(e-\beta)(s,Y_s)ds],\ \beta \in \mathcal{S}'\}.$$

where δ *is measurable satisfying* $\forall x, \delta(.,x)$ *is càdlàg and* $x \mapsto \delta(s,x)$ *is uniformly Lipschitz on* $[0,1]$ *taking its values in* $]0,1[$.
(2) A strategy $\alpha^* \in \mathcal{S}$ *is* **optimal** *if*

$$\alpha^* \in argmax\{\alpha \mapsto E[\int_0^1 (V - P_s)\alpha(s,Y_s,V)ds/\sigma(V)],\ \alpha \in \mathcal{S}\}.$$

Remark that the coefficient δ is an "impatience coefficient", it means that it delays the agent's profit. Think of [5] where entrepreneur and financiers are differently impatient.

Besides, in [37], the insider's terminal wealth is $W_0 + \int_0^1 X_s dP_s + (V - P_1)X_1$, but using Ito formula this is the same since in our case $\langle X, P\rangle = 0$.
The market maker's aim is to discover a pricing rule H satisfying (38) and such that optimal strategies exist in $\mathcal{S}, \mathcal{S}'$. Moreover, the price has to be rational (P is a $\mathcal{F}^Y$-martingale):

$$P_t = H(t,Y_t) = E[V/\mathcal{F}_t^Y],\ t \in [0,1[.$$

Non necessarily V is equal to P_1 (cf. [3] or [37]), it could be $V = P_{1+}$.

Remark that, as in filtering theory, we can introduce the innovation process which is a $\mathcal{F}^Y-$Brownian motion, i.e.

$$dI_t = dB_t + \sigma^{-1}(\alpha + e - \beta - \tilde{\alpha})_t dt \text{ where } \tilde{\alpha}_t = E_{\mathbb{P}}[\alpha(t,Y_t,V)/\mathcal{F}_t^Y] + (e-\beta)(t,Y_t),$$

in other words $dY_t = \sigma dI_t + \tilde{\alpha}_t dt$.

5.4 Risk neutral agents, $u = Id$

Bellman's principle is to optimize between t and terminal time 1 supposing we know how to optimize between 0 and t (for instance look at [13] p. 95 et sq). Let the value functions:

$$J^\alpha(t) = \int_0^t (V - H(s,Y_s))\alpha(s,Y_s,V)ds + \text{ess sup}\{E[\int_t^1 (V-P_s)\gamma(s,Y_s,V)ds/\mathcal{F}_t],\ \gamma \in \mathcal{S},\ \gamma\mathbf{1}_{[0,t]} = \alpha\mathbf{1}_{[0,t]}\}. \quad (44)$$

$$\begin{aligned} J'^{\beta}(t) &= E[\int_0^t (\delta(s,Y_s)V - P_s)(e-\beta)(s,Y_s)ds/\mathcal{F}_t^Y] \\ &+ \text{ess sup}\{E[\int_t^1 (\delta(s,Y_s)V - P_s)(e-\zeta)(s,Y_s)ds)/\mathcal{F}_t^Y],\ \zeta \in \mathcal{S}',\ \zeta\mathbf{1}_{[0,t]} = \beta\mathbf{1}_{[0,t]}\} \end{aligned} \quad (45)$$

Remark that, in the first term above, V can be get out the integral and replaced by P_t, its conditional expectation with respect to $\mathcal{F}_t^Y$.

The following is a consequence of Proposition 5.3.

Proposition 5.8 $\forall \gamma \in \mathcal{S},\ \zeta \in \mathcal{S}'$, *there exist measurable functions* f_γ, g_ζ *such that :*

$$E[\int_t^1 (V - H(s,Y_s))\gamma(s,Y_s,V)ds/\mathcal{F}_t] = f_\gamma(t,Y_t,V),$$

$$E[\int_t^1 (\delta(s,Y_s)V - P_s)(e-\zeta)(s,Y_s)sds)/\mathcal{F}_t^Y] = g_\zeta(t,Y_t).$$

We now denote the value functions:

$$W^\alpha(t,Y_t,V) = \text{ess sup}\{E[\int_t^1 (V - H(s,Y_s))\gamma(s,Y_s,V)ds/\mathcal{F}_t], \gamma \in \mathcal{S}, \gamma\mathbf{1}_{[0,t]} = \alpha\mathbf{1}_{[0,t]}\}, \quad (46)$$

and

$$U^\beta(t,Y_t) = \text{ess sup}\{E[\int_t^1 (\delta(s,Y_s)V - P_s)(e-\zeta)(s,Y_s)sds)/\mathcal{F}_t^Y], \zeta \in \mathcal{S}', \zeta\mathbf{1}_{[0,t]} = \beta\mathbf{1}_{[0,t]}\}. \quad (47)$$

Remark that

$$J^\alpha(t) = \int_0^t (V - H(s,Y_s))\alpha ds + W^\alpha(t,Y_t,V),$$

and

$$J'^{\beta}(t) = \int_0^t -H(s,Y_s)(e-\beta)(s,Y_s)ds + H(t,Y_t)\int_0^t \delta(s,Y_s)(e-\beta)(s,Y_s)ds + U^\beta(t,Y_t). \quad (48)$$

Moreover, we have boundary conditions : $W^\alpha(1,x,v) = 0,\ U^\beta(1,x) = 0,\ \forall (v,x) \in \mathbb{R}^2$.

Let us use Nicole El Karoui's result (above Lemma 1.7): $\forall \alpha,\ \forall \beta,\ J^\alpha$ is a $(\mathcal{F},\mathbb{P})$ super-martingale and J'^{β} is a $(\mathcal{F}^Y,\mathbb{P})$ super-martingale. Moreover, $\alpha^* \in \mathcal{S}, \beta^* \in \mathcal{S}'$ are optimal if and only if J^α is a local $(\mathcal{F},\mathbb{P})$ martingale and J'^{β} is a local $(\mathcal{F}^Y,\mathbb{P})$ martingale. So we get a tool to manage the existence of a couple of optimal strategies. With Ito's derivation formula -dropping α and β for simplicity and denoting $\delta_t(e-\beta)_t$ instead of $\delta(t,Y_t)(e-\beta)(t,Y_t)$- we get:

$$
\begin{aligned}
dJ^{\alpha}(t) = \quad & (V - H(t,Y_t))\alpha dt + \partial_t W dt + \mathcal{A}^{\alpha,\beta} W dt + \sigma \partial_x W dB_t. \\
dJ'^{\beta}(s) = \quad & -H(t,Y_t)(e-\beta)_t dt + H(t,Y_t)\delta_t(e-\beta)_t dt + \\
& \int_0^t \delta_s(e-\beta)_s ds[\partial_t H(t,Y_t)dt + \partial_x H(t,Y_t)[\alpha + e - \beta]_t dt + \frac{1}{2}\partial^2_{xx} H\sigma^2 dt] + \\
& \partial_t U dt + \mathcal{A}^{\alpha,\beta} U dt + \sigma[\partial_x U + \int_0^t \delta_s(e-\beta)_s ds \times \partial_x H(t,Y_t)] dB_t.
\end{aligned} \tag{49}
$$

We have to write the process J'^{β} with respect to the $(\mathcal{F}^Y, \mathbb{P})$-Brownian motion I:

$$
\begin{aligned}
dJ'^{\beta}(s) \;=\; & -H(t,Y_t)(e-\beta)_t dt + H(t,Y_t)\delta_t(e-\beta)_t dt + \\
& \int_0^t \delta_s(e-\beta)_s ds[\partial_t H + \partial_x H \tilde{\alpha}_t + \frac{1}{2}\partial^2_{xx} H \sigma^2](t,Y_t) dt + \\
& [\partial_t U + \partial_x U \tilde{\alpha}_t + \frac{1}{2}\partial^2_{xx} U \sigma^2](t,Y_t) dt + \sigma[\partial_x U + \int_0^t \delta_s(e-\beta)_s ds \times \partial_x H(t,Y_t)] dI_t.
\end{aligned} \tag{50}
$$

First of all, the super-martingale property implies the two following inequalities $\forall \alpha, \forall \beta$:

$$
\begin{aligned}
- \quad & H(t,Y_t)(e-\beta)_t + H(t,Y_t)\delta_t(e-\beta)_t \\
+ \quad & [\partial_t H + \partial_x H \tilde{\alpha}_t + \frac{1}{2}\partial^2_{xx} H \sigma^2](t,Y_t). \int_0^t \delta_s(e-\beta)_s ds \\
+ \quad & [\partial_t U + \partial_x U \tilde{\alpha}_t + \frac{1}{2}\partial^2_{xx} U \sigma^2](t,Y_t) \leq 0, \\
& (V - H(t,Y_t))\alpha + \partial_t W + \mathcal{A}^{\alpha,\beta} W \leq 0.
\end{aligned} \tag{51}
$$

The first inequality has to be satisfied $\forall \alpha, \forall \beta$, so it has to be $\forall \tilde{\alpha}$. But this expression is linear with respect to $\tilde{\alpha}$. So we get:

$$
\int_0^t \delta_s(e-\beta)_s ds \times \partial_x H(t,Y_t) + \partial_x U(t,Y_t) = 0, \tag{52}
$$

and the optimality of β is equivalent to:

$$
(\delta_t - 1)(e-\beta)_t H(t,Y_t) + \int_0^t \delta_s(e-\beta)_s ds[\partial_t H + \frac{1}{2}\partial^2_{xx} H \sigma^2](t,Y_t) + (\partial_t U + \frac{1}{2}\sigma^2 \partial^2_{xx} U)(t,Y_t) = 0.
$$

Remark that this system implies that dJ'^{β} is identically null, so J'^{β} is a constant on time, so it seems that the optimal strategy could be anything. So we get the following proposition.

Proposition 5.9 *Non insider's optimal strategy is not to invest.*

The proof stays on some lemmas.

Lemma 5.10 *There exist $f, g \in \mathcal{C}^1[0,T]$ such that the non insider's value function U satisfies:*

$$U(t,x) = [f(t) - f(0)]H(t,x) + g(t) \quad \textit{with} \quad f(1) - f(0) = g(1) = 0.$$

Proof : Equation (52) implies $dt \otimes d\mathbb{P}$ almost surely (recall that by hypothesis $\partial_x H > 0$):

$$\int_0^t \delta_s(e-\beta)_s ds = -\frac{\partial_x U(t,Y_t)}{\partial_x H(t,Y_t)}, \tag{53}$$

Differentiate this equation with respect to time :

$$\delta_t(e-\beta)_t dt = \tag{54}$$
$$\left(-\frac{\partial^2_{xt}U}{\partial_x H} + \frac{\partial_x U \partial^2_{xt} H}{(\partial_x H)^2}\right)(t,Y_t)dt - \partial_x\left(\frac{\partial_x U}{\partial_x H}\right)(t,Y_t)dY_t - \frac{1}{2}\sigma^2\partial^2_{xx}\left(\frac{\partial_x U}{\partial_x H}\right)(t,Y_t)dt.$$

So the local martingale part is null, that is to say, since Y is a Brownian diffusion,

$$\forall t, \forall x,\ 0 = \partial_x(\frac{\partial_x U}{\partial_x H})(t,x), \tag{55}$$

and consequently $\frac{\partial_x U(t,x)}{\partial_x H(t,x)}$ does not depend on x so

$$\forall t,\ d\mathbb{P}\ \ a.s.,\ (\delta(e-\beta))(t,Y_t) = -\partial_t(\frac{\partial_x U}{\partial_x H})(t,Y_t)\ (\text{denoted as }\ -f'(t)) \tag{56}$$

only depends on time, so does the function $\delta(e-\beta)$. Yields from (55) that $\partial_x U/\partial_x H$ only depends on time and from (53):

$$\partial_x U(t,x) = [f(t) - f(0)]\partial_x H(t,x),$$

and since $U(1,x) = 0\ \forall x$ we can conclude that

$$U(t,x) = [f(t) - f(0)]H(t,x) + g(t) \quad \text{with} \quad f(1) - f(0) = g(1) = 0.$$

Function f is differentiable by definition, thus g is too, since H, f, U are differentiable. •

Lemma 5.11 *Functions f and g are linked by the following:*

$$\forall (t,x), \quad \delta^{-1}(t,x)f'(t)H(t,x) + g'(t) = 0. \tag{57}$$

Proof : We deduce from (56) $\delta_t(e-\beta)_t = -f'(t)$, that:

$$(e-\beta)_t = -\delta_t^{-1} f'(t) \ \text{ and } \int_0^t \delta_s(e-\beta)_s ds = f(0) - f(t). \tag{58}$$

We now use Equation (53) and the results in (58), $\forall t,\ \forall x$:

$$0 = -(\delta_t - 1)\delta_t^{-1} f'(t) H(t,x) + (f(0) - f(t))(\partial_t H + \frac{1}{2}\sigma^2 \partial^2_{xx} H)(t,x)$$

$$+(f(t) - f(0))(\partial_t H + \frac{1}{2}\sigma^2 \partial^2_{xx} H)(t,x) + f'(t)H(t,x) + g'(t)$$

and after cancellations, $\forall t,\ \forall x$ yields the result (57).

Lemma 5.12 *A necessary and sufficient condition for the existence of an optimal α^* is this system admits a solution (W,H):*

$$\begin{aligned} \partial_x W(s,x,V) &= H(s,x) - V,\ \partial^2_{xx} W(s,x,V) = \partial_x H(s,x), && (59)\\ \partial_t W(s,x,V) &= -(e-\beta)_t \partial_x W(s,x,V) - \frac{1}{2}\sigma^2 \partial^2_{xx} W(s,x,V). && (60) \end{aligned}$$

Proof : Recalling that $\delta^{-1}(t,Y_t) f'(t) = -(e-\beta)(t,Y_t)$, this yields the optimal strategy for the non informed agent:

$$(e-\beta^*)(t,Y_t) H(t,Y_t) = g'(t). \tag{61}$$

By the way, as in the non insider's case, the super-martingale property :

$$\forall \alpha,\ (V - H(t,Y_t))\alpha + \partial_t W + (\alpha + e - \beta)_t \partial_x W + \frac{1}{2}\sigma^2 \partial^2_{xx} W \leq 0, \tag{62}$$

induces a linear expression with respect to α, so once again we get:

$$0 = V - H(s,x) + \partial_x W(s,x,V) \tag{63}$$

and there exists optimal α^* is equivalent to:

$$0 = \partial_t W(s,x,V) + \frac{1}{2}\sigma^2 \partial^2_{xx} W(s,x,V) + (e-\beta)_t \partial_x W \tag{64}$$

with boundary condition $W(1,x,V) = 0$.

In such a case, we get the announced necessary and sufficient condition for the existence of an optimal α^*. •

Proof of the proposition 5.9:
Using the expression (57) for $(e-\beta)$ yields:

$$\partial_t W = -H^{-1}(s,x) g'(t)(H(s,x) - V) - \frac{1}{2}\sigma^2 \partial_x H(s,x) = -g'(t) + g'(t) H^{-1} V - \frac{1}{2}\sigma^2 \partial_x H.$$

We differentiate this last equation with respect to x and (59) with respect to time t, so we get : $\partial_t H = \partial^2_{xt} W = -g'(t)H^{-2} V \partial_x H - \frac{1}{2}\sigma^2 \partial^2_{xx} H$
and a nonlinear differential equation:

$$\partial_t H = -g'(t)H^{-2} V \partial_x H - \frac{1}{2}\sigma^2 \partial^2_{xx} H. \tag{65}$$

But this one is depending on V, so necessarily g' is identically null and thus, recalling (58), $f' = 0$ implies that the optimal strategy is $(e - \beta)(t, Y_t) = 0$, that is to say

the better the noise trader has to do is not to invest!! (66)

•

So we now came back to Cho's paper [9]. Using Itô formula, we get $H(t, Y_t)$ as a semi-martingale, but by definition, $t \mapsto H(t, Y_t)$ is a $(\mathcal{F}^Y, \mathbb{P})-$martingale so it has to be driven by the innovation process I:

$$dH(t, Y_t) = [\partial_t H + \tilde{\alpha}_t \partial_x H + \frac{1}{2}\sigma^2 \partial^2_{xx} H](t, Y_t)dt + \sigma \partial_x H(t, Y_t) dI_t,$$

thus $t \mapsto H(t, Y_t)$ is a $(\mathcal{F}^Y, \mathbb{P})-$martingale is equivalent to a new partial differential equation

$$\partial_t H + \tilde{\alpha}_t \partial_x H + \frac{1}{2}\sigma^2 \partial^2_{xx} H = 0. \tag{67}$$

The comparison with the previous one (65) shows that actually if α is optimal, $\tilde{\alpha}$ has to be null, i.e. a result shown by Cho. Remark that in such a case $dY_t = \sigma dI_t$ and $\sigma^{-1} Y$ is a $(\mathcal{F}^Y, \mathbb{P})$ Brownian motion.

5.5 Modification

So far we have discussed the problem in its simplified situation, that is: we supposed the market maker observes the sum $Y_t = X_t + Z_t$ to make price. In other words we developed the discussion by treating the intensity of both traders' activities, insider and non-informed, with equal weight. But there may be an idea saying that such situation may not be close to the reality. Let us consider for example an extreme case where the activity X_t of the insider is very small and almost negligeable before the overwhelming activity of majority noise traders. Then we wonder if our result can still hold true for such case.

As a first step toward the amelioration in this point of our model, we try to take into acount the portion between the intensities of these two traders, say A ($\in]0, 1[$) and we suppose that for the price making the market maker observes the amount,

$$Y_t = AX_t + (1 - A)Z_t.$$

Combining this with the equations (39),(41) we find the following equation for the Y_t, instead of the equation (43).

$$dY_t = [A\alpha(t, Y_t, V) + (1 - A)\{e(t, Y_t) - \beta(t, Y_t)\}]dt + (1 - A)\sigma dB_t, \quad Y_0 \neq 0. \tag{68}$$

Now with this model we like to repeat the discussion. But this is quite easy because we need not to do other things but to follow the same discussion only by changing the coefficients $\alpha(t, y, v)$, $(e-\beta)(t, y)$, σ to the, $A\alpha(t, y, v)$, $(1-A)(e-\beta)(t, y)$, $(1-A)\sigma$ respectively. By consequence we readily find that our result about the optimal policy for noise trader (66) still holds true in this modified model, but the equation (67) for the price function should be modified into the following form;

$$\partial_t H + A\tilde{\alpha}_t \partial_x H + \frac{1}{2}(1-A)^2\sigma^2\partial^2_{xx} H = 0. \tag{69}$$

Recall that $\tilde{\alpha}_t = E[\alpha(t, Y_t, V)/\mathcal{F}^Y_t]$ so actually $A\tilde{\alpha}_t$ is to be null. Either $A = 0$ or $\tilde{\alpha}_t = 0$. The first case $A = 0$ means that there is no insider traders and it is another problem. The alternative is $\tilde{\alpha}_t = 0$ and we come back to the previous section and Cho's paper [9].

Finally, what happens if A goes to 1 ? meaning that the percent of insider traders is increasing. Then since $dY_t = A\tilde{\alpha}_t dt + (1-A)\sigma dI_t$ and the insider's optimal strategy satisfies $\tilde{\alpha}_t = 0$, Y_t goes to be a constant when A goes to 1. Recalling that $dY_t = A\alpha_t dt + (1-A)\sigma dB_t$ it means that also α should be null. This could mean that the existence of noise traders is indispensable for the trading to be done in the market. On the other hand, if A goes to 1, the consequence in Equation (69) is that $\partial_t H$ goes to 0, the price H becomes a constant, and these two facts (constant price and no trading) are consistent.

6 Another type of equilibrium, C. Hillairet [25] 2005

This paper concerns a quite different equilibrium, but the common facts are the existence of differently informed agents and the existence of an “equilibrium" between them, so that the trading occurs. But here the prices are exogeneous and a pricing rule is not looked for. Moreover, the author considers non continuous price process.

6.1 The model, notations and definitions

The filtered probability space is the product of canonical spaces of a pair (W, N), W being a $m-$Brownian motion and N a n-point process on $(\Omega, (\mathcal{F}_t, t \in [0,T]], \mathbb{P})$, $d = m+n$. The point process admits the positive intensity κ such that the compensated process $M_t = N_t - \int_0^t \kappa_s ds$ is a $(\mathcal{F}, \mathbb{P})$-martingale. There exists a riskless assets, the bond $P_0(t) = P_0(0) \exp \int_0^t r_s ds$ and d risky assets:

$$dP_t^i = P_{t-}^i [b_t^i dt + \sum_j \sigma_j^i(t) dW_j(t) + \sum_j \sigma_j^i(t) dN_j(t)], \; i = 1, ...d. \tag{70}$$

The processes $r,\ b,\ \sigma$ are such that there exists a unique strong solution to (70).

The author considers three types of information: initial or progressive strong information, meaning that the kth agent's information is

$$\mathcal{G}_t^k = \mathcal{F}_t \vee \mathcal{H}_t^k,$$

$\mathcal{H}^k$ denotes his private information.

Definition 6.1 *The agents's strategies are $\mathcal{G}^k$-adapted portfolio-consumption (π^k, c^k), satisfying $c^k \geq 0$,*

$$\int_0^T (c_t^k + \|\tilde{\sigma}_t \pi_t^k\|^2) dt < \infty, \quad \mathbb{P}a.s.$$

The kth agent has his own endowment ε^k, $\mathcal{G}^k$-adapted.

Let $\beta_t = (P_0(t))^{-1}$ be the deflation process. Then assuming that the strategy is self-financing, the discounted wealth could be (here $\mathbf{1}_d$ is a vector in $\mathbb{R}^d$ with all components equal to 1):

$$\beta_t X_t^k = \int_0^t \beta_s (\varepsilon_s^k - c_s^k) ds + \int_0^t \beta_s \tilde{\pi}_s (b_s - r\mathbf{1}_d) ds + \int_0^t \beta_s \tilde{\pi}_{s-} \sigma_s d(W, N)_s. \tag{71}$$

But actually, the stochastic integrals are not well defined with such integrand processes, not $\mathcal{F}-$adapted. Below, we add some hypotheses such that the enlargement of filtration can be used.

As it is usual, the kth agent's aim is to optimise his strategy, given his utility function U^k:

$$(\pi, c) \mapsto E[U(X_A^k)/\mathcal{G}_0^k].$$

Some notations:

$$\begin{array}{lcl} \Theta_t & & \text{the first } m \text{ lines of } \sigma_t^{-1}(b_t - r_t\mathbf{1}), \qquad (72) \\ \kappa_t q_t & & \text{the last } n \text{ lines of } \sigma_t^{-1}(b_t - r_t\mathbf{1}), \text{ supposed to be } > 0, \\ \hat{W}_t & = & W_t + \int_0^t \Theta_s ds, \\ \hat{M}_t & = & N_t - \int_0^t q_s \kappa_s ds, \\ \hat{S} & = & \hat{W}, \hat{M}. \qquad (73) \end{array}$$

6.2 Enlargement of filtration

To make valid the stochastic integrals in the wealth process, the following assumption is done:
For all agent k, there exists $(Z_t^k,\ t \in [0, A],\ A < T)$, a non negative $\mathcal{G}^k-$martingale, such that $E[Z_A^k] = 1$ satisfying

$$dZ_t^k = Z_{t-}^k[\rho_t^k dW_t + (\tau_t^k - I_n)dM_t]. \qquad (74)$$

We denote Q^k the equivalent probability

$$dQ^k = (Z_A^k)^{-1}\mathbb{P}. \qquad (75)$$

A consequence of this assumption is the following:

Corollary 6.2 *On the filtered probability space* $(\Omega, \mathcal{G}^k, \mathbb{P})$

$$\tilde{W}_t^k = W_t - \int_0^t \rho_s^k ds$$

is a Brownian motion,

$$\tilde{M^k}_t = N_t - \int_0^t \kappa_s \tau_s^k ds$$

is the compensated point process of N.

Thus, the wealth equation (71) is meaningfull on the filtered probability space $(\Omega, \mathcal{G}^k, \mathbb{P})$.

Definition 6.3 *Let* Y^k *the Doléans exponential of the process*

$$-(\Theta_t + \rho_t^k).d\tilde{W^k}_t + \frac{q_t}{\tau_t^k}.d\tilde{M^k}_t.$$

Such a process is a positive $(\mathcal{G}^k, \mathbb{P})$-local martingale. To make the work easier, let us assume that Y^k is a martingale.
Examples of $\mathcal{H}^k_t$, initial or progressive strong information:

$$\sigma(L^k) \ ; \ \sigma(h(L^k, B^k_s)).$$

In the third exemple, page 11:

$$\xi^k_t(L^k) = \frac{d\nu^k}{d\hat{\mathbb{P}}_{L^k}} = \frac{d\mathbb{P}^{\nu^k}}{d\hat{\mathbb{P}}}.$$

$$Z^k_t = E_{\hat{\mathbb{P}}}[\xi^k_t(L^k)/\mathcal{F}_t].$$

6.3 Optimization of consumption

pages 6-12
Here we only consider a logarithmic utility. The aim is to optimize in the set of admissible strategies:

$$(\pi^k, c^k) \mapsto E_P[\int_0^A \log(\beta_t X^k_t) dt/\mathcal{G}^k_0]. \tag{76}$$

Proposition 6.4 *Under the hypotheses that*

$$\beta_t \varepsilon^k_t \text{ is deterministic and } E_{\mathbb{P}}[\int_0^A Y^k_t \beta_t \varepsilon^k_t dt/\mathcal{G}^k_0] > 0 \ a.s. \tag{77}$$

Given

$$\lambda^k = \frac{A}{E_{\mathbb{P}}[\int_0^A Y^k_t \beta_t \varepsilon^k_t dt/\mathcal{G}^k_0]},$$

the optimal strategy is

$$\hat{c}^k_t = \frac{1}{\lambda^k \beta_t Y^k_t}, \tag{78}$$

$$\hat{\pi}^k_t = (\tilde{\sigma}_t)^{-1} \left(\frac{A-t}{A\beta_t} \int_0^A \beta_t \varepsilon^k_t) dt \right) \frac{l^k_t}{Y^k_{t-}}. \tag{79}$$

In such a case, the optimal wealth is

$$\hat{X}^k_t = (\beta_t Y^k_t)^{-1} \left(\frac{A-t}{\lambda^k} - E_{\mathbb{P}}[\int_t^A Y^k_s \beta_s \varepsilon^k_s ds/\mathcal{G}^k_t] \right).$$

6.4 Equilibrium

pages 12-15, section 4.

Definition 6.5 *An equilibrium is done when there exists optimal strategies* $(\hat{c}^k, \hat{\pi}^k)$ *clearing the market, meaning that:*

$$\sum_{k=1}^{K} \hat{\pi}_t^k = 0_d, \; dt \otimes d\mathbb{P} \; a.s. \; on \; [0, A] \times \Omega.$$

"This setting is not usual for an equilibrium model in the sense that we assume that the price processes are exogeneous. The considered agents are price takers. We can think for example at a small closed structure of agents trying to set agents trying to set transactions such that the stock market clears in their 'local' structure. Their transactions do not affect the price processes, that are fixed by an external market. It is a competitive dynamic equilibrium.... Definition 6.5 means that the transactions can occur if the endowments and the agents' informations are well-balanced. It can interpreted as
"the more informed an agent is, the less weight he must invest" (not to be discovered).

Using the explicit form of optimal portfolios in the logarithmic utility case, yields the theorem:

Theorem 6.6 *Under the hypotheses (74,77), there exists an equilibrium (6.5) if and only if the processes* $(Y^k, \; k = 1, ...K)$ *satisfy the equilibrium relation*

$$dt \otimes d\mathbb{P} \; a.s. \; \sum_{k=1}^{K} \frac{a_k}{Y_t^k} = 1 \; where \; a_k = \frac{\int_0^A \beta_t \varepsilon_t^k dt}{\sum_{j=1}^K \int_0^A \beta_t \varepsilon_t^j dt}. \tag{80}$$

The coefficient a_k can be seen as the kth agent's weight in the market. The process Y^k summarizes the kth agent's information and determines his optimal behaviour.
cf. Imkeller-Schweizer or [2]

page 14: "Insider's additional expected logarithmic utility can be considered as the relative entropy of his own probability measure w.r.t. the risk neutral probability of a non insider trader."

Corollary 6.7 *Under the hypotheses (74,77), given the processes* $(Y^k, \; k = 1, ...K - 1,$ *then there exists an equilibrium if and only if*

$$\sum_{k=1}^{K-1} a_k(1 - 1/Y^k) \quad > \quad -a_K,$$

$$(Y^K)^{-1} = 1 + \sum_{k=1}^{K-1} \frac{a_k}{a_K}(1 - 1/Y^k). \tag{81}$$

EXAMPLE (page 18 4.3) with $K = 2$.

7 Some other points of view

Here are gathered the summaries of some communications in the AMaMeF workshop (Toulouse, January 2007) concerning some equilibria.

7.1 L. Campi and U. Çetin: Insider trading in an equilibrium with default, a passage from reduced-form to structural modelling

The equilibrium model is Back's model [3] concerning the pricing of a defaultable zero coupon bond issued by a firm. Recall that the market consists of a risk-neutral informed agent, noise traders and a market maker who sets the price using a total order. When the insider doesn't trade, the default time possesses a default intensity in market's view as in reduced-form credit risk models. However, the authors show that, in the equilibrium, the modelling becomes “structural": this means that the default time becomes the first time that some continuous observation process falls below a certain barrier. Interestingly, the firm value is still not observable. They also establish the no expected theorem that the insider's trades are inconspicuous.

7.2 E. Barucci et al., A market model with insider's trades and no information transmission

The authors consider a continuous extension of Kyle's model [36]. They show that in equilibrium, the insider trades but no private information are transmitted. The main difference between their setting and Kyle's one is that the insider is risk adverse and that the risky assets pay a continuous time dividend stream, which is described by a mean reverting process. The dividend realisation is observed by all agents, but the insider knows the dividend growth rate (which is itself mean reverting) while the market maker does not. Noise trader's order flow is described by a stochastic differential equation. Here is define a *Bayesian-Nash equilibrium.*

Definition 7.1 *A Bayesian-Nash equilibrium....*

In equilibrium the insider trades but no information is transmitted because the order flow is uninformative for the market maker.

7.3 E. Jouini and C. Napp, Are more risk-adverse agents more optimistic? Insights from a simple rational expectations equilibrium model.

The authors analyse the link between pessimism and risk-aversion. They consider a model of partially revealing, competitive rational expectations equilibrium with diverse informations, in which the distribution of risk-aversion across individuals is unknown. They show that when a high individual level of risk-aversion, more risk-averse agents are more optimistic. This correlation between individual risk-aversion and optimism leads to a pessimistic "consensus belief", hence to an increase in the market price of risk. Risk-sharing schemes and welfare implications are analysed. Finally they show that agent's welfare may increase upon the receipt of more precise information.

7.4 Guillaume Lassere [37, 38]

7.5 R. MONTE and B. TRIVELLATO, [40]

An equilibrium model of insider trading in continuous time,
Decision Econ. Finan., 2009, 32(2), 83-128.

7.6 M. Schweizer and U. Gruber, Mechanism for price formation in the presence of a large trader.

This one is slightly different from the previous works, since it concerns an asymmetric information with a large investor.

The authors present an approach to the derivation and study of a class of models where there is interaction between underlying assets prices and the actions of a large investor. More precisely, they start with a simple discrete time situation and provide a precise description of a model for the way that a large investor affects prices when trying to hedge a given option that he has sold in a self-financing manner. Under suitable assumptions, they can show that this leads to a class of generalized correlated random walks, for which they are able to obtain a diffusion limit. The resulting continuous model contains several special cases that have been discussed in the literature. Their approach allows them to see more clearly whre the micoeconomic price formation mechanism comes in.

7.7 C. T. WU

Look at his PHD: "Construction of Brownian Motions in Enlarged Filtrations and Their Role in Mathematical Models of Insider Trading".
In this thesis, we study Gaussian processes generated by certain linear transformations of two Gaussian martingales. This class of transformations is motivated by financial equilibrium

models with heterogeneous information. In Chapter 2 we derive the canonical decomposition of such processes, which are constructed in an enlarged ltration, as semimartingales in their own filtration. The resulting drift is described in terms of Volterra kernels. In particular we characterize those processes which are Brownian motions in their own filtration. In Chapter 3 we construct new orthogonal decompositions of Brownian filtrations.

In Chapters 4 to 6 we are concerned with applications of our characterization results in the context of mathematical models of insider trading. We analyze extensions of the financial equilibrium model of Kyle [42] and Back [7] where the Gaussian martingale describing the insider information is specified in various ways. In particular we discuss the structure of insider strategies which remain inconspicuous in the sense that the resulting cumulative demand is again a Brownian motion.

On large investors, also look at Grorud-Pontier [21].

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