Ulaan Baator, september 2008 **Mongolia University**

Introduction to STOCHASTIC CALCULUS,
APPLICATIONS to FINANCE

To manage with Mathematical Finance we need some stochastic tools, especially stochastic calculus. Indeed, the financial market is supposed to offer some assets, the price of them being stochastic processes: they depend both time and alea. We suppose they are known continuously. We also suppose that the nature space, Ω, is infinite, that the information on market and trading are available on any time, meaning "continuous trading". So the time index is $t, t \in [0,T]$ or $\mathbb{R}^+$. We now introduce some stochastic tools, (which anyway can model very different other situations, outside Finance!)

0.1 Plan

i) Wiener process or Brownian motion is such that its small increments well fit the noise, the alea, the physical measure error and so on. The first chapter 1 proves the existence of such a process, explicitely building it. Some of the most useful properties are given.
ii) Itô calculus (Chapter 2) gives more sophisticated stochastic processes using stochastic integration.
iii) Itô formula (Chapter 3) is a differentiation formula of a stochastic process function.
iv) These tools allow us to introduce the so called stochastic diferential equations (SDE), we look only on linear SDE which are the basis of Black-Scholes model, and perhaps a little more complicate one: Ornstein-Uhlenbeck SDE, useful to model the rate behaviour.
v) Chapter 5 presents two problems: change of probability measure and martingale problem. Indeed, in mathematical Finance it is easier to consider the market under the risk neutral probability measure (if it exists !) meaning that under this measure, all the price processes are martingales. The existence of such a measure forbids "arbitrage opportunity" (meaning to win money on the financial market with a null initial wealth). Thus we present:

- Girsanov theorem to know changing the probability measure,

- martingale problem= to find a probability measure under which all the price processes are martingales,

- martingale representation property, meaning that under convenient hypotheses, any random variable, $\mathcal{F}_T$-mesurable (a contingent claim) is the T-value of a martingale; thus we obtain a hedging portfolio.

vi) Finally, we use all these tools to model some financial assets markets, to exhibit of an optimal portfolio, to solve the valuation and the hedging of an option.

0.2 Basic definitions.

- probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with σ-algebra $\mathcal{A}$,
- Borelian σ-algebra on R, R^d.
- random variable,
- filtration, filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t), \mathbb{P})$.
- random process,
- right continuous trajectory (càd), left limited (làg),
- $(\mathcal{F}_t)$ adapted process.

0.3 Convergences

Definition 0.1 *Let $(\mathbb{P}_n)$ be a probability measure sequence on a metric space (E, d) endowed with Borelian σ-algebra $\mathcal{B}$, let $\mathbb{P}$ be a probability measure on $\mathcal{B}$. The sequence $(\mathbb{P}_n)$ is said to* **weakly converge** *to $\mathbb{P}$ if $\forall f \in \mathcal{C}_b(E)$, $\mathbb{P}_n(f) \to \mathbb{P}(f)$.*

Definition 0.2 *Let (X_n) be a random variable sequence $\forall n$ defined on $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$ and taking its values in $(E, d, \mathcal{B})$. The sequence (X_n) is said to* **converge in law** *to X if the measure sequence $(\mathbb{P}_n X_n^{-1})$ weakly converges to $\mathbb{P}X^{-1}$, meaning $\forall f \in \mathcal{C}_b(E)$, $\mathbb{P}_n(f(X_n)) \to \mathbb{P}(f(X))$.*

Recall:

- convergence in L^p,
- $\mathbb{P}$ almost sure convergence,
- probability convergence.

Proposition 0.3 *$\mathbb{P}$ almost sure convergence yields probability convergence.*

Proposition 0.4 *L^p convergence yields probability convergence.*

- Lebesgue theorems: monotoneous convergence, bounded convergence.
- lim sup $= \cap_k \cup_{n \geq k} A_n$, lim inf $= \cup_k \cap_{n \geq k} A_n$, .

Theorem 0.5 *Fatou: For all event set (A_n),*

$$\mathbb{P}(\liminf_n A_n) \leq \liminf_n \mathbb{P}(A_n) \leq \limsup_n \mathbb{P}(A_n) \leq \mathbb{P}(\limsup_n A_n).$$

Theorem 0.6 *Borel-Cantelli: $\sum_n \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(\limsup A_n) = 0$.*
Suppose that the events A_n are independent and $\sum_n \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(\limsup A_n) = 1$.

Definition 0.7 *Let* $\{U_\alpha, \alpha \in A\}$ *be an event family, it is said* **uniformly integrable** *if* $\lim_{n\to\infty} \sup_\alpha \int_{\{|U_\alpha|\geq n\}} |U_\alpha| d\mathbb{P} = 0$.

Theorem 0.8 *The following are equivalent.*

(i) The family $\{U_\alpha, \alpha \in A\}$ *is uniformly integrable,*

(ii) $\sup_\alpha E[|U_\alpha|] < \infty$ *and* $\forall \varepsilon, \exists \delta > 0 : A \in \mathcal{A}$ *and* $\mathbb{P}(A) \leq \delta \Rightarrow E[|U_\alpha| 1_A] \leq \varepsilon$.

RECALL : An almost surely convergent sequence which is a uniformly integrable family is convergent in L^1.

0.4 Conditional expectation

Definition 0.9 *Let* X *in* $L^1(\Omega, \mathcal{A}, \mathbb{P})$, $\mathcal{B}$ *a sub-*σ*-algebra of* $\mathcal{A}$. $E_{\mathbb{P}}(X/\mathcal{B})$ *is the unique random variable in* $L^1(\mathcal{B})$ *such that*

$$\forall B \in \mathcal{B}, \int_B X d\mathbb{P} = \int_B E_{\mathbb{P}}(X/\mathcal{B}) d\mathbb{P}.$$

Corollary 0.10 *Let* $X \in L^2(\mathcal{A})$, $\|X\|_2^2 = \|E_{\mathbb{P}}(X/\mathcal{B})\|_2^2 + \|X - E_{\mathbb{P}}(X/\mathcal{B})\|_2^2$.

0.5 Stopping time

It is a notion related to a filtered probability space.

Definition 0.11 *A random variable* $T : (\Omega, \mathcal{A}, (\mathcal{F}_t), \mathbb{P}) \to (\mathbb{R}^+, \mathcal{B})$ *is a* **stopping time** *if* $\forall t \in \mathbb{R}^+$, *the event* $\{\omega/T(\omega) \leq t\} \in \mathcal{F}_t$.

Examples:

- a constant,
- let O be an open set in $\mathcal{A}$ and X be a continuous process, then

$$T_O(\omega) = \inf\{t, X_t(\omega) \in O\}$$

is a stopping time, so called "hitting time".

Definition 0.12 *Let* T *be a* $\mathcal{F}_.$*-stopping time. The set*

$$\mathcal{F}_T = \{A \in \mathcal{A}, A \cap \{\omega/T \leq t\} \in \mathcal{F}_t\}$$

is called the T**-stopped** σ**-algebra**.

Definition 0.13 *The process* $X_{.\wedge T}$ *is called "* T*-stopped process" generally denoted as* X^T.

0.6 Martingales

(cf. [23] pages 8 à 12 ; [13] pages 11 à 30.)

Definition 0.14 *Let X be an adapted real process. It is a* **martingale** *(resp supra/sub) if*

(i) $X_t \in L^1(\Omega, \mathcal{A}, \mathbb{P}), \forall t \in \mathbb{R}^+$,

(ii) $\forall s \leq t, E[X_t/\mathcal{F}_s] = X_s$. *(resp $\leq, \geq$.)*

Lemma 0.15 *Let X be a martingale, φ a convex function such that $\forall t\ \phi(X_t) \in L^1$, then $\varphi(X)$ is a sub-martingale ; if φ is a concave function, $\varphi(X)$ is a super-martingale.*

Proof exercise.

Definition 0.16 *The martingale X is* **closed** *by $Y \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ if $X_t = E[Y/\mathcal{F}_t]$.*

Proposition 0.17 *Every martingale admits a càdlàg modification (cf. [23]).*

Theorem 0.18 *Martingale convergence: let X be a càd super (or sub)-martingale such that $\sup_t E[|X_t|] < \infty$. Then $\lim_{t\to\infty} X_t$ exists almost surely and belongs to $L^1(\Omega, \mathcal{A}, \mathbb{P})$. If X is closed by Z, it is closed by $\lim_{t\to\infty} X_t$ denoted as X_∞ which equals $E[Z/\vee_{t\geq 0}\mathcal{F}_t]$.*

Corollary 0.19 *A bounded from below supermartingale converges almost surely when t goes to infinity.*

Theorem 0.20 *Let X be a uniformly integrable càd martingale; then the almost sure limit Y of X_t when t goes to infinity exists and belongs to L^1. Moreover $X_t = E[Y/\mathcal{F}_t]$.*

Theorem 0.21 *Let X be a martingale. X is uniformly integrable iff*

(i) X_t converges almost surely to Y (which belongs to L^1) when t goes to infinity and $\{X_t, t \in \overline{\mathbb{R}^+}\}$ is a martingale.

or

(ii) X_t converges to Y in L^1 when t goes to infinity.

Theorem 0.22 *Doob: Let $(X_n, n \in \mathbb{N})$ be a $\mathcal{F}$-martingale, S and T $\mathcal{F}$-stopping times such that:*

(i) $E[|X_S|],\ E[|X_T|] < \infty$,

(ii) $\lim_{n\to+\infty} \int_{\{T>n\}} |X_n| d\mathbb{P} = \lim_{n\to+\infty} \int_{\{S>n\}} |X_n| d\mathbb{P} = 0$,

(iii) $S \leq T < \infty$ almost surely.
Then $E[X_T/\mathcal{F}_S] = X_S$ $\mathbb{P}-$ almost surely.

Let X be a càd sub-martingale with terminal value X_∞, let two stopping times S and T satisfying (i)(ii)(iii). Then:

$$X_S \leq E[X_T/\mathcal{F}_S] \; \mathbb{P}- \text{ almost surely.}$$

Proof: pages 19-20 [13].

Definition 0.23 *The increasing process $\langle M \rangle$ (or $\langle M, M \rangle$) is defined on time t as following (π are $[0,t]$ partitions):*

$$\langle M \rangle_t = \lim_{|\pi| \to 0} proba \sum_{t_i \in \pi} (M_{t_i} - M_{t_{i-1}})^2.$$

Next chapter: if M is Brownian motion B then $\langle B \rangle_t = t$.

Remark 0.24 *The martingales in L^2 admit a bracket.*

Proposition 0.25 *$\langle M \rangle_t$ is the unique continuous process which is adapted increasing such that*

$$M_t^2 - \langle M \rangle_t \text{ is a martingale.}$$

Very often this proposition is the bracket definition and then 0.23 is a consequence.

0.7 Some definitions and theorems about processes

Definition 0.26 *Let X and Y be two processes: X is a* **modification** *of Y if :*

$$\forall t \geq 0, \mathbb{P}\{X_t = Y_t\} = 1.$$

X and Y are **indistinguable** *if almost surely their trajectories coincide:*

$$\mathbb{P}\{X_t = Y_t, \forall t \geq 0\} = 1.$$

Remark 0.27 *The second notion is stronger than the first.*

Definition 0.28 *A process X is "progressively measurable" with respect to the filtration $(\mathcal{F}_t, t \geq 0)$ if $\forall t \geq 0, \forall A \in \mathcal{B}(R)$:*

$$\{(s,\omega)/0 \leq s \leq t \; ; \; X_s(\omega) \in A\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t,$$

meaning that the application $([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t) : (s,\omega) \mapsto X_s(\omega)$ is measurable.

Proposition 0.29 *(cf [13], 1.12) If X is an adapted measurable process, it admits a progressively measurable modification.*

Proof: cf. Meyer 1966, page 68.

Proposition 0.30 *Let X be a $\mathcal{F}$-progressively measurable process and T a $(\mathcal{F}_t)$ stopping time. Then*
(i) the application $\omega \mapsto X_{T(\omega)}(\omega)$ is $\mathcal{F}_T$-measurable
(ii) the process $t \mapsto X_{t\wedge T}$ is $\mathcal{F}$-adapted.

Proof: (i) X is progressively measurable yields that $\forall A$ Borelian set,

$$\forall t,\ \{(s,\omega),\ 0 \leq s \leq t,\ X_s(\omega) \in A\} \in \mathcal{B}_{[0,t]} \otimes \mathcal{F}_t,$$

$\{\omega : X_{T(\omega)}(\omega) \in A\} \cap \{\omega : T(\omega \leq t\} = \{\omega : X_{T(\omega)\wedge t}(\omega) \in A\} \cap \{T \leq t\}.$

Using that T is a $\mathcal{F}$-stopping time, the second event belongs to $\mathcal{F}_t$, and since progressive measurability holds, the first one too belongs to $\mathcal{F}_t$.

(ii) The second assertion moreover shows that X^T is $\mathcal{F}$-adapted.

Proposition 0.31 *(cf [13], 1.13) If X is an adapted measurable process admitting càd or càg trajectories, it is progressively measurable.*

Proof: Define

$$X_s^{(n)}(\omega) = X_{(k+1)t2^{-n}}(\omega),\ s \in]\frac{kt}{2^n}, \frac{(k+1)t}{2^n}],\ X_0^{(n)}(\omega) = X_0(\omega)\ ;\ k = 0, \cdots, 2^n - 1.$$

Then the application $(s,\omega) \mapsto X_s^{(n)}(\omega)$ is $\mathcal{B}([0,t])\otimes\mathcal{F}_t$-measurable. Right continuity shows that the sequence $(X_s^{(n)}(\omega))$ converges to $X_s(\omega)$ $(\forall s, \omega)$ thus the limit is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable.

Definition 0.32 *(page 33 [23]) Let X an adapted càdlàg process: it is a* **local martingale** *if there exists a sequence of stopping times T_n increasing to infinity, such that $\forall n$ the process X^{T_n} is a martingale.*

Suitably stopping a process is a useful tool to get uniformly integrable martingales which are easier to manage with: some results are obtained $\forall n$, then we go to the limit, for instance using Lebesgue theorems. We denote the local martingale set as $\mathcal{M}_{loc}$.

Theorem 0.33 *(cf [23], th. 44, page 33) Let $M \in \mathcal{M}_{loc}$ and a stopping time T such that M^T is uniformly integrable.*

(i) $S \leq T \Rightarrow M^S$ is uniformly integrable.

(ii) $\mathcal{M}_{loc}$ is a real vectorial set.

(iii) M^S and M^T uniformly integrable imply that $M^{S\wedge T}$ is uniformly integrable.

Notation :

$$M_t^* = \sup_{0\leq s\leq t} |M_s| \ ; \ M^* = \sup_{0\leq s} |M_s|.$$

Theorem 0.34 *(cf [23], th. 47, page 35) If $M \in \mathcal{M}_{loc}$ is such that $E[M_t^*] < \infty \forall t$, then M is a 'true' martingale.*
Moreover suppose $E[M^*] < \infty$, then M is uniformly integrable.

Proof:
(i) $\forall s \leq t, |M_s| \leq M_t^*$ belonging to L^1, the sequence $T_n \wedge t$ increases to t and

$$E[M_{T_n\wedge t}/\mathcal{F}_s] = M_{T_n\wedge s}.$$

Using almost sure convergence in this equality, Lebesgue Theorem allows us to go to the L^1 limit. Thus M is a martingale.
(ii) M^* belongs to L^1. The martingale convergence theorem shows the almost sure convergence of M_t to M_∞. Finally we prove the uniform integrability (using the equivalent definition of uniform integrability).

0.8 Exercises

1. Let $\mathcal{G}$ be a sub-σ algebra of $\mathcal{A}$ and an almost surely positive random variable X. Prove that the conditional expectation $E[X/\mathcal{G}]$ is also strictly positive.
Prove that the reciprocal is false given a contra-example (for instance use the trivial σ-algebra $\mathcal{G}$).

2. Let $\mathcal{G} \subset \mathcal{H} \subset \mathcal{A}$ and $X \in L^2(\Omega, \mathcal{A}, \mathbb{P})$. Prove (Pythagore Theorem):

$$E[(X - E[X/\mathcal{G}])^2] = E[(X - E[X/\mathcal{H}])^2] + E[(E[X/\mathcal{H}] - E[X/\mathcal{G}])^2].$$

3. Let O be an open set in $\mathcal{A}$ and a $\mathcal{F}-$adapted continuous process X. One notes

$$T_0 = \inf\{t \ : \ X_t \in O\}.$$

Prove that T_O is a stopping time.

4. Lemma 0.15: Let X be a martingale, φ a convex function such that $\forall t \ \phi(X_t) \in L^1$, then $\varphi(X)$ is a sub-martingale ; if φ is a concave function $\varphi(X)$is a super-martingale.

5. Theorem 0.18 Martingale convergence: Let X be a càd super (or sub)-martingale such that $\sup_t E[|X_t|] < \infty$. Then $\lim_{t\to\infty} X_t$ exists almost surely and belongs to $L^1(\Omega, \mathcal{A}, \mathbb{P})$. If X is closed by Z, it is closed by $\lim_{t\to\infty} X_t$ denoted as X_∞ which equals $E[Z/\vee_{t\geq 0} \mathcal{F}_t]$.

6. Let X be a martingale. Prove the following are equivalent
(i) X is uniformly integrable
(ii) X_t converges almost surely to Y (which belongs to L^1) when t goes to infinity and

$\{X_t, t \in \overline{\mathbb{R}^+}\}$ is a martingale.
(iii) X_t converges to Y in L^1 when t goes to infinity.

Indication: $(i) \to (iii) \to (ii) \to (i)$

7. Proposition 0.30: Let X be a $\mathcal{F}$-progressively measurable process and T a $(\mathcal{F}_t)$ stopping time. Then
(i) the application $\omega \mapsto X_{T(\omega)}(\omega)$ is $\mathcal{F}_T$-measurable
(ii) the process $t \mapsto X_{t\wedge T}$ is $\mathcal{F}$-adapted.

8. Proposition 0.31: If X is an adapted measurable process admitting càd or càg trajectories, it is progressively measurable.

9. Theorem 0.34
If $M \in \mathcal{M}_{loc}$ is such that $E[M_t^*] < \infty \forall t$, then M is a 'true' martingale.
Moreover suppose $E[M^*] < \infty$, then M is uniformly integrable.

1 Brownian motion, Wiener process

([13] pages 21-24 ; [23] pages 17-20.)

Firstly this was a way to model the irregular motion of pollen particules in water, phenomena observed by Robert BROWN 1828). So, there is a dispersion of micro-particules in water, or a pollen "diffusion" in water. Actually, this motion is used in many other modelizations of dynamic phenomenae:

- suspension of micro-particules in water,
- stock prices in financial markets,
- physical measure errors,
- waiting queue asymptotical behaviour,
- any dynamical behaviour with some random part (stochastic differential equations).

Definition 1.1 **Brownian motion** *or* **Wiener process** *is a process on a filtered probability space* $(\Omega, \mathcal{A}, (\mathcal{F}_t), \mathbb{P})$ *which is* $\mathcal{F}$*-adapted, continuous, taking its values in a vectorial space and:*

(i) $B_0 = 0$, $\mathbb{P}$*-almost surely on* Ω*,*

(ii) $\forall s \leq t, B_t - B_s$ *is independent of* $\mathcal{F}_s$*, with Gaussian law* $\mathcal{N}(0, (t-s)I_d)$.

Thus let us consider a real sequence $0 = t_0 < t_1 < \cdots < t_n < \infty$, the sequence $(B_{t_i} - B_{t_{i-1}})_i$ is a centered Gaussian vector with variance-covariance diagonal matrix, the diagonal being $t_i - t_{i-1}$. B is said to be an **independent increment process**.

The first point to be solved is the existence of such process. There exist some classical constructions; here we present a trajectorial construction.

1.1 Existence based on a trajectorial construction, Kolmogorov lemma

(You can find some details in [13] 2.2 ; [23] pages 17-20.)

Roughly speaking, avoiding technical, baresome, long proofs, we do as following. Let $\Omega = \mathcal{C}(\mathbb{R}^+, \mathbb{R}^d)$, $B(t, \omega) = \omega(t)$ the "coordinate" maps, called **trajectories**. We endow Ω with the smaller σ-algebra $\mathcal{A}$ such that $\{B_t, t \in \mathbb{R}^+\}$ is measurable and with the "natural" filtration generated by the process B : $\mathcal{F}_t = \sigma\{B_s, s \leq t\}$. On $(\Omega, \mathcal{A})$, the existence of a unique probability measure $\mathbb{P}$ is proved, s. t.

$$\forall n \in \mathbb{N}, \forall t_1, \cdots, t_n \in \mathbb{R}^+, \forall B_1, \cdots, B_n \ \ \mathbb{R}^d \text{ Borelian sets:}$$

$$\mathbb{P}\{\omega/\omega(t_i) \in B_i \quad \forall i = 1, \cdots, n\} =$$
$$\int_{B_1} \cdots \int_{B_n} p(t_1, 0, x_1) p(t_2 - t_1, x_1, x_2) \cdots p(t_n - t_{n-1}, x_{n-1}, x_n) dx_1..dx_n,$$

where $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$.

We have to show:

- $\mathbb{P}$ is actually a probability measure on $\mathcal{A}$,

- under this probability measure, the process $t \mapsto \omega(t)$ is a Brownian motion as it is defined above.

Actually, this definition only concerns the Borelian events of the space: $\Omega' = \mathcal{A}(\mathbb{R}^+, \mathbb{R}^d)$. But, unhappily, Ω is not a Borelian set. So $\Omega = \mathcal{A}(\mathbb{R}^+, \mathbb{R}^d)$ is chosen and Kolmogorov theorem is used (1933) :

Definition 1.2 *A "consistent familly" of finite dimensional distributions* $(Q_t, t$ *n-uple* $\mathbb{R}^+)$ *is a measure familly on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ *such that*

- If $s = \sigma(t), s$ *and* $t \in (\mathbb{R}^+)^n$, σ *permutation of* $\{1, \cdots, n\}$, $A_1, \cdots, A_n \in \mathcal{B}(\mathbb{R}^d)$, *then* $Q_t(A_1, \cdots, A_n) = Q_s(A_{\sigma(1)}, \cdots, A_{\sigma(n)})$

- If $u = (t_1, \cdots, t_{n-1})$, $t = (t_1, \cdots, t_{n-1}, t_n)$, *then* $\forall t_n, Q_t(A_1, \cdots, A_{n-1}, \mathbb{R}) = Q_u(A_1, \cdots, A_{n-1})$.

Theorem 1.3 *(cf [13] page 50 : Kolmogorov, 1933) Let* $(Q_t, t \in (\mathbb{R}^+)^n)$ *be a consistent familly of finite dimensional distributions.*
Then there exists a probability measure $\mathbb{P}$ *on* $(\Omega, \mathcal{B}(\Omega))$ *such that* $\forall B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R}^d)$,

$$Q_t(B_1, \cdots, B_n) = \mathbb{P}\{\omega/\omega(t_i) \in B_i, i = 1, \cdots, n\}.$$

This theorem is applied to the familly of measures

$$Q_t(A_1, \cdots, A_n) = \int_{\Pi_i A_i} p(t_1, 0, x_1) \cdots, p(t_n - t_{n-1}, x_{n-1}, x_n) dx.$$

Then the existence of a continuous modification of the 'coordinate applications' process of Ω (Kolmogorov-Centsov, 1956) is proved. Finally one proves the existence of a continuous modification of the canonical process:

Theorem 1.4 *(Kolmogorov-Centsov, 1956, cf. [13] page 53, [23] page 171) Let* X *be a real random process on* $(\Omega, \mathcal{A}, \mathbb{P})$ *satisfying:*

$$\exists \alpha, \beta, C > 0 : E|X_t - X_s|^\alpha \leq C|t - s|^{1+\beta},\ 0 \leq s, t \leq T,$$

then X *admits a continuous modification* $\tilde{X}$ *which is locally* $\gamma-$*Hölder continuous:*

$$\exists \gamma \in]0, \frac{\beta}{\alpha}[, \exists h \text{ random variable } > 0, \exists \delta > 0 \ :$$
$$\mathbb{P}\{\sup_{0<t-s<h; s,t\in[0,T]} |\tilde{X}_t - \tilde{X}_s| \leq \delta |t - s|^\gamma\} = 1.$$

Remark that this theorem is also true for processes $(X_t,\ t \in \mathbb{R}^d)$.

1.2 Brownian motion trajectories properties

1.2.1 Gaussian process

Definition 1.5 *A process X is said to be* **a Gaussian process** *when* $\forall d, \forall (t_1, \cdots, t_d) \in \mathbb{R}_+^d$, $(X_{t_1}, \cdots, X_{t_d})$ *is a Gaussian vector. When* $(X_{t+t_i}; i = 1, \cdots, d)$ *law is independent of* t*, the process is said to be* **stationnary**.

The **covariance** *of the vector X is the matrix*

$$\rho(s,t) = E[(X_s - E(X_s))(X_t - E(X_t))^T], \ s, t \geq 0$$

with A^T denotes the transposed matrix.

Proposition 1.6 *The Brownian motion B is a continuous Gaussian process with covariance $\rho(s,t) = s \wedge t$.*

Reciprocally, any centered continuous Gaussian process with covariance $\rho(s,t) = s \wedge t$ is a Brownian motion.

The Brownian motion converges "in mean" to zero:

$$\frac{B_t}{t} \to 0$$

almost surely when t goes to infinity.

Proof exercise. The third point is similar to a "large number law". •

Other Brownian motions can be obtained using standard transformations, with perhaps also change of filtration.

(i) scaling: $(\frac{1}{\sqrt{c}} B_{ct}, \mathcal{F}_{ct})$.

(ii) time inversion: $(Y_t, \mathcal{F}_t^Y)$, $Y_t := tB_{\frac{1}{t}}$ if $t \neq 0$, $Y_0 = 0$ and $\mathcal{F}_t^Y = \sigma\{Y_s, s \leq t\}$.

(iii) time returning: $(Z_t, \mathcal{F}_t^Z)$, $Z_t := B_T - B_t$ and $\mathcal{F}_t^Z = \sigma\{Z_s, s \leq t\}$.

(iv) symmetry: $(-B_t, \mathcal{F}_t)$.

In each case, we have to verify that the new process is continuous, adapted, satisfying the Brownian motion characteristic property or that it is a Gaussian process with covariance $\rho(s,t) = s \wedge t$. The most difficult to prove is (ii) (use Proposition 1.6).

Proposition 1.7 *B is a martingale with respect to its own filtration and it is a Markov process.*

Proof: exercise, as an application of the definition.

1.2.2 Variations of the trajectories

(cf. [13] pb 9.8 p. 106 and 125)
Notation: $\pi_n = (t_0 = 0, \cdots, t_n = t)$ is a "partition" of $[0,t]$, let us denote $\|\pi_n\| = \sup_i\{t_i - t_{i-1}\}$, called "step" of π_n.

Theorem 1.8 *(cf. [23] 28 p. 18)*
Let (π_n) be a partition sequence of the interval $[0,t]$ such that $\pi_n \subset \pi_m$ when $n \leq m$ and $\|\pi_n\|$ goes to zero when n goes to infinity. We set

$$\pi_n(B) = \sum_{t_i \in \pi_n} (B_{t_{i+1}} - B_{t_i})^2.$$

Then when n goes to infinity, $\pi_n(B)$ goes to t in $L^2(\Omega)$,
and almost surely if moreover $\sum_n \parallel \pi_n \parallel < \infty$.

Proof: Let $z_i = (B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)$; $\sum_i z_i = \pi_n(B) - t$. (z_i) is a centered independent random variable sequence since $B_{t_{i+1}} - B_{t_i}$ admits a Gaussian law with null mean and variance $t_{i+1} - t_i$. We compute the expectation of z_i^2 :

$$E[z_i^2] = E[(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)]^2 = E[(B_{t_{i+1}} - B_{t_i})^4 - 2(B_{t_{i+1}} - B_{t_i})^2(t_{i+1} - t_i) + (t_{i+1} - t_i)^2].$$

Using the Gaussian law moments, yields

$$E[z_i^2] = 2(t_{i+1} - t_i)^2.$$

The independence of (z_i) shows that $E[(\sum_i z_i)^2] = \sum_i E[(z_i)^2] \leq 2t\|\pi_n\|$, going to zero when n goes to infinity. Yields the convergence in $L^2(\Omega)$ (so probability convergence) of sequence $(\pi_n(B))$ to t.

Let $\varepsilon > 0$. Moreover suppose that $\sum_n \parallel \pi_n \parallel < \infty$; $\mathbb{P}\{|\pi_n(B) - t| > \varepsilon\} \leq \frac{1}{\varepsilon^2} 2 \parallel \pi_n \parallel t$. Then the series $\sum_n \mathbb{P}\{|\pi_n(B) - t| > \varepsilon\}$ converges and Borel-Cantelli lemma shows:

$$\mathbb{P}[\overline{\lim}_n\{|\pi_n(B) - t| > \varepsilon\}] = 0,$$

meaning that:

$$\mathbb{P}[\cap_n \cup_{m \geq n}\{|\pi_m(B) - t| > \varepsilon\}] = 0 \Leftrightarrow \forall \varepsilon > 0, \text{ almost surely } \cup_n \cap_{m \geq n}\{|\pi_m(B) - t| \leq \varepsilon\} = \Omega,$$

that is the almost sure convergence of $\pi_n(B)$ to t. •

Theorem 1.9 *to admit... (cf [13] 9.18, p.110 : Paley-Wiener-Zygmund, 1933)*

$$\mathbb{P}\{\omega : \exists t_0 \; t \mapsto B_t(\omega) \text{ differentiable in } t_0\} = 0.$$

To go farer....let us denote

$$D^+f(t) = \overline{\lim}_{h\to 0}\frac{f(t+h)-f(t)}{h}\ ;\ D_+f(t) = \underline{\lim}_{h\to 0}\frac{f(t+h)-f(t)}{h},$$

there exists an event F with probability measure 1 which subset of:

$$\{\omega\ :\forall t,\ D^+B_t(\omega) = +\infty \text{ or } D_+B_t(\omega) = -\infty\}.$$

This result can be understood as following: almost surely w.r.t. ω, the trajectory $t \mapsto B_t(\omega)$ is continuous but no differentiable.

Definition 1.10 *Let a function f on the interval $[a, b]$: the* **variation of f on this interval** *is*

$$Var_{[a,b]}(f) = \sup_\pi \sum_{t_i\in\pi} |f(t_{i+1}) - f(t_i)|$$

where π belongs to the partition of $[a, b]$ set.

Theorem 1.11 *(cf. [23] p.19-20) Let $a,\ b$ in $\mathbb{R}^+$:*

$$\mathbb{P}\{\omega : Var_{[a,b]}(B) = +\infty\} = 1.$$

Proof : Let a and b be fixed in $\mathbb{R}^+$, a partition π of $[a, b]$.

$$(1)\qquad \sum_{t_i\in\pi} |B(t_{i+1}) - B(t_i)| \geq \frac{\sum_{t_i\in\pi} |B(t_{i+1}) - B(t_i)|^2}{sup_{t_i\in\pi}|B(t_{i+1}) - B(t_i)|}.$$

The numerator is the quadratic variation of B, we know that it goes to $b - a$. Then, $s \mapsto B_s(\omega)$ is continuous (so uniformly continuous) on the interval $[a, b]$:

$$\forall\varepsilon, \exists\eta, \parallel \pi \parallel < \eta \Rightarrow sup_{t_i\in\pi}|B(t_{i+1}) - B(t_i)| < \varepsilon.$$

Thus the fraction (1) goes to infinity. •

1.3 Computation of $2\int_0^t B_s dB_s$ (exercise)

B trajectories are not differentiable, but we look for understanding this integral. Logically it could be B_t^2, but it is not. To stress the difference, we split B_t^2 as a sum of increments along the interval $[0, t]$ partition: $t_i = it/n$, then using the identity $x^2 - y^2 = 2y(x - y) + (x - y)^2$:

$$B_t^2 = \sum_i (B_{t_{i+1}}^2 - B_{t_i}^2) = \sum_i 2B_{t_i}[B_{t_{i+1}} - B_{t_i}] + \sum_i [B_{t_{i+1}} - B_{t_i}]^2.$$

The first term "naturally" converges to: $2\int_0^t B_s dB_s$ (this will be proved below, Section 2.1). It could be believed that the second term converges to 0, but here it is the paradox.

Using Brownian motion definition, this second term is the sum of n squarred Gaussian variables, independent centered with variance t/n: Central Limit Theorem proves that it is a random variable with law $t/n\chi_n^2$. Its mean is t, its variance is $t^2/nVar\chi_1^2$: so this term converges in L^2 (so also probability convergence) to the mean t. Thus we will prove below more precisely that

$$B_t^2 = 2\int_0^t B_s dB_s + t.$$

1.4 Exercises

1. Proposition 1.6: The Brownian motion B is a continuous Gaussian process with covariance $\rho(s,t) = s \wedge t$.
Reciprocally, any centered continuous Gaussian process with covariance $\rho(s,t) = s \wedge t$ is a Brownian motion.

2. Let $Y_t = t.B_{1/t}$; $Y_0 = 0$ and $\mathcal{F}_t^Y$ the natural filtration linked to process Y. Prove that$(Y_t, \mathcal{F}_t^Y)$ is a Brownian motion. (use 1. and exercise 5).

3. Proposition 1.7: Prove that B is a martingale with respect to its own filtration and that it is a Markov process.

4. Let $\mathcal{G}_t = \sigma(B_s, s \leq t) \vee \mathcal{N}, t \geq 0$. Prove that this filtration is càd, i.e. $\mathcal{G}_{t^+} = \cap_{s>t}\mathcal{G}_s$. *Indication: use that the conditional characteristic function of vector (B_u, B_z), $z, u > t$ given $\mathcal{G}_{t^+}$, is the limit of the conditional characteristic function of the same vector given $\mathcal{G}_w$ when w decrease to t, and it coincides with the conditional characteristic function of vector (B_u, B_z), $z, u > t$, given $\mathcal{G}_t$. Yields any conditional law given $\mathcal{G}_{t^+}$ coincides with this given $\mathcal{G}_t$, then any random variable $\mathcal{G}_{t^+}$-measurable is $\mathcal{G}_t$-measurable, conclude.*

5. Let B be a real Brownian motion.
a) Prove that the sequence $\frac{B_n}{n}$ goes to 0 almost surely.

b) Use that B is a martingale and a Doob inequality to get the majoration

$$E[\sup_{\sigma \leq t \leq \tau} (\frac{B_t}{t})^2] \leq \frac{4\tau}{\sigma^2}.$$

c) Now let $\tau = 2\sigma = 2^{n+1}$, give a majoration of $\mathbb{P}\{\sup_{2^n \leq t \leq 2^{n+1}} |\frac{B_t}{t}| > \varepsilon\}$ to prove this sequence convergence, finally apply Borel Cantelli lemma.

d) Yields $\lim_{t\to\infty} \frac{B_t}{t} = 0$ almost surely (i.e. large numbers law, cf. Problem 9.3, pages 124-125, Karatzas-Schreve.)

2 Ito calculus

The aim of this chapter is to give sense to the notion of process integral with respect to Brownian motion or more generally with respect to a martingale. If we think of the aim of this course, stochastic calculus applied to Finance, the motivation of stochastic integral could be the following: let us suppose a market model where the assets price could be modelled as a martingale, M_t at time t. If we have $X(t)$ of these asssets and if we trade on times t_k, finally the wealth could increase as following:

$$\sum_k X(t_{k-1})(M_{t_k} - M_{t_{k-1}}).$$

If we trade on continuous time, at any time t, this expression is a sum which could converge to a Stieltjes integral. But when $M = B$ we know that the differential B' doesn't exist!! and since the variation $V(B)$ is infinite, there is not a "deterministic" limit, the "naïve" stochastic integral is impossible (cf. Protter [23] page 40) as the following result proves it.

Theorem 2.1 *Let $\pi = (t_k)$ be a partition of $[0, T]$. Let us suppose that for all continuous function x: $\lim_{|\pi|\to 0} \sum_k x(t_{k-1})(f(t_k) - f(t_{k-1}))$ exists, then f admits a finite variation. (cf. Protter [23], th. 52, page 40)*

Reciprocally, if $V(f) = \sum_k |f(t_k) - f(t_{k-1})| = +\infty$, then the limit doesn't exist, for instance, think of the Brownian motion $f : t \mapsto B_t$. So we need other tools. Itô's idea was to restrain the integrands to the processes which can't"see" the future increments, meaning the adapted processes, so that $x(t_{k-1})$ and $(B_{t_k} - B_{t_{k-1}})$ could be independent; thus, we can't do anything trajectorially but we work "in probability".

Here is the scheme: we introduce the problem and some notations (2.1.1), we first (2.1.2) define the integral of "simple" processes (denoting their set as $\mathcal{S}$, defined below) then in 2.1.3 we give the properties of this integral on $\mathcal{S}$ which allows us an extension of this operator on the closure $\mathcal{S}$ with respect to a suitable topology to obtain a large enough integrands set.

The section 2.2 is devoted to the quadratic variation, which is a time variance. Among others, this tool will be usefull to define the integral of an integral.

2.1 The stochastic integral

2.1.1 Introduction and notations

Let M be a continuous martingale, square-integrable, its bracket, increasing process being denoted as $\langle M \rangle$, on a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, where for instance $\mathcal{F}_t$ is the

Brownian motion natural filtration, completed with negligible events. Let a measurable process X, $n \in \mathbb{N}$, time t, we define:

$$I_n(X,t) = \sum_j X(\frac{j-1}{2^n} \wedge t)(M_{\frac{j}{2^n}\wedge t} - M_{\frac{j-1}{2^n}\wedge t}).$$

As we saw that in Theorem 2.1, the limit could not exist. So we only consider the adapted measurable processes X, almost surely square-integrable with respect to $d\langle M\rangle$.

The construction of $I(X,t)$ was done by Itô (1942) in case of Brownian motion M and by Kunita et Watanabe (1967) in case of square-integrable martingales. We now use the Introduction prerequisites to define a topology on integrands set:

. let π be partitions of interval $[0,t]$, the process $\langle M\rangle$ is defined at time t:

$$\langle M\rangle_t = \lim_{|\pi|\to 0} proba \sum_{t_i\in\pi}(M_{t_i} - M_{t_{i-1}})^2,$$

. for instance, if $M = B$ then $\langle B\rangle_t = t$,

. the square-integrable martingales M admit a bracket $\langle M\rangle$,

. $\langle M\rangle_t$ is the unique increasing continuous adapted process such that $M_t^2 - \langle M\rangle_t$ is a martingale.
This property, equivalent to the definition, is sometimes used as the definition.

Notation: on the σ-algebra $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$, we define the measure

$$\mu_M(A) = E[\int_0^\infty 1_A(t,\omega)d\langle M\rangle_t(\omega)].$$

In the Brownian case, $\mu_B(A) = E[\int_0^\infty 1_A(t,\omega)dt]$.
For any adapted process X, we denote

$$[X]_T^2 = E[\int_0^T X_t^2 d\langle M\rangle_t].$$

The processes X and $Y$are said to be "equivalent" if $X = Y\mu_M$ $a.s.$, let us remark that X and Y are equivalent if and only if $[X-Y]_T^2 = 0$ $\forall T \geq 0$.

Let us define the following set:

$$\mathcal{L} = \{ \text{ classes of } \mathcal{F}\text{-adapted measurable processes } X \text{ such that } \forall T,\ [X]_T < +\infty\}$$

endowed with the metric:

$$d(X,Y) = \sum_n 2^{-n} 1 \wedge [X-Y]_n. \tag{2}$$

Concerning the general case, we introduce the subset

$$\mathcal{L}^* = \{X \in \mathcal{L},\ \text{progressively measurable}\}$$

and we have to restrain the integral to this subset. But in the case where the martingale M is such that its increasing process $\langle M \rangle$ is absolutely continuous with respect to Lebesgue measure (meaning there exists a measurable function f such that $\langle M \rangle_t(\omega) = \int_0^t f(\omega, s)ds$), using that any element of $\mathcal{L}$ admits a modification in $\mathcal{L}^*$, in this case, we can manage only with $\mathcal{L}$; the Browian motion satisfies this property, the only case that we here completely study. Thus, we will not use the set $\mathcal{L}^*$.

Proposition 2.2 *Let $\mathcal{L}_T$ be the set of adapted measurable processes X on $[0, T]$ such that:*

$$[X]_T^2 = E[\int_0^T X_s^2 d\langle M \rangle_s] < +\infty.$$

$\mathcal{L}_T^*$ denotes the set of $\mathcal{L}_T$ progressively measurable elements. This set is closed in $\mathcal{L}_T$ and complete for the norm $[.]_T$.

Let a sequence $[X - X^n]_T \to 0$ in L^2 which is a complete set. Thus, $X \in \mathcal{L}_T$ and L^2 convergence imply the existence of a subsequence which converges almost surely. Let Y be this almost sure limit in $\Omega \times [0, T]$, meaning that $A = \{(\omega, t) :\ \lim_n X_t^n(\omega, t) \text{ exists}\}$ has a probability measure equal to 1,

$$Y(\omega, t) = X(\omega, t) \text{ if } (\omega, t) \in A,\ 0 \text{ ifnot }.$$

The fact that $\forall n,\ X^n \in \mathcal{L}_T^*$ shows that $Y \in \mathcal{L}_T^*$ and Y, X are equivalent. •

2.1.2 Integral of simple processes

Definition 2.3 *The following process X is said to be* **simple***:*

$$X_t = \xi_0 \mathbf{1}_0(t) + \sum_{i=0}^{\infty} \xi_i \mathbf{1}_{]t_i t_{i+1}]}(t),$$

ξ_i being a bounded $\mathcal{F}_{t_i}-$measurable random variable and (t_i) being a non negative real sequence going to infinity, $t_0 = 0$. We denote $\mathcal{S}$ their set.

We admit the following inclusions $\mathcal{S} \subset \mathcal{L}^* \subset \mathcal{L}$.
Exercise: Compute $[X]_T^2$ when $X \in \mathcal{S}$.

Definition 2.4 *Let $X \in \mathcal{S}$. The stochastic integral of X along M is*

$$I_t(X) := \sum_{i=1}^{\infty} \xi_i (M_{t\wedge t_{i+1}} - M_{t\wedge t_i}).$$

Exercise: Compute $E[(I_T(X))^2]$ when $X \in \mathcal{S}$. Do the comparison with $[X]_T^2$.

The next step is to extend this definition to a larger class of integrands.

Lemma 2.5 *For any* **bounded** *process $X \in \mathcal{L}$ there exists a sequence of processes $X_n \in \mathcal{S}$ such that* $\sup_{T\geq 0} \lim_n E[\int_0^T (X_n - X)^2(t) d\langle M\rangle_t] = 0$.

Preuve to be admitted.

Proposition 2.6 *In the case when $\mathbb{P}$-almost surely $\langle M\rangle_t$ is absolutely continuous with respect to Lebesgue measure dt, then $\mathcal{S}$ is dense in the metric space $(\mathcal{L}, d)$, d was defined above (2).*

This proposition thus proves that $\mathcal{S}$ is dense in $\mathcal{L}$ as soon as M satisfies the hypothesis

(H) the increasing process $\langle M\rangle_t$ is absolutely continuous with respect to dt.

The Brownian motion satisfies this hypothesis. Henceforth, we only manage with martingales satisfying (H) so we only consider the space $\mathcal{L}$ and no $\mathcal{L}^*$.

Remark 2.7 *useful: the metric (2) implies the following equivalent topology: $d(X_n, X) \to 0$ when n goes to the infinity if and only if*

$$\forall T > 0, E[\int_0^T |X_n(t) - X(t)|^2 d\langle M\rangle_t] \to 0.$$

2.1.3 Basic properties and extension of stochastic integral

Recall the simple X stochastic integral

$$I_t(X) := \sum_{i=1}^{\infty} \xi_i (M_{t\wedge t_{i+1}} - M_{t\wedge t_i}).$$

We note $I_t(X)$ or $\int_0^t X_s dM_s$ or $(X.M)_t$ to stress that the integrator is M. Elementary properties to show as an exercise:
Exercise. Let $\mathcal{S}$ the set of simple processes endowed with the stochastic integral with respect to M :

$$I_t(X) = \sum_j \xi_j (M_{t_{j+1}\wedge t} - M_{t_j\wedge t}).$$

Prove that I_t satisfies the following properties:

(i) I_t is a linear application on $\mathcal{S}$.

(ii) $I_t(X)$ is square-integrable.

(iii) $E[I_t(X)] = 0$.

(iv) $t \mapsto I_t(X)$ is a continuous martingale.

(v) $E[I_t(X)]^2 = E[\int_0^t X_s^2 d\langle M\rangle_s]$, meaning $[X]_t^2$.

(vi) $E[(I_t(X) - I_s(X))^2/\mathcal{F}_s] = E[\int_s^t X_u^2 d\langle M\rangle_u/\mathcal{F}_s] = E[(I_t(X))^2 - (I_s(X))^2/\mathcal{F}_s]$.

(vii) $\langle I_.(X)\rangle_t = \int_0^t X_s^2 d\langle M\rangle_s$.

These properties allow us to extend the set of integrands outside the set of simple processes, using the density results in Proposition 2.6. Moreover, we will show that the extended operator satisfies all properties (i) to (vii).

Proposition 2.8 *Let $X \in \mathcal{L}$ and a sequence of simple processes X^n going to X with the distance d. Then, the sequence $I_t(X^n)$ is a Cauchy sequence in $L^2(\Omega)$, and the limit doesn't depend on the chosen sequence, it is denoted as $I_t(X)$ or $\int_0^t X_s dM_s$ or $(X.M)_t$, stochastic integral of X with respect to the martingale M.*

Proof: to do as an exercise.

Definition 2.9 *This limit defines the stochastic integral $I_t(X)$.*

Now we prove the properties, 'going to the limit': I_t is an isometry from $(\mathcal{L}, d)$ to $L^2(\Omega)$, the simple processes are dense in $(\mathcal{L}, d)$.

Proposition 2.10 *Let $X \in \mathcal{L}$ then: I_t satisfies the following properties:*

(i) I_t is a linear application on $\mathcal{S}$.

(ii) $I_t(X)$ is square-integrable.

(iii) $E[I_t(X)] = 0$.

(iv) $t \mapsto I_t(X)$ is a continuous martingale.

(v) $E[I_t(X)]^2 = E[\int_0^t X_s^2 d\langle M\rangle_s]$, meaning $[X]_t^2$.

(vi) $E[(I_t(X) - I_s(X))^2/\mathcal{F}_s] = E[\int_s^t X_u^2 d\langle M\rangle_u/\mathcal{F}_s] = E[(I_t(X))^2 - (I_s(X))^2/\mathcal{F}_s]$.

(vii) $\langle I_.(X)\rangle_t = \int_0^t X_s^2 d\langle M\rangle_s$.

2.2 Quadratic variation

(cf. [13], pages 141-145; [23], pages 58-60.)

Let us consider the martingales M and N satisfying (H). We recall that $\langle M\rangle_t$ is the limit in probability $\lim_{|\pi|\to 0}$ proba $\sum_{t_i\in\pi}(M_{t_{i+1}}-M_{t_i})^2$. Similarly we define the quadratic covariation of two square-integrable continuous martingales M and N: let π be the partitions of $[0,t]$ we define

$$\langle M,N\rangle_t := \lim_{|\pi|\to 0} \text{proba} \sum_{t_i\in\pi}(M_{t_{i+1}}-M_{t_i})(N_{t_{i+1}}-N_{t_i})$$

or, equivalently:

$$4\langle M,N\rangle_t := \langle M+N\rangle_t - \langle M-N\rangle_t.$$

Thus when X and $Y \in \mathcal{L}(M)$, we have to study the "bracket" $\langle I(X), I(Y)\rangle$. Firstly some useful recalls concerning the square-integrable continuous martingales.

Proposition 2.11 *Let M et N two square-integrable continuous martingales, then:*

(i) $|\langle M,N\rangle_t|^2 \leq \langle M\rangle_t\langle N\rangle_t$*;*

(ii) $M_tN_t - \langle M,N\rangle_t$ *is a martingale.*

Corollary 2.12 $\forall s, \forall t,\ s\leq t,\ E[(M_t-M_s)(N_t-N_s)/\mathcal{F}_s] = E[(\langle M,N\rangle_t - \langle M,N\rangle_s/\mathcal{F}_s]$.

Preuve *Exercise*: (i) is a Cauchy inequality; since $M+N$ is a square-integrable continuous martingale, the difference $(M+N)^2-\langle M+N\rangle_t$ is a martingale, yields (ii) .

•

Proposition 2.13 *Let T be a stopping time and M and N two square-integrable continuous martingales, then:* $\langle M^T, N\rangle = \langle M, N^T\rangle = \langle M,N\rangle^T$.

Preuve: cf. Protter [23] th. 25, page 61, to admit. •

Theorem 2.14 *(Kunita-Watanabe inequality, cf. [13] Prop. 2.14 page 142.) Let M et N two square-integrable continuous martingales, $X\in\mathcal{L}(M)$ and $Y\in\mathcal{L}(N)$; then almost surely :*

$$(3)\qquad (\int_0^t |X_sY_s|d\langle M,N\rangle_s)^2 \leq \int_0^t |X_s|^2 d\langle M\rangle_s \int_0^t |Y_s|^2 d\langle N\rangle_s.$$

Proposition 2.15 *Let M and N be two square-integrable continuous martingales, $X\in\mathcal{L}(M)$ and $Y\in\mathcal{L}(N)$; then:*

$$(4)\qquad \langle X.M, Y.N\rangle_t = \int_0^t X_uY_u d\langle M,N\rangle_u,\ \forall t\in\mathbb{R},\ \mathbb{P}\ p.s.$$

and

$$(5)\qquad E[\int_s^t X_u dM_u \int_s^t Y_u dN_u/\mathcal{F}_s] = E[\int_s^t X_uY_u d\langle M,N\rangle_u/\mathcal{F}_s],\ \forall s\leq t,\ \mathbb{P}\ p.s.$$

The case $M = N$ is nothing else but the stochastic integral property (vii), applied to $I_t(X+Y)$. Besides it is easy to see that (4) and (5) are equivalent.

Proposition 2.16 *The stochastic integral is "associative" as following: let $H \in \mathcal{L}(M)$ and $G \in \mathcal{L}(H.M)$, then $GH \in \mathcal{L}(M)$ and:*

$$G.(H.M) = GH.M$$

Preuve *Exercise*: cf. [23] Th. 19 page 55 or [13] Corollary 2.20, page 145. •

2.3 Exercises

Let M square integrable martingale on $(\Omega, (\mathcal{F}_t; t \in \mathbb{R}^+), P)$ with bracket $\langle M \rangle_t$ absolutely continuous with respect to Lebesgue measure.

1. Let the simple process X:

$$X_t = \xi_0 \mathbf{1}_0(t) + \sum_{i=0}^{\infty} \xi_i \mathbf{1}_{]t_i t_{i+1}]}(t),$$

Compute $[X]_T^2$ when $X \in \mathcal{S}$ and $E[(I_t(X))^2]$.

2. Let $\mathcal{L}_T(M)$ be the set

$$\{X \text{adapted measurable process on } [0,T] \text{ such that: } [X]_T^2 = E[\int_0^T X_s^2 d < M >_s] < +\infty\}.$$

Prove that $(\mathcal{L}_T(M), d)$ is a metric space with distance d:

$$d(X,Y) = \sqrt{[X-Y]_T^2}.$$

3. Prove the following are equivalent:

$$\lim_{n\to\infty} \sum_{j\geq 1} 2^{-j} \inf(1, [X - X_n]_j) = 0 \Longleftrightarrow \forall T,\ [X - X_n]_T \longrightarrow_{n\to\infty} 0.$$

4. Let $\mathcal{S}$ the set of simple processes endowed with stochastic integral w.r.t. M:

$$I_t(X) = \sum_j X_j (M_{t_{j+1}\wedge t} - M_{t_j \wedge t}).$$

Prove that I_t (also denoted as $(X.M)_t$) satisfies the following properties:
(i) I_t is a linear map on $\mathcal{S}$.
(ii) $I_t(X)$ is square-integrable.
(iii) $E[I_t(X)] = 0$.
(iv) $I_t(X)$ is a continuous martingale.
(v) $E([I_t(X)]^2) = E[\int_0^t X_s^2 d < M >_s]$.
(vi) $E[(I_t(X) - I_s(X))^2/\mathcal{F}_s] = E[I_t^2(X) - I_s^2(X)/\mathcal{F}_s] = E[\int_s^t X_u^2 d < M >_u /\mathcal{F}_s]$.
(vii) $< I_.(X) >_t = \int_0^t X_s^2 d < M >_s$.

5. Proposition 2.11:
Let M et N two square-integrable continuous martingales, then:

(i) $|\langle M, N\rangle_t|^2 \leq \langle M\rangle_t \langle N\rangle_t$;

(ii) $M_t N_t - \langle M, N\rangle_t$ is a martingale.

6. Proposition 2.16:
Prove that the stochastic integral is associative i.e.: if H is M-integrable, the integral being denoted as $H.M$, and if G is $H.M$-integrable, then the product GH is M-integrable and:

$$G.(H.M) = (GH).M.$$

3 Ito formula

(cf. [13], pages 149-156, [23], pages 70-83.)

This new tool allows us "integrodifferential" calculus, namely "Itô calculus". It concerns calculus about process trajectories, meaning what happens when alea ω occurs.

We first recall what is the integral along finite variation processes.

Definition 3.1 *Let A a continuous process.It is said to be* **with finite variation** *if* $\forall t$, *considering partitions π of $[0,t]$ yields:*

$$\lim_{|\pi|\to 0} \sum_{t_i \in \pi} |A_{t_{i+1}} - A_{t_i}| < \infty \ \mathbb{P} \ a.s.$$

Such processes, for one fixed ω, induce Stieltjes integrals. For instance if A is a non decreasing process, for any partition π of $[0,t]$, $V_\pi(A) = A_t - A_0$.

Theorem 3.2 *(cf. Protter [23], Th. 31 page 71) Let A a finite variation continuous process, and a C^1 class function f. Then, $f(A_.)$ is a finite variation continuous process and:*

$$f(A_t) = f(A_0) + \int_0^t f'(A_s) dA_s.$$

This is the plain Taylor formula, order 1.

These processes, joined with the continuous local martingales generate a large set of integrators which we consider now.

Definition 3.3 *A* **continuous semi-martingale** *is a process X on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, defined $\mathbb{P}$ a.s. as following:*

$$X_t = X_0 + M_t + A_t, \ \forall t \geq 0,$$

X_0 is $\mathcal{F}_0$-measurable, M is a continuous local martingale and $A = A^+ - A^-$, A^+ and A^- being adapted finite variation nondecreasing continuous processes.

Remark 3.4 *A finite variation continuous process admits a null quadratic variation: $\langle A \rangle_t = 0 \ \forall t$.*

Proof: *exercise,* first use an nondecreasing process A, then apply to $A = A^+ - A^-$.

This notion is particularly important because of arbitrage opportunity absence hypothesis (Arbitrage opportunity means to obtain a nonnegative wealth with a positive probability starting with a null wealth).
Indeed, Delbaen-Schachermayer proved [5] that under AOA Hypothesis the price processes are necessarily semi-martingales.

Definition 3.5 *If X is the continuous real semi-martingale $X_0 + M + A$, one notes $\langle X\rangle$ the limit in probability of increments square sum (similarly martingale case); actually, one shows that this limit is $\langle M\rangle$. Similarly, let two continuous semi-martingales X et Y, one notes $\langle X, Y\rangle$ their martingale part bracket.*

3.1 Itô rule, or variable change formula

Theorem 3.6 *(Itô, 1944 and Kunita-Watanabé, 1967) Let $f \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$ and X a continuous semi-martingale. Then $\mathbb{P}$ a.s. $\forall t \geq 0$:*

$$df(X_s) = f'(X_s)dX_s + \frac{1}{2}f"(X_s)d\langle X\rangle_s,$$

meaning:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dA_s + \frac{1}{2}\int_0^t f"(X_s)d\langle M\rangle_s,$$

here the first integral is a stochastic integral, the two others are Stieltjes integral.

We already saw this formula in Chapter 1, Section 1.3 for the function $f : x \mapsto x^2$ and $X = B : B_t^2 = 2\int_0^t B_s dB_s + t$.

Differential notation: $f(X_t)$ stochastic differential is

$$df(X_s) = f'(X_s)dX_s + \frac{1}{2}f"(X_s)d\langle X\rangle_s,$$

thus we can manage with a stochastic differential calculus. Think of this formula as somethink like a Taylor formula to the order 2.
Preuve: four steps:

localisation to be in bounded case,

Taylor development of function f to the order 2,

studying the term which will be the stochastic integral,

studying the term which will be the quadratic variation.

1. Let the stopping time

$$\begin{aligned} T_n &= 0 \text{ si } |X_0| \geq n, \\ &\quad \inf\{t \geq 0; |M_t| \geq n \text{ or } |A_t| \geq n \text{ or } \langle M\rangle_t \geq n\} \\ &\quad +\infty \text{ if not.} \end{aligned}$$

This stopping times sequence is almost surely increasing to $+\infty$. The property to be proved is true along trajectories, thus we only need to prove it on $\{(t,\omega),\ t \leq T_n(\omega)\}$

meaning that we stop at time T_n. Then n goes to infinity. This allows us to suppose bounded the processes $M, A, \langle M\rangle$, the random variable X_0, the process X, and we can suppose f to have a compact support: $f(X_.), f'(X_.), f"(X_.)$ are bounded.

2. To obtain Ito formula, particularly the stochastic integral term, as we did in Chapter 1 to study B_t^2, we cut the interval $[0,t]$ as a partition $\pi = (t_i, i = 1, ..., n)$ and we study the increments of $f(X_.)$ along this partition:

$$
\begin{aligned}
(6) \qquad f(X_t) - f(X_0) &= \sum_{i=0}^{n-1}(f(X_{t_{i+1}}) - f(X_{t_i})) = \\
\sum_{i=0}^{n-1} f'(X_{t_i})(M_{t_{i+1}} - M_{t_i}) &+ \sum_{i=0}^{n-1} f'(X_{t_i})(A_{t_{i+1}} - A_{t_i}) + \frac{1}{2}\sum_{i=0}^{n-1} f"(\eta_i)(X_{t_{i+1}} - X_{t_i})^2,
\end{aligned}
$$

where $\eta_i = \lambda X_{t_i} + (1-\lambda)X_{t_{i+1}}$.
The second term obviously goes to the Stieltjes integral of $f'(X_s)$ with respect to A,without any stochastic notion.

3. Concerning the first term, we consider the simple process associated to the partition π :

$$Y_s^\pi = f'(X_{t_i}) \text{ if } s \in]t_i, t_{i+1}].$$

Here $f'(X_{t_i})$ is $\mathcal{F}_{t_i}$-measurable, thus $Y^\pi \in \mathcal{S}$ and this first term is equal to $\int_0^t Y_s^\pi dM_s$ by definition. But,

$$\int_0^t |Y_s^\pi - f'(X_s)|^2 d\langle M\rangle_s = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |f'(X_{t_i}) - f'(X_s)|^2 d\langle M\rangle_s.$$

The application $s \mapsto f'(X_s)$ is continuous, so the above integrand almost surely converges to zero. Using the boundness of f' and Lebesgue Theorem, yields Y_s^π L^2-converges to $f'(X_s)$ in the metric space $(\mathcal{S}, d)$, with the metric d defined in Chapter 2: stochastic integral definition implies that the first term L^2-converges to

$$\int_0^t f'(X_s)dM_s.$$

4. **Quadratic variation term** : we cut it in three terms :

$$
\begin{aligned}
(7) \qquad \sum_{i=0}^{n-1} f"(\eta_i)(X_{t_{i+1}} - X_{t_i})^2 &= \sum_{i=0}^{n-1} f"(\eta_i)(M_{t_{i+1}} - M_{t_i})^2 \\
+2\sum_{i=0}^{n-1} f"(\eta_i)(M_{t_{i+1}} - M_{t_i})(A_{t_{i+1}} - A_{t_i}) &+ \sum_{i=0}^{n-1} f"(\eta_i)(A_{t_{i+1}} - A_{t_i})^2
\end{aligned}
$$

The last term is less than $\|f"\| \sup_i |\Delta_i A| \sum_{i=0}^{n-1} |\Delta_i A|$, where the first and the third factors are bounded by hypothesis; $\sup_i |\Delta_i A|$ goes to zero almost surely since A is continuous.

We bound the second term by $\|f"\| \sup_i |\Delta_i M| \sum_{i=0}^{n-1} |\Delta_i A|$ which also converges almost surely to zero since M is continuous and $\sum_{i=0}^{n-1} |\Delta_i A| \leq V_t(A) < \infty$.

The first term of (7) is "near" of

$$\sum_{i=0}^{n-1} f"(X_{t_i})(M_{t_{i+1}} - M_{t_i})^2.$$

Indeed :

$$\sum_{i=0}^{n-1}(f"(\eta_i) - f"(X_{t_i}))(\Delta_i M)^2 \leq \sup_i |f"(\eta_i) - f"(X_{t_i})| \sum_{i=0}^{n-1}(\Delta_i M)^2$$

where the first factor goes almost surely to zero using $f"$ continuity and the second factor goes to $\langle M\rangle_t$ in probability by definition; thus a subsequence converges almost surely. Then the product goes to zero in L^2 using Lebesgue Theorem. Finally we have to study

$$\sum_{i=0}^{n-1} f"(X_{t_i})(M_{t_{i+1}} - M_{t_i})^2$$

to be compared to $\sum_{i=0}^{n-1} f"(X_{t_i})(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i})$. The L^2 limit of that is $\int_0^t f"(X_s)d\langle M\rangle_s$ since

- by continuity the simple process $\sum_i f"(X_{t_i})\mathbf{1}_{]t_i,t_{i+1}]}$ almost surely converges to $f"(X_s)$;

- Lebesgue Theorem allows to conclude.

Denoting $\langle M\rangle_s^t = \langle M\rangle_t - \langle M\rangle_s$, let the difference :

$$\sum_{i=0}^{n-1} f"(X_{t_i})[(M_{t_{i+1}} - M_{t_i})^2 - \langle M\rangle_{t_i}^{t_{i+1}}],$$

we study the L^2 limit; inside the square expectation, let us consider the terms:

$$i < k \ : \ E[f"(X_{t_i})f"(X_{t_k})(\Delta_i M^2 - \langle M\rangle_{t_i}^{t_{i+1}})(\Delta_k M^2 - \langle M\rangle_{t_k}^{t_{k+1}})]$$

Using $\mathcal{F}_{t_i}$ conditional expectation, we deduce that these terms are null, since $M^2 - \langle M\rangle$ is a martingale. Finally look at the square terms:

$$\begin{aligned}\sum_i E[(f"(X_{t_i}))^2(\Delta_i M^2 - \langle M\rangle_{t_i}^{t_{i+1}})^2] &\leq 2\|f"\|_\infty^2 \sum_i [E(\Delta_i M^4) + E((\langle M\rangle_{t_i}^{t_{i+1}})^2)] \\ &\leq 2\|f"\|_\infty^2 E[(\sup_i \Delta_i M^2 \sum_i \Delta_i M^2) + \sup_i(\langle M\rangle_{t_i}^{t_{i+1}})\langle M\rangle_t]\end{aligned}$$

In the majoration, $\sup_i \Delta_i M^2$ and $\sup_i(\langle M\rangle_{t_i}^{t_{i+1}})$ are bounded and converge almost surely to zero by continuity; $\sum_i \Delta_i M^2$ converges to $\langle M\rangle_t$ in probability by definition; globally, Lebesgue Theorem yields the L^1-convergence to zero of a subsequence.

Finally the sequence of sums (6) converges in probability to the Theorem expression; we conclude using the almost sure convergence of a subsequence.

3.2 Extension and applications

We can extend this result to functions of vectorial semi-martingales, also depending on time.

Proposition 3.7 *Let M be a d-dimension vector of continuous local martingales, and let A be a d-dimension vector of adapted continuous processes with finite variation, X_0 a random $\mathcal{F}_0$-measurable. Let $X_t = X_0 + M_t + A_t$. Then, $\mathbb{P}$ almost surely :*

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \partial_t f(s, X_s)ds + \int_0^t \partial_i f(s, X_s)dM_s^i + \int_0^t \partial_i f(s, X_s)dA_s^i \\ &\quad + \frac{1}{2}\int_0^t \sum_{ij} \partial^2_{ij} f(s, X_s)d\langle M^i, M^j\rangle_s \end{aligned}$$

Preuve: *exercise*, but very long and tedious....

When f and its derivatives are bounded and M is a square integrable martingale, the above stochastic integral term is a "true" martingale, null at time $t = 0$ and:

$$f(t, X_t) - f(0, X_0) - \int_0^t \partial_t f(s, X_s)ds - \int_0^t \partial_i f(s, X_s)dA_s^i - \frac{1}{2}\int_0^t \partial^2_{ij} f(s, X_s)d\langle M^i, M^j\rangle_s \in \mathcal{M}$$

For instance if $A = 0$ and $X = M$ is the Brownian motion, yields:

$$f(t, X_t) - f(0, X_0) - \int_0^t \mathcal{L}f(s, X_s)ds \in \mathcal{M}$$

where the differential operator $\mathcal{L} = \partial_t + \frac{1}{2}\sum_i \partial^2_{ii}$ and $\mathcal{M}$ denotes the martingales set.

On another hand, Ito formula allows to deduce the solution of the so called "equation de la chaleur", meaning the partial derivatives equation:

$$f \in C^{1,2}(\mathbb{R}^+, \mathbb{R}^d),\ \partial_t f = \sum_i \frac{1}{2}\partial^2_{ii} f \text{ and } f(0, x) = \varphi(x)$$

where $\varphi \in C^2_b(\mathbb{R}^d)$ and its unique solution is

$$f(t, x) = E[\varphi(x + B_t)].$$

We can easily see that actually this function is a solution, using Ito formula; uniqueness is a consequence of PDE theory.
cf. exercise 5 below: develop $\varphi(x + X_t)$ using Itô formula, take the expectation to get $f(t, x)$, differentiate w.r.t. t and x.

The next corollary is very easy to prove and very useful.

Corollary 3.8 *Let two real continuous semi-martingales X and Y; then:*

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \langle X, Y\rangle_t.$$

This is the **integration by parts formula**.

Preuve: *exercise*; apply Ito formula to the function on $\mathbb{R}^2$, $(x, y) \mapsto xy$.

3.3 Exercises

Let a square integrable martingale M on the filtered probability space $(\Omega, \mathcal{F}_t, P)$ such that $d\langle M\rangle_t$ is absolutely continuous with respect to dt.
1. Let M be a continuous martingale and $X \in \mathcal{L}(M)$. Let $s < t$ and Z a $\mathcal{F}_s$-measurable bounded random variable; prove:

$$\int_s^t ZX_u dM_u = Z \int_s^t X_u dM_u.$$

indication: use the property (vi) of stochastic integral, Proposition 2.11 and its corollary to compute $E[\int_s^t ZX_u dM_u - Z\int_s^t X_u dM_u]^2$.

2. Remark 3.4.
A finite variation continuous process admits a null quadratic variation: $\langle A\rangle_t = 0\ \forall t$.

3. Let two semi-martingales $X = X_0 + M + A$ and $Y = Y_0 + N + C$. Use Ito formula to:

a) prove that $\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \langle X, Y\rangle_t$. i.e. the *integration by parts formula.*

b) develop the following processes

$$t \mapsto X_t^{-1}\ ;\ t \mapsto \exp(X_t)\ ;\ t \mapsto X_t.Y_t^{-1}.$$

4. **Lévy Theorem**: Let X be a continuous semi-martingale, $X_0 = 0$ almost surely. This process is a real Brownian motion iff it is a continuous local martingale with bracket $\langle X\rangle_t = t$.
Indication: firstly, using Ito formula, compute the conditional characteristic function of $X_t - X_s$ *given* $\mathcal{F}_s$, $\forall s \leq t$.

5. Prove that the unique solution in $\mathcal{C}^{1,2}(R^+, R^d)$ of the PDE (Heat Equation)

$$\partial_t f = \frac{1}{2}\Delta f, f(0,x) = \varphi(x),\ \forall x \in R^d,$$

$\varphi \in \mathcal{C}^2(R^d)$ is $f(t,x) = E[\varphi(x + B_t)]$ where B is the Brownien $d-$dimensional motion.

6. Let M be a $d-$dimensional vector of continuous martingales, A be a $d-$dimensional vector ofadapted continuous processes with finite variation, X_0 a $\mathcal{F}_0$-measurable random variable; let $f \in C^{1,2}(\mathbb{R}^+, \mathbb{R}^d)$. Let $X_t = X_0 + M_t + A_t$. Then prove that $\mathbb{P}$ almost surely:

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \partial_t f(s, X_s) ds + \int_0^t \partial_i f(s, X_s) dM_s^i + \int_0^t \partial_i f(s, X_s) dA_s^i \\ &+ \frac{1}{2}\int_0^t \partial_{ij}^2 f(s, X_s) d\langle M^i, M^j\rangle_s \end{aligned}$$

4 Stochastic exponentials, examples of stochastic differential equations

There exists other applications of Ito formula: Brownian motion is very useful to model some additive noise, measure error, in a differential equation. For instance, let us suppose the dynamics:

$$\dot{x}(t) = a(t)x(t),\ t \in [0,T],\ x(0) = x.$$

Unhappily, this is not exact, there is some noise which is added to the speed, so we get:

$$dX_t = a(t)X_t dt + b(t)dB_t,\ t \in [0,T],\ X_0 = x,$$

f a so called **stochastic differential equation, SDE**. *cf. exercises.*

Here we don't study the more general SDE, but we give a second example below.

4.1 Stochastic exponential

Let us consider the C^∞ class function $f : x \mapsto e^x$, and a continuous semi-martingale $X,\ X_0 = 0$, we apply the Ito formula to the semi-martingale $X_t - \frac{1}{2}\langle X\rangle_t$ and the exponential function, meaning that we compute the "stochastic differential" of the process $t \mapsto Z_t = \exp(X_t - \frac{1}{2}\langle X\rangle_t)$. Yields:

$$Z_t = 1 + \int_0^t [\exp(X_s - \frac{1}{2}\langle X\rangle_s)(dX_s - \frac{1}{2}d\langle X\rangle_s) + \frac{1}{2}\exp(X_s - \frac{1}{2}\langle X\rangle_s)d\langle X\rangle_s].$$

After some simplifications (remind that two semimartingales with the same martingale part have same bracket) :

$$Z_t = 1 + \int_0^t \exp(X_s - \frac{1}{2}\langle X\rangle_s)dX_s,$$

or using the differential notation: $dZ_s = Z_s dX_s, Z_0 = 1$. Here is the linear stochastic differential equation.

Theorem 4.1 *Let X be a continuous semi-martingale, $X_0 = 0$. Then there exists a unique continuous semi-martingale which is solution of the stochastic differential equation:*

$$Z_t = 1 + \int_0^t Z_s dX_s \tag{8}$$

admitting the closed form:

$$Z_t(X) = \exp(X_t - \frac{1}{2}\langle X\rangle_t).$$

The Ito formula proves that this process is actually solution of the equation. Here, we admit the uniqueness (*exercise*).

Definition 4.2 *Let X be a continuous semi-martingale, $X_0 = 0$. The* **stochastic exponential** *of X, denoted $\mathcal{E}(X)$, is the unique solution of the differential equation (8).*

Example : Let $X = aB$ with a a real number and B the Brownian motion; then $\mathcal{E}_t(aB) = \exp(aB_t - \frac{1}{2}a^2 t)$, so called "geometric Brownian motion".

Here are some results about these stochastic exponentials.

Theorem 4.3 *(cf [23], th. 37) Let X and Y two continuous semi-martingales, $X_0 = Y_0 = 0$. Then*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + \langle X, Y\rangle).$$

Proof as an exercise: Let $U_t = \mathcal{E}_t(X)$ et $V_t = \mathcal{E}_t(Y)$ and apply the formula (3.8):

$$U_t V_t - 1 = \int_0^t U_s dV_s + V_s dU_s + d\langle U, V\rangle_s$$

Putting $W = UV$ and using the differential definition of the stochastic exponential yield the result. •

Corollary 4.4 *Let X a continuous semi-martingale, $X_0 = 0$. Then the inverse $\mathcal{E}_t^{-1}(X) = \mathcal{E}_t(-X + \langle X\rangle)$*

Proof as an *exercise.*

We now consider a little bit more general stochastic differential equations.

Theorem 4.5 *(cf [23], th. 52, page 266.) Let Z and H be two real continuous semi-martingales, $Z_0 = 0$. Then the unique solution of the stochastic differential equation:*

$$X_t = H_t + \int_0^t X_s dZ_s$$

$$\textit{is: } \mathcal{E}_H(Z)_t = \mathcal{E}_t(Z)(H_0 + \int_0^t \mathcal{E}_s^{-1}(Z)(dH_s - d\langle H, Z\rangle)_s).$$

Once again the proof is an exercise using Ito formula...

The example quoted on the beginning of this chapter is important since it is often used in Finance (for instance to model the rate dynamics): **Ornstein-Uhlenbeck** equation (cf. [13], page 358 and exercises below) :

$$dX_t = a(t)X_t dt + b(t)dB_t,\ t \in [0, T],\ X_0 = x$$

where a and b are $\mathcal{F}-$adapted processes, a is almost surely time integrable and $b \in L^2(\Omega \times [0,T], d\mathbb{P} \otimes dt)$. When these processes are constants (α et σ), yields the solution:

$$X_t = e^{-\alpha t}(x + \int_0^t \sigma e^{\alpha s} dB_s).$$

In this case we also can show:

$$\begin{aligned} m(t) &= E(X_t) = m(0)e^{-\alpha t} \\ V(t) &= Var(X_t) = \frac{\sigma^2}{2\alpha} + (V(0) - \frac{\sigma^2}{2\alpha})e^{-2\alpha t} \\ \rho(s,t) &= cov(X_s, X_t) = [V(0) + \frac{\sigma^2}{2\alpha}(e^{2\alpha(t\wedge s)} - 1)]e^{-\alpha(t+s)} \end{aligned}$$

4.2 Link with PDE

(cf. [13] 5.7 pages 363 et sq.)
In this section we use a SDE solution with initial condition $X_t = x$:

$$X_s^{t,x} = x + \int_t^s b(u, X_u)du + \sigma(u, X_u)dW_u \tag{9}$$

and the following assumptions:
(i) the coefficients are continuous, increasing at most linearly in the space,
(ii) there exists a unique solution in law, i.e. **weak solution**: there exists a probability measure $\mathbb{P}_x$ on the Wiener space $(\Omega, \mathcal{F})$ under which

. X is continuous $\mathcal{F}-$adapted, takes its values in $\overline{\mathbb{R}}$,

. if $S_n = \inf\{t : |X_t| > n\}$, X^{S_n} satisfies the existence conditions for strong solutions (meaning trajectorial solutions).
The increasing limit of times S_n is called the explosion time. $\mathbb{P}_x$-almost surely, $\forall n$

$$X_{t\wedge S_n} = x + \int_t^{t\wedge S_n} b(u, X_u)du + \int_t^{t\wedge S_n} \sigma(u, X_u)dW_u$$

4.2.1 Dirichlet problem

Let D an open set in $\mathbb{R}^d$.

Definition 4.6 *An order 2 differential operator $\mathcal{A} = \sum_{i,j} a_{i,j}(x)\partial^2_{ij}$ is said to be x-**elliptic** if*

$$\forall \xi \in \mathbb{R}^d_*, \ \sum_{i,j} a_{i,j}(x)\xi_i\xi_j > 0.$$

If $\mathcal{A}$ is elliptic for any point in D, it is said to be elliptic in D.
If there exists $\delta > 0$ such that

$$\forall x \in D, \ \forall \xi \in \mathbb{R}^d, \ \sum_{i,j} a_{i,j}(x)\xi_i\xi_j \geq \delta\|\xi\|^2,$$

A is said to be uniformly elliptic.

The **Dirichlet problem** *is to find a C^2 -class function u and satisfying:*

$$\mathcal{A}u - ku = -g, \forall x \in D;\ u(x) = f(x),\ \forall x \in \partial D,$$

$\mathcal{A}$ is elliptic, $k \in \mathcal{C}(\bar{D}, \mathbb{R}^+)$, $g \in \mathcal{C}(\bar{D}, \mathbb{R})$, $f \in \mathcal{C}(\partial D, \mathbb{R})$.

Proposition 4.7 *(Proposition 7.2, page 364 [13])*
Let u be a solution of Dirichlet problem $(\mathcal{A}, D)$ and X solution of (9) with the operator $\mathcal{A} = \frac{1}{2}\sum_{i,j,l} \sigma_l^i \sigma_l^j(x)\partial_{ij}^2 + \nabla.b(x)$; T_D the hitting time of X out of D. If

$$E_x(T_D) < \infty,\ \forall x \in D, \tag{10}$$

then $\forall x \in \bar{D}$,

$$u(x) = E_x[f(X_{T_D})\exp(-\int_0^{T_D} k(X_s)ds) + \int_0^{T_D} g(X_t)\exp(-\int_0^t k(X_s)ds)dt).$$

Proof as an exercise (problem 7.3 in [13], corrected page 393).
First of all the X continuity implies $X_{T_D} \in \partial D$.
Indication : prove that

$$M : t \mapsto u(X_{t\wedge T_D})\exp(-\int_0^{t\wedge T_D} k(X_s)ds) + \int_0^{t\wedge T_D} g(X_s)\exp(-\int_0^s k(X_u)du)ds, t \geq 0$$

is a uniformly $\mathbb{P}_x$ integrable martingale: one compute $E_x(M_0) = E_x(M_\infty)$; on $\{t < T_D\}$, we do the Ito differential of M and we use that on $D, \mathcal{A}u - ku + g = 0$. $M_0 = u(x)$ since $X_0 = x$ $\mathbb{P}_x$ almost surely,

$$dM_t = \exp(-\int_0^{t\wedge T_D} k(X_s)ds)[\mathcal{A}u(X_{t\wedge T_D})dt + \nabla u(X_{t\wedge T_D})\sigma(t, X_{t\wedge T_D})dW_t + g(X_{t\wedge T_D}) - (k.u)(X_{t\wedge T_D})dt],$$

the functions ∇u and σ are continuous thus bounded on the compact set $\bar{D}$. So the second term above is a martingale, moreover the other terms cancel since $\mathcal{A}u - ku + g = 0$ and $\forall t, E_x[M_t] = u(x)$.

This martingale is uniformly integrable since it is L^2-bounded, so let t go to infinity and apply stopping Doob theorem, available thanks to $E_x[T_D] < \infty$. •

Remark 4.8 *(Friedman, 1975)*
A sufficient condition to get the hypothesis (10) is: $\exists l, \exists \alpha\ :\ a_{l,l}(x) \geq \alpha > 0$. This condition implies ellipticity but it is weaker than the uniform ellipticity in D.

Let:

$$b^* = \max\{|b_l(x)|, x \in \bar{D}\}, q = \min\{x_l, x \in \bar{D}\},$$

choose $\nu > 4b^*/\alpha, h(x) = -\mu \exp(\nu x_l), x \in D$, μ will be chosen later. Then h is C^∞, $-\mathcal{A}h(x)$ can be computed and bounded below:

$$-\mathcal{A}h(x) = (\frac{1}{2}\nu^2 a_{ll} + \nu b_l(x))\mu e^{\nu x_l} \geq (\frac{8(b^*)^2}{\alpha} - \frac{4b^*}{\alpha}b^*)\mu e^{\nu x_l} \geq \frac{4(b^*)^2}{\alpha}\mu e^{\nu q} \geq 1.$$

Then we choose μ great enough so that $-\mathcal{A}h(x) \geq 1$; $x \in D$, h and its derivatives are bounded in D, we apply Itô formula to h:

$$h(X_t^{T_D}) = h(x) + \int_0^{t\wedge T_D} \mathcal{A}h(X_s)ds + \int_0^{t\wedge T_D} \nabla h(X_s)\sigma(X_s)dW_s.$$

Yields

$$t \wedge T_D \leq h(x) - h(X_t^{T_D}) = -\int_0^{t\wedge T_D} \mathcal{A}h(X_s)ds + M_t,$$

M being a uniformly integrable martingale. Thus $E_x[t \wedge T_D] \leq 2\|h\|_\infty$, we get the result with t going to infinity.

4.2.2 Feynman-Kac formula

Let us consider the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,\ X_0 = x, \tag{11}$$

the associated infinitesimal generator $\mathcal{L}_t = b(t,.)\partial_x + \frac{1}{2}\sigma(t,.)\partial^2_{xx}$, the rate coefficient r defined on $[0,T] \times \mathbb{R}$ taking its values in $\mathbb{R}^+_*$ and the partial derivatives equation:

$$rv - \partial_t v - \mathcal{L}_t v = f \text{ on } [0,T] \times \mathbb{R},\ v(T,.) = g. \tag{12}$$

This problem is known as the Cauchy problem. This problem admits a solution using this of (11).

Theorem 4.9 *Let v be a $C^{1,2}([0,T] \times \mathbb{R})$ solution of (12). Suppose that $\partial_x v$ is bounded. Then v admits the representation*

$$v(t,x) = E\left[\int_t^T e^{-\int_t^s r(u,X_u^{t,x})du} f(s, X_s^{t,x})ds + e^{-\int_t^T r(u,X_u^{t,x})du} g(X_T^{t,x})\right],\ \forall (t,x).$$

4.3 Black-Scholes model

This model is exactly the stochastic exponential, with constant coefficients: the risky assets is solution of the SDE

$$dS_t = S_t b dt + S_t \sigma dW_t, S_o = s,$$

the coefficient b is called the "trend" and σ the "volatility". Using everything above, this SDE admits the explicit unique solution:

$$S_t = s\exp[\sigma W_t + (b - \frac{1}{2}\sigma^2)t].$$

We remark that $\log S_t$ law is Gaussian, thus its support is all $\mathbb{R}$, and so the exponential support is all $\mathbb{R}^+ - \{0\}$.

Definition 4.10 *We call* **risk neutral probability measure** *any probability Q, equivalent to $\mathbb{P}$, such that all prices are $(\mathcal{F}, Q)-$martingales.*
A market is called **viable** *if AOA hypothesis is satisfied. A sufficient condition is there exists at least one risk neutral probability measure.*
A market is called **complete** *as soon as, $\forall X \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$, there exists a process θ, integrable with respect to the prices vector and such that $X = E(X) + \int_0^T \theta_t dS_t$.*

In Chapter 6 we will see that the Black and Scholes model is viable and complete, with the unique risk neutral probability measure

$$Q = L_T\mathbb{P}, dL_t = -L_t\sigma^{-1}(b-r)dW_t, t \in [0,T], L_0 = 1.$$

Definition 4.11 *A "call option " is the following contract: the buyer pays at time 0 a sum q which allows him to buy at time 1 the assets at price K, but it is not an obligation. If at final time $T, S_T > K$, he buys and he wins $S_T - K - q$. If not, he does nothing and losses q. Globally he wins $(S_T - K)^+ - q$.*

A "put option " is the following contract: the buyer pays at time 0 a sum q which allows him to sell at time 1 the assets at price K, but it is not an obligation. If at final time $T, S_T < K$, he sells and he wins $K - S_T - q$. If not, he does nothing and losses q. Globally he wins $(K - S_T)^+ - q$.

Then the problem is to find a *fair price*, q, between the buyer an the seller of this contract. This is the aim of **Black-Scholes formula**
To do that, we assume that the hedging portfolio θ of this contingent claim is such that there exists a $C^{(1,2)}$-class function C such that the value V satisfies:

$$V_t(\theta) = C(t, S_t). \tag{13}$$

On another hand, θ is (a, d) and we get

$$V_t(\theta) = a_t S_t^0 + d_t S_t \tag{14}$$

Under the hypothesis "the portfolio θ is self-financing", we get

$$V_t(\theta) = \langle \theta_0, p_0 \rangle + \int_0^t a_s dS_s^0 + \int_0^t d_s dS_s. \tag{15}$$

Using this strategy θ, the option seller could "hedge" the option with initial price $q = V_0$.

We can compute the V differential by two ways, then we will identify them; starting with (13) and Ito formula :

$$dV_t(\theta) = \partial_t C(t, S_t)dt + \partial_x C(t, S_t)dS_t + \frac{1}{2}\partial^2_{x^2} C(t, S_t)S_t^2\sigma^2 dt,$$

starting with (15) :

$$dV_t(\theta) = ra_t S_t^0 dt + d_t S_t(bdt + \sigma dW_t).$$

The identification implies two equations (dt coefficient and dW_t coefficient) and (14)which is merely $C(t, S_t))$:

$$\begin{aligned} &\partial_t C(t, S_t) + bS_t\partial_x C(t, S_t) + \frac{1}{2}\partial^2_{x^2} C(t, S_t)S_t^2\sigma^2 = ra_t S_t^0 + d_t S_t b \\ &\partial_x C(t, S_t)S_t\sigma = d_t S_t\sigma. \end{aligned} \tag{16}$$

Yields the portfolio:

$$d_t = \partial_x C(t, S_t)\ ;\ a_t = \frac{C(t, S_t) - S_t\partial_x C(t, S_t)}{S_t^0}. \tag{17}$$

To obtain the hedging portfolio explicitely, we need the function C, solution of the PDE which could be obtained using the first equation of (16) and (17).
Remark: indeed we can replace S_t by $x > 0$ since it is a lognormal random variable thus with a support $= \mathbb{R}^+_*$:

$$\partial_t C(t, x) + rx\partial_x C(t, x) + \frac{1}{2}\partial^2_{x^2} C(t, x)x^2\sigma^2 = rC(t, x),$$

$$C(T, x) = (x - K)^+, x > 0,\ t \in [0, T].$$

This is a Cauchy problem with $D =]0, T[\times\mathbb{R}^+_*$ and the operator $\frac{1}{2}x^2\sigma^2\partial^2_{x^2} + \partial_t + rx\partial_x$, $g = 0,\ k(x) = r,\ f(x) = (x - K)^+$. Let Y be solution to

$$dY_s = Y_s(rds + \sigma dW_s), Y_t = x.$$

Then $Y_s = x\exp[\sigma(W_s - W_t) + (s - t)(-\frac{1}{2}\sigma^2 + r)]$ and

$$C(t, x) = E[e^{-r(T-t)}(Y_T - K)^+/Y_t = x]$$

is the waited solution, the portfolio is given via the equations (17).

The famous Black-Scholes formula allows an explicit computation of this function: let Φ be the distribution function of the standard Gaussian law.

$$C(t, x) = x\Phi\left(\frac{\log(x/K) + (T - t)(r + \frac{1}{2}\sigma^2)}{\sigma\sqrt{T - t}}\right) - Ke^{-r(T-t)}\Phi\left(\frac{\log(x/K) + (T - t)(r - \frac{1}{2}\sigma^2)}{\sigma\sqrt{T - t}}\right). \tag{18}$$

The option initial price q is then $C(0,x)$.

Actually, one solve it using (variable,function) change:

$$x = e^y, y \in \mathbb{R} \ ; \ D(t,y) = C(t, e^y),$$

so one goes to a simpler Cauchy problem

$$\partial_t D(t,y) + r\partial_y D(t,y) + \frac{1}{2}\partial^2_{y^2} D(t,y)\sigma^2 = rD(t,y), y \in \mathbb{R},$$

$$D(T,y) = (e^y - K)^+, y \in \mathbb{R},$$

linked to the stochastic differential equation :

$$dX_s = rds + \sigma dW_s, s \in [t,T], X_t = y.$$

This one is exactly the one in Proposition 4.7, with $g = 0$, $f(x) = (e^x - k)^+$, $k(x) = r$. *Thus,*

$$D(t,y) = E[e^{-r(T-t)}(e^{X_T} - K)^+ / X_t = y],$$

and yields the explicit formula since X_T *law is a Gaussian law and get (18): the price at time* t *is* $C(t, S_t) = E_Q e[^{-r(T-t)}(e^{X_T} - K)^+ / \mathcal{F}_t]$. *This is easy to compute:* X_T *law conditionally to* $\mathcal{F}_t$ *is a Gaussian law* $(S_t + r(T-t), \sigma^2(T-t))$.

4.4 Exercises

Let a filtered probability space $(\Omega, \mathcal{F}_t, P)$.

1. Let B be the real Brownian motion; using Ito formula prove that $B_t^2 = 2\int_0^t B_s dB_s + t$.
2. Let $X \in \mathcal{L}(B)$, prove that $(X.B)_t^2 = 2\int_0^t (X.B)_s X_s dB_s + \int_0^t X_s^2 ds$.
3. Let $Z_t = \exp((X.B)_t - \frac{1}{2}\int_0^t X_s^2 ds)$. Prove that Z is solution of the SDE:

$$Z_t = 1 + \int_0^t Z_s X_s dB_s.$$

Prove that $Y = Z^{-1}$ is solution of

$$dY_t = Y_t(X_t^2 dt - X_t dB_t),\ Y_0 = 1.$$

Using Ito formula applied to the function $(x, y) \mapsto x/y$, prove the uniqueness of the solution of the linear SDE.

4. Prove that

$$(\exp \int_0^t a_s ds)(x + \int_0^t b_s \exp(-\int_0^s a_u du) dB_s)$$

is solution of the SDE $dX_t = a_t X_t dt + b_t dB_t,\ t \in [0, T],\ X_0 = x$. Take care to justify all the integrals which appear in this formula (i.e. precise the useful hypotheses on the parameters a and b.)
This equation is the Ornstein-Uhlenbeck or Vasicek equation, this model often is used to model rate behaviour.

5. The following is a particular case of Orstein Uhlenbeck SDE:

$$dX_t = -\alpha X_t dt + \sigma dB_t,\ X_0 = x,$$

where x is a random variable in $L^1(\mathcal{F}_0)$.
(i) Prove that the following process is solution of this SDE:

$$X_t = e^{-\alpha t}(x + \int_0^t \sigma e^{\alpha s} dB_s).$$

(ii) Prove that $t \mapsto m(t) = E[X_t]$ is solution of the ordinary differential equation which is obtained by integrating the following: $X_t = x - \alpha \int_0^t X_s ds + \sigma B_t$. Deduce that $m(t) = m(0)e^{-\alpha t}$.
(iii) Prove that the variance function satisfies

$$V(t) = Var[X_t] = \frac{\sigma^2}{2\alpha} + (V(0) - \frac{\sigma^2}{2\alpha})e^{-2\alpha t}.$$

(iv) If x is a $\mathcal{F}_0$-measurable random variable with the law $\mathcal{N}(0, \frac{\sigma^2}{2\alpha})$, show that X is a Gaussian process with the covariance function $\rho(s,t) = \frac{\sigma^2}{2\alpha} e^{-\alpha|t-s|}$.

6. Let W a standard Brownian motion, ε a number in $[0,1]$, and $\Pi = (t_0, \cdots, t_m)$ a partition of $[0,1]$ with $0 = t_0 < \cdots < t_m = t$). Consider the approximating sum :

$$S_\varepsilon(\Pi) = \sum_{i=0}^{m-1} [(1-\varepsilon)W_{t_i} + \varepsilon W_{t_{i+1}}](W_{t_{i+1}} - W_{t_i})$$

for the stochastic integral $\int_0^t W_s dW_s$. Show that :

$$\lim_{|\Pi| \to 0} S_\varepsilon(\Pi) = \frac{1}{2} W_t^2 + (\varepsilon - \frac{1}{2})t,$$

where the limit is in probability. The right hand of the last limit is a martingale if and only if $\varepsilon = 0$, so that W is evaluated at the left-hand endpoint of each interval $[t_i, t_{i+1}]$ in the approximating sum ; this corresponds to the Ito integral.

With $\varepsilon = \frac{1}{2}$ we obtain the Stratonovitch integral, which obeys the usual rules of calculus such as $\int_0^t W_s \circ dW_s = \frac{1}{2} W_t^2$.
Indication: stress one approximation of the $\int_0^t W_s \circ dW_s$ and of W quadratic variation. Then apply Ito formula to $t \mapsto W_t^2$. Alternative: write $S_\varepsilon(\Pi)$ as a combination of $W_{t_{i+1}}^2 - W_{t_i}^2$ and $(W_{t_{i+1}} - W_{t_i})^2$.

5 Probability measure change and martingale problem

Motivation: martingales and local martingales are powerful tools, so it could be interesting to model real world so that the processes could be martingales, at least local martingales. Moreover, in such a case, we avoid "arbitrage opportunities". Thus, to apply stochastic calculus to Finance, real data are modelled as price processes evolving on financial market as semi-martingales. To avoid artitrage opportunities, for the moment we assume there exists at least one risk neutral probability measure, meaning there exists Q such that all the prices processes are $Q-$martingales. This is a sufficient condition for AOA. Thus yields the two problems in this chapter:

- how to change from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to another one $(\Omega, \mathcal{F}, Q)$, does there exist a density of probability measure $\frac{d\mathbb{P}}{dQ}$? in this case, how is the Brownian motion changed? this is Girsanov theorem, Section 5.1,

- then we deal with the so-called "representation property", meaning how to express the contingent claims using the assets prices, meaning to find a portfolio (an hedging portfolio) which allows us to realize the terminal time contingent claim, cf. Section 5.3.

The last important question will be studied in Chapter 6: let a family of adapted processes on the filtered probabilisable space $(\Omega, (\mathcal{F}_t))$, does there exist a probability $\mathbb{P}$ such that all these processes could be $(\Omega, (\mathcal{F}_t), \mathbb{P})$-martingales? This is the so-called martingale problem: cf. the seminal book [12] on this topic.

Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space endowed with a d-dimensional Brownian motion $B,\ B_0 = 0$. The filtration is the one generated by the Brownian motion and we denote $\mathcal{M}(\mathbb{P})$ the set of $(\Omega, (\mathcal{F}_t), \mathbb{P})$ martingales. Let us recall the notion of **local martingale**, the set of which is denoted as $\mathcal{M}_{loc}(\mathbb{P})$, meaning an adapted process M such that there exists an increasing sequence of stopping times (T_n), going to infinity, such that $\forall n$ the T_n stopped process M^{T_n} is a 'true' martingale.

5.1 Girsanov Theorem

([13] 3.5, p 190-196 ; [23] 3.6, p 108-114) Let X be an adapted measurable process, $X \in \mathcal{P}(B)$ meaning:

$$\mathcal{P}(B) := \{X \text{ adapted measurable process: } \forall T,\ \int_0^T \parallel X_s \parallel^2 ds < +\infty \ \mathbb{P}\ a.s.\}$$

This set contains $\mathcal{L}(B) = L^2(\Omega \times \mathbb{R}^+, d\mathbb{P} \otimes dt)$.
More generally, for any martingale M we define the set $\mathcal{P}(M)$ which contains $\mathcal{L}(M) = L^2(\Omega \times \mathbb{R}^+, d\mathbb{P} \otimes d\langle M \rangle)$:

$$\mathcal{P}(M) := \{X \text{ adapted measurable process: } \forall T,\ \int_0^T \parallel X_s \parallel^2 d\langle M \rangle_s < +\infty \ \mathbb{P}\ a.s.\}$$

For such processes X, the stochastic integral $X.M$ is only a "local" martingale.

Thus we could define the local martingale $X.B$ and its Doléans exponential (stochastic exponential) as soon as $\forall t$, $\int_0^t \| X_s \|^2 ds < +\infty$ $\mathbb{P}$ $a.s.$:

$$\mathcal{E}_t(X.B) = \exp[\int_0^t (\sum_i X_s^i dB_s^i - \frac{1}{2} \| X_s \|^2 ds)],$$

solution of the SDE

(19) $$dZ_t = Z_t \sum_i X_t^i dB_t^i \ ; \ Z_0 = 1,$$

this is also a local martingale since $\int_0^t Z_s^2 \| X_s \|^2 ds < +\infty$ $\mathbb{P}$ $a.s.$ using the integrand continuity on $[0, t]$.

Under suitable conditions, $\mathcal{E}_.(X.B)$ is a "true" martingale, then $\forall t$, $E[Z_t] = 1$, which allows the probability change on σ-algebra $\mathcal{F}_t$:

$$Q = Z_t.\mathbb{P} \text{ meaning: if } A \in \mathcal{F}_t, \ Q(A) = E_{\mathbb{P}}[1_A Z_t].$$

Since $Z_t > 0$, the two probabilities are equivalent and $\mathbb{P}(A) = E_Q[Z_t^{-1}\mathbf{1}_A]$. Moreover, $\lim_{t\to\infty} Z_t$ exists and $\forall$ S stopping time, $\forall A \in \mathcal{F}_S$, $\mathbb{P}(A) = E_Q[Z_S^{-1}\mathbf{1}_A]$.

Theorem 5.1 *(Girsanov, 1960; Cameron-Martin, 1944) If the process $Z = \mathcal{E}(X.B)$, solution to (19), belongs to $\mathcal{M}(\mathbb{P})$, and if Q is the probability defined on $\mathcal{F}_T$ as $Z_T.\mathbb{P}$ then:*

$$\tilde{B}_t = B_t - \int_0^t X_s ds, \ t \leq T$$

is a $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, Q)$ Brownian motion.

Thus, look at Black-Scholes model

$$dS_t = S_t(bdt + \sigma dB_t),$$

using $\tilde{B}_t = B_t + \sigma^{-1}bt$, and $Q = \mathcal{E}(X.B)\mathbb{P}$ with $X_t = -\sigma^{-1}bt$,
we obtain that S is a $Q-$martingale.

Actually, we need the discounted price $\tilde{S}_t = \frac{S_t}{S_t^0}$ could be a local martingale, thus a risk neutral probability measure is $Q = \mathcal{E}(X.B)\mathbb{P}$ with $X_t = -\sigma^{-1}(b - r)t$. The quantity $\sigma^{-1}(b - r)$ is said to be the "risk premium".

The proof needs a lemma. Below E_Q denotes the Q-expectation and $E_{\mathbb{P}}$ denotes the $\mathbb{P}$-expectation.

Lemma 5.2 *Let $T \geq 0$, $Z \in \mathcal{M}(\mathbb{P})$ and $Q = Z_T\mathbb{P}$. Let $0 \leq s \leq t \leq T$ and Y a $\mathcal{F}_t-$measurable random variable in $L^1(Q)$, then*

$$E_Q(Y/\mathcal{F}_s) = \frac{E_{\mathbb{P}}(YZ_t/\mathcal{F}_s)}{Z_s}.$$

This could be seen as a Bayes formula.
Preuve (exercise 1 below) : Let $A \in \mathcal{F}_s$, yields:

$$E_Q(1_A \frac{E(YZ_t/\mathcal{F}_s)}{Z_s}) = E(1_A E(YZ_t/\mathcal{F}_s))$$

using $\mathcal{F}_s$, $Q = Z_s\mathbb{P}$. Then: $E[1_A E(YZ_t/\mathcal{F}_s)] = E(1_A YZ_t)$
using conditional expectation definition, Q definition and the fact that 1_AY is $\mathcal{F}_t$-measurable, $\forall A \in \mathcal{F}_s$,

$$E(1_AYZ_t) = E_Q(1_AY),$$

so we can identify $\frac{E_{\mathbb{P}}(YZ_t/\mathcal{F}_s)}{Z_s}$ as the expected conditional expectation. •

Proposition 5.3 *Assuming Girsanov Theorem hypotheses, for any continuous local $\mathbb{P}$-martingale M, the processus N, defined below, is a local $Q-$martingale:*

$$N = M - \int_0 \sum_i X_s^i d\langle M, B^i\rangle_s.$$

Proof: exercise 2 below.

•

As a corollary, $\tilde{B}$ is a Q-martingale with bracket t. The fact that it is a Q-Brownian motion is a consequence of one of the following:
- either it is a process admitting $Q-$Gaussian independent increments,
- or it is a $Q-$Gaussian process.

We now look at that problem on another point of view: i.e. given a set of equivalent probability measures, looking for a link between the martingales with respect these different probability measures, (but always with the same filtration).

Proposition 5.4 *Let $\mathbb{P}$ and Q two equivalent probability measures on $(\Omega, \mathcal{F})$ and the uniformly integrable continuous martingale $Z_t = E[\frac{dQ}{d\mathbb{P}}/\mathcal{F}_t]$. Then*

$$M \in \mathcal{M}_{loc}(Q) \Leftrightarrow MZ \in \mathcal{M}_{loc}(\mathbb{P}).$$

Preuve : Let a stopping time sequence (T_n) to localize M: Lemma 5.2 yields $\forall s \leq t$:

$$E_Q[M_{t\wedge T_n}/\mathcal{F}_s] = \frac{E_{\mathbb{P}}[Z_t M_{t\wedge T_n}/\mathcal{F}_s]}{Z_s} \tag{20}$$

then the fact that $M^{T_n} \in \mathcal{M}(Q)$ yields $(MZ)^{T_n} \in \mathcal{M}(\mathbb{P})$.
Reciprocally, it is enough to consider a stopping time sequence (T_n) localizing ZM and to apply (20) once again. •

Theorem 5.5 *Girsanov-Meyer: Let $\mathbb{P}$ and Q two equivalent probability measures, $Z_t = E[\frac{dQ}{d\mathbb{P}}/\mathcal{F}_t]$ and X a semi-martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ with the decomposition $X = M + A$. Then, X is also a semi-martingale on $(\Omega, \mathcal{F}, Q)$ with the decomposition $X = N + C$, where*

$$N = M - \int_0^t Z_s^{-1} d\langle Z, M\rangle_s \ ; \ C = A + \int_0^t Z_s^{-1} d\langle Z, M\rangle_s.$$

Preuve : (i) C is a finite variation process as the sum of two finite variation processes.

(ii) Proposition 5.4 is applied to N, we compute the product NZ using Itô formula under the probability $\mathbb{P}$.

$$d(NZ)_t = N_t dZ_t + Z_t dM_t - Z_t Z_t^{-1} d\langle Z, M\rangle_t + d\langle Z, N\rangle_t$$

But N is a $\mathbb{P}$-semi-martingale, with martingale part M: the bracket $\langle Z, N\rangle = \langle Z, M\rangle$ thus NZ is a $\mathbb{P}$-martingale so N is a Q-martingale. •

5.2 Novikov Condition

(cf [13] pages 198-201.)
The above subsection is based on the hypothesis that the process $\mathcal{E}(X.B)$ is a 'true' martingale. We now have to give some conditions on X so that this hypothesis is satisfied. Generally, at least, $\mathcal{E}(X.B)$ is a local martingale with a localizing sequence, for instance:

$$T_n = \inf\{t \geq 0, \ \int_0^t \| \mathcal{E}_s(X.B) X_s \|^2 ds > n\}$$

Lemma 5.6 *$\mathcal{E}(X.B)$ is a supermartingale, it is a martingale if and only if $\forall t \geq 0$, $E[\mathcal{E}_t(X.B)] = 1$.*

Proof: There exists an increasing stopping times sequence T_n such that $\forall n$, $\mathcal{E}(X.B)^{T_n} \in \mathcal{M}(\mathbb{P})$, thus $\forall s \leq t$ yields

$$E[\mathcal{E}_{T_n \wedge t}(X.B)/\mathcal{F}_s] = \mathcal{E}_{T_n \wedge s}(X.B).$$

If n goes to infinity, Fatou lemma and this equality yield that $\mathcal{E}(X.B)$ is a supermartingale (any positive local martingale is a supermartingale). Since $E[\mathcal{E}_0(X.B)] = 1$, it is sufficient that $\forall t \geq 0$, $E[\mathcal{E}_t(X.B)] = 1$ to prove that $\mathcal{E}(X.B)$ is a martingale. •

Proposition 5.7 *Let M a continuous local $\mathbb{P}$ -martingale and $Z = \mathcal{E}(M)$ such that $E[\exp \frac{1}{2}\langle M\rangle_t] < \infty$ $\forall t \geq 0$. Then $\forall t \geq 0, E[Z_t] = 1$.*

Corollary 5.8 *(Novikov, 1971) : Let X an adapted measurable process such that:*

$$E[\exp \frac{1}{2} \int_0^t \| X_s \|^2 ds] < \infty \textit{ for all } t \geq 0$$

then $\mathcal{E}(X.B) \in \mathcal{M}(\mathbb{P})$.

To end this subsection, let an example of process $X \in \mathcal{P}(B)$ which doesn't satisfy Novikov condition, such that $\mathcal{E}(X.B) \in \mathcal{M}^c_{loc}(\mathbb{P})$ but not being a "true" martingale: exercise 3 below.
Let the stopping time $T = \inf\{1 \geq t \geq 0, t + B_t^2 = 1\}$ *and*

$$X_t = -\frac{2}{(1-t)^2} B_t 1_{\{t \leq T\}} \ ; \ 0 \leq t < 1, \ X_1 = 0.$$

(i) Prove that almost surely $T < 1$ *so almost surely* $\int_0^1 X_t^2 dt < \infty$.

(ii) Apply Itô formula to the process $t \to \frac{B_t^2}{(1-t)^2}$; $0 \leq t < 1$ *to show that:*

$$\int_0^1 X_t dB_t - \frac{1}{2}\int_0^1 X_t^2 dt = -1 - 2\int_0^T [\frac{1}{(1-t)^4} - \frac{1}{(1-t)^3}] B_t^2 dt < -1.$$

(iii) The local martingale $\mathcal{E}(X.B)$ *is not a martingale (no up to time 1!): we deduce from (ii) that its expectation is less than* $\exp(-1) < 1$ *and this fact contradict Lemma 5.6. Nevertheless we can show that* $\forall n \geq 1$ *and* $\sigma_n = 1 - (1/\sqrt{n})$, *the process* $\mathcal{E}(X.B)^{\sigma_n}$ *is a martingale.*

5.3 Martingale representation Theorem

(cf. Protter [23], pages 147-157.)
The aim of this subsection is to show that a wide class of local martingales can be written (to be "represented" by) $X.B$: there exists $X \in \mathcal{P}(B)$ such that $M_t = M_0 + \int_0^t X_s dB_s$.

This tool will allow us to solve the option hedging problem.

NOTATIONS :
$\mathcal{M}^{2,c}$ continuous martingales set in L^2;
$\mathcal{M}^{2,c}_{loc}$ continuous local martingales set in L^2.

5.3.1 Stable subspace , definitions

To be skipped in a first lecture.

Let us consider the martingales in $\mathcal{M}^{2,c}$ which equal 0 at time $t = 0$ and satisfies $\langle M\rangle_\infty \in L^1$. Then $\sup_t E[M_t^2] = \sup_t E[\langle M\rangle_t] = E[\langle M\rangle_\infty] < \infty$. These martingales are uniformly integrable, there exists M_∞ such that $\forall t \leq 0, \ M_t = E[M_\infty/\mathcal{F}_t]$. One notes $\mathcal{H}_0^2$ their set.

$$\mathcal{H}_0^2 = \{M \in \mathcal{M}^{2,c}, M_0 = 0, \langle M\rangle_\infty \in L^1\}.$$

Definition 5.9 *A vectorial* $\mathcal{H}_0^2$ *subspace* F *is called* **stable subspace** *if* $\forall M \in F$ *and for any stopping time* T *then* $M^T \in F$.

Definition 5.10 *Let $\mathcal{A}$ a $\mathcal{H}_0^2$ subset. One notes $S(\mathcal{A})$ the smaller stable closed vectorial subspace which contains $\mathcal{A}$.*

Definition 5.11 *Let $\mathcal{A} \subset \mathcal{H}_0^2$, $\mathcal{A}$ is said to have the* **predictable representation property** *if:*

$$\mathcal{I} = \{X = \sum_{i=1}^{n} H^i M^i, \ M^i \in \mathcal{A}, H^i \in \mathcal{L}^*(M^i) \cap L^2(d\mathbb{P} \otimes d\langle M^i \rangle)\} = \mathcal{H}_0^2.$$

Definition 5.12 *Let $\mathcal{A} \subset \mathcal{H}_0^2(\mathbb{P})$. One notes $\mathcal{M}(\mathcal{A})$ the set of probability measures on $\mathcal{F}_\infty$ which are absolutely continuous with respect to $\mathbb{P}$ and equal to $\mathbb{P}$ on $\mathcal{F}_0$ and such that $\mathcal{A} \subset \mathcal{H}_0^2(Q)$.*

Lemma 5.13 *$\mathcal{M}(\mathcal{A})$ is convex.*

Proof: exercise.

Definition 5.14 *$Q \in \mathcal{M}(\mathcal{A})$ is said to be* **extremal** *if*

$$Q = aQ_1 + (1-a)Q_2, a \in [0,1], Q_i \in \mathcal{M}(\mathcal{A}) \Rightarrow a = 0 \ ou\ 1.$$

Theorem 5.15 *Let $\mathcal{A} = (M^1, \cdots, M^n) \subset \mathcal{H}_0^2(\mathbb{P})$ with $M^i \dagger M^j, i \neq j$. The fact that $\mathbb{P}$ is extremal in $\mathcal{M}(\mathcal{A})$ implies that $\mathcal{A}$ has the predictable representation property.*

5.3.2 Fondamental theorem

Theorem 5.16 *Let a d-dimensional Brownian motion B on $(\Omega, (\mathcal{F}_t), \mathbb{P})$. Then $\forall M \in \mathcal{M}_{loc}^{c,2}$, there exists a unique vector of processes $(H^j \in \mathcal{P}(B^j), j = 1, \cdots, d)$ such that:*

$$M_t = M_0 + \sum_{j=1}^{d} \int_0^t H_s^j dB_s^j.$$

Proof (exercise): apply Theorem 5.15 to the Brownian motion components, prove that $\mathbb{P}$ is the unique element of $\mathcal{M}(B)$. Then, localize the martingale M.

Corollary 5.17 *Under the same hypotheses, let $Z \in L^1(\mathcal{F}_\infty, \mathbb{P})$, then there exists a unique vector H such that $H^j \in \mathcal{P}(B^j), j = 1, \cdots, d$, and: $Z = E[Z] + \sum_{j=1}^{d} (H^j.B^j)_\infty$.*

Proof: apply the theorem to the martingale $M_t = E[Z/\mathcal{F}_t]$ and do t going to infinity.

•

Then let us remark that, if $\mathbb{P}$ and Q are two equivalent probability measures, denoting Z the $\mathbb{P}-$integrable variable $\frac{dQ}{d\mathbb{P}}$, then the martingale $Z_t = E_{\mathbb{P}}[Z/\mathcal{F}_t]$ is an exponential martingale: there exists a process ϕ such that $dZ_t = Z_t \phi_t dB_t$.

5.3.3 Application: finding a risk neutral probability measure

Let us assume that the underlying assets prices $S^i, i = 1, ...n$ are positive semi-martingales as

$$dS_t^i = S_t^i b_t^i dt + S_t^i \sum_j \sigma_j^i(t) dB_t^j.$$

Besides let the equivalent probability measure $Q = \mathcal{E}(\phi.B)\mathbb{P} = Z\mathbb{P}$. Using Girsanov theorem, $\forall j$:

$$\tilde{B}_t^j = B_t^j - \int_0^t \phi_s^j ds$$

is a Q-Brownian motion. So actually, S^i are also Q-semi-martingales satisfying:

$$dS_t^i = S_t^i(b_t^i + \sum_j \sigma_j^i(t)\phi_t^j)dt + S_t^i \sum_j \sigma_j^i(t) d\tilde{B}_t^j.$$

Thus the problem is now to find a vector ϕ in $\mathcal{L}(B)$ satisfying (for instance) Novikov condition and such that $\forall i = 1, ...n$

$$b_t^i + \sum_j \sigma_j^i(t)\phi_t^j = 0,$$

this is a system with n equations and d unknown.
exercise : system to be solved when $n = d = 1$, then when $n = d$. What is to do when $n \neq d$?

5.3.4 Application: option hedging

In case of a complete market, using the representation theorem, we can manage the so called "hedging" option.

Definition 5.18 *A "call option" is the following contract: at time* 0, *the buyer pays q so that he has the right at time* 1 *to buy the asset at price K even if $S_1 > K$. It is not an obligation, only a right.... When at time* 1 *$S_1 > K$ he buys, so that he wins $S_1 - K - q$. In the other case, and if he does nothing, he loses q. Globally, he wins $(S_1 - K)^+ - q$.*

A "put option " is the following contract: at time 0, *the buyer pays q so he has the right at time* 1 *to sell the asset at price K even if $S_1 < K$. It is not an obligation, only a right.... When at time* 1 *$S_1 < K$ he buys, so he wins $K - S_1 - q$. In the other case, and if he does nothing, he loses q. Globally, he wins $(K - S_1)^+ - q$.*

K is the option strike and T the maturity.

This means that on time $t = 0$ we buy the right to buy the assets on price K even if S_T is above it (call) or the right to sell the assets on price K even if S_T is below it (put). But to find the 'fair price' of this contract, the seller has to hedge the contract, thus he has to have a portfolio such that his initial wealth will be this price, and its terminal wealth would be what he have to pay to his buyer, at least in expectation.

Definition 5.19 *The so called "fair price" of the contingent claim H is the smaller $x \geq 0$ such that there exists an admissible self-financing strategy π to hit the wealth X^π, the discounted price of it being $e^{-rT} X_T^\pi = H, X_0^\pi = x$.*

The seller look for a hedging strategy. Here the useful tool will be the martingale representation theorems..... Let r be the discounting rate (for instance bond rate), $e^{-rT} X_T$ is the discounted contingent claim. Let us suppose a market as defined in 5.3.3, [$n = d$, σ invertible, process ϕ satisfying Novikov condition] thus the market admits a risk neutral probability measure on $\mathcal{F}_T$: $Q = \mathcal{E}_T(\phi.B)\mathbb{P}$. Using the fondamental theorem there exists a vector θ such that

$$e^{-rT} X_T = E_Q[e^{-rT} X_T] + \int_0^T \sum_j \theta_t^j d\tilde{B}_t^j. \tag{21}$$

But if we use the definition of the Q-Brownian motion $\tilde{B}$ above:

$$dS_t^i = S_t^i \sum_j \sigma_j^i(t) d\tilde{B}_t^j$$

Let $\forall j$

$$d\tilde{B}_t^j = (S_t^i)^{-1} (\sigma^{-1})_i^j(t) dS_t^i$$

to put in (21):

$$e^{-rT} X_T = E_Q[e^{-rT} X_T] + \int_0^T \sum_{i,j} \theta_t^j (S_t^i)^{-1} (\sigma^{-1})_i^j(t) dS_t^i.$$

This allows us to identify the hedging portfolio

$$\pi_t^i = (S_t^i)^{-1} \sum_j \theta_t^j (\sigma^{-1})_i^j(t)$$

then the fair price is:

$$q = E_Q[e^{-rT} X_T].$$

5.4 Exercises

1. Let Q be a $\mathbb{P}-$equivalent probability defined by $Q = Z.P$ où $Z \in L^1(\Omega, \mathcal{F}_T, P)$ i.e. Q restrained to the $\sigma-$algebra $\mathcal{F}_t$ is $Z_t.P,\ Z_t = E[Z/\mathcal{F}_t]$.
Prove that $\forall t$ and $\forall Y \in L^1(\Omega,\ \ \mathcal{F}_t, P), E_P[YZ_t/\mathcal{F}_s] = Z_s E_Q[Y/\mathcal{F}_s]$.
Indication: compute $\forall A \in \mathcal{F}_s$, *the expectations* $E[1_A Y Z_t]$ *and* $E[1_A Z_s E_Q[Y/\mathcal{F}_s]]$.

2. Let M be a $\mathbb{P}$-martingale, $X \in \mathcal{L}(B)$ such that $Z = \mathcal{E}(X.B)$ is a martingale; we recall $Z_t = Z_t X_t dB_t,\ Z_0 = 1$. One put $Q = Z_T\mathbb{P}$ an $\mathbb{P}-$equivalent probability.
(i) Prove that $d\langle M, Z\rangle = ZXd\langle M, B\rangle$.
(ii) Using Ito formula to develop $M_t Z_t - M_s Z_s$, compute $E_{\mathbb{P}}[M_t Z_t/\mathcal{F}_s]$.
(iii) Use Ito formula between s and t to the process $Z_. \int_0^. X_u d\langle M, B\rangle_u$.
(iv) Deduce that $M_. - \int_0^. X_u d\langle M, B\rangle_u$ is a $Q-$martingale.

3. Let $T = \inf\{1 \geq t \geq 0, t + B_t^2 = 1\}$ be a stopping time and

$$X_t = -\frac{2}{(1-t)^2} B_t 1_{\{t \leq T\}}\ ;\ 0 \leq t < 1,\ X_1 = 0.$$

(i) Prove that $T < 1$ almost surely, thus $\int_0^1 X_t^2 dt < \infty$ almost surely.
(ii) Apply Itô formula to the process $t \to \frac{B_t^2}{(1-t)^2}\ ;\ 0 \leq t < 1$ and prove:

$$\int_0^1 X_t dB_t - \frac{1}{2}\int_0^1 X_t^2 dt = -1 - 2\int_0^T \frac{t}{(1-t)^4} B_t^2 dt < -1.$$

(iii) The local martingale $E(X.B)$ is not a martingale: (ii) yields that its expectation is bounded above $\exp(-1) < 1$ and this fact contradicts the martingale property $E(M_t) = M_0$ (which could be 1!....)
Nevertheless, prove that $\forall n \geq 1$ and $\sigma_n = 1 - (1/\sqrt{n})$, the process $E(X.B)^{\sigma_n}$ is a martingale.

4. Let n processes modelling prices, driven by the following SDEs:

$$dS_t^i = S_t^i \sum_{j=1}^d \sigma_j^i(t) dB_t^j,\ S_0 = s \in \mathbb{R}^n$$

where $\sigma^*\sigma$ is a definite positive matrix. Give a sufficient condition such that there exists a hedging strategy of the contingent claim $H = (S_T^1 - K)^+$, i.e. such that the market could be 'complete'.

6 Market model, continuous time, continuous prices.

Among others, look at [6] Chapters 12.1-12.5 or [13] Section 5.8, pages 371 et sq. Let us suppose AOA hypothesis, thus the prices processes are semi-martingales.

6.1 The model

We choose a finite horizon: $t \in [0, T]$, the market, denoted as S, offers $N + 1$ assets, the prices are continuous semi-martingales, we can buy or sell some real quantities of them, there is no transaction or change costs. These price processes are continuous, defined on a filtered probability space, Wiener space, $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F}_t)$, endowed with a d-dimensional Brownian motion denoted as B. Moreover, we suppose that $\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_T = \mathcal{A}$.

Hypothesis on the market S: The first assets is riskless, so called the 'bond':

$$dS_t^0 = S_t^0 r dt,\ r > 0,\ S_0^0 = 1,$$

meaning that $S_t^0 = e^{rt}$.
Then N risky assets on the market are positive semi-martingales satisfying $\forall n = 1, ..., N$, there exists a semi-martingale x^n such that:

$$S_t^n = \mathcal{E}_t(x^n),\ t \in [0, T],$$

meaning that

$$dx_t^n = \sigma_j^n(t) dB_t^j + b^n(t) dt, n = 1, \cdots, N; dx_t^0 = r dt.$$

6.2 Equilibrium price measure or risk neutral probability

Definition 6.1 *Let $(S^0, \cdots, S^N)$ be a price system, an* **equilibrium price measure** *or risk neutral probability on $(\Omega, \mathcal{F}_t)$ is a probability Q, which is equivalent to $\mathbb{P}$ such that the discounted prices $e^{-rt} S^n$, denoted as $\tilde{S}^n$, are local Q-martingales.*

We note $\mathcal{Q}_S$ the set of such probability measures.

Let us assume that $\mathcal{Q}_S$ **is non empty**, and choose $Q \in \mathcal{Q}_S$; this one is not unique, but generally the results don't depend on the one chosen in the set $\mathcal{Q}_S$.
This hypothesis yields the absence of arbitrage (Definition 6.6 and Theorem 6.8 below). Nevertheless, despite what it is too often written, it is not an equivalent condition: sufficient but not necessary condition. But, this hypothesis is equivalent to the condition NFLVR: no free lunch with vanishing risk (cf. [5]).

Exercise: in this context, translate the main model hypothesis, i.e. the existence of a risk neutral probability Q, so that the discounted prices $\tilde{S}^n$ are martingales.

A good tool will be Ito formula:

$$(22)\ d\tilde{S}_t^n = e^{-rt}dS_t^n - rS_t^n e^{-rt}dt = \tilde{S}_t^n(dx_t^n - rdt) = \tilde{S}_t^n[\sum_j \sigma_j^n(t)dB_t^j + (b^n(t) - r)dt].$$

The aim is to find a $\mathbb{P}$-equivalent probability measure Q and a Q Brownian motion $\tilde{B}$ such that $dx_t^n - rdt = \sigma d\tilde{B}$. Here we use Girsanov Theorem, denoting $Z_t = E_{\mathbb{P}}[\frac{d\mathbb{P}}{dQ}/\mathcal{F}_t]$ which is a martingale. So Z can be 'represented' with respect to the d-dimensional Brownian motion B: there exists a vector process $X \in \mathcal{P}(B)$ such that $dZ_t = Z_t \sum_{j=1}^d X^j dB_t^j$. To find Q risk neutral probability measure is equivalent to find X.
As an example, end the exercise assuming that

. the matrix ${}^t\sigma.\sigma$ has the rank d, thus it is invertible,

. there exists a Novikov condition on vector $v_. = ({}^t\sigma_.\sigma_.)^{-1} \times^t \sigma_.(b_. - r_.\mathbf{1})$ where $\mathbf{1} = (1, \cdots, 1)$.
More generally, discuss the existence of risk neutral probability measure in the cases $d =, <, > N$.

6.3 Financial strategies

Notation: below, $\langle x, y\rangle$ notes the scalar product between the two vectors x and y, not to be confused with the stochastic bracket between two martingales or semi-martingales!

A strategy is a portfolio θ, $\mathcal{F}$-adapted process taking its values in $\mathbb{R}^{N+1}$, θ^n denoting the portfolio component which is invested in the n-th assets. The suitable conditions are those which allows the real process $\int\langle\theta_s, dS_s\rangle$ to be well defined:

θ has to be integrable on $[0, t]$, $\forall t$, respectively with respect to martingale part and to finite variation part of the semimartingale, the discounted price $\tilde{S}^n$. This quantity $\int_0^t\langle\theta_s, dS_s\rangle$ represents the profit obtained between 0 and t, $\int_0^t\langle\theta_s, d\tilde{S}_s\rangle$ represents the discounted profit obtained between 0 and t.

Definition 6.2 *An* **admissible strategy** *is an adapted process, taking its values in $\mathbb{R}^{N+1}$, defined on $(\Omega, \mathcal{F}_t, Q)$, stochastically integrable (cf. Section 2.1) with respect to the price vector S.*

Definition 6.3 *A strategy is* **self-financing** *if, moreover, $\forall t \in \mathbb{R}^+$ the portfolio value satisfies:*

$$V_t(\theta) = \langle\theta_t, S_t\rangle = \langle\theta_0, S_0\rangle + \int_0^t\langle\theta_s, dS_s\rangle.$$

Remark: The interpretation could be the following: there is no external endowment, only the portfolio variation makes the wealth evolving.

Look at that in a discrete model:

$$
\begin{aligned}
(23) \qquad V_{t+1} - V_t &= \langle \theta_{t+1}, S_{t+1} \rangle - \langle \theta_t, S_t \rangle = \langle \theta_{t+1}, S_{t+1} - S_t \rangle \\
\Longleftrightarrow & \quad \langle \theta_{t+1}, S_t \rangle = \langle \theta_t, S_t \rangle.
\end{aligned}
$$

The portfolio is done between t and $t+1$ with an internal distribution between the assets.

Here we assume the prices to be stochastic exponentials so that they are strictly positive.

Theorem 6.4 *Let θ an admissible strategy. It is self-financing if and only if the discounted portfolio value $\tilde{V}_t(\theta) = e^{-rt} V_t(\theta)$ satisfies:*

$$\tilde{V}_t(\theta) = V_0(\theta) + \int_0^t \langle \theta_s, d\tilde{S}_s \rangle$$

where the scalar product belongs to $\mathbb{R}^N$ instead of $\mathbb{R}^{N+1}$ since $d\tilde{S}^0_s = 0$.

Proof: exercise, using Ito formula to develop $e^{-rt} \times V_t(\theta)$ and equation (22).

Corollary 6.5 *Let Q a risk neutral probability measure. For any self-financing strategy θ which belongs to $\mathcal{P}(\tilde{S})$, the discounted portfolio value is a local $Q-$martingale.*

Preuve: exercise.

Definition 6.6 *The process θ is said to be an* **arbitrage strategy** *if it is admissible, self-financing and satisfies one of the three following properties:*

$$
\begin{aligned}
& \langle \theta_0, S_0 \rangle \leq 0 \text{ et } \langle \theta_T, S_T \rangle \geq 0 \text{ almost surely and } \neq 0 \text{ with probability } > 0, \\
& \langle \theta_0, S_0 \rangle < 0 \text{ et } \langle \theta_T, S_T \rangle \geq 0 \text{ almost surely,} \\
(24) \qquad & \langle \theta_0, S_0 \rangle = 0 \text{ et } \langle \theta_T, S_T \rangle \geq 0 \text{ almost surely and } \neq 0 \text{ with probability } > 0.
\end{aligned}
$$

If there exists such a strategy, it is said that there exists an "arbitrage opportunity".

There exists an arbitrage strategy is equivalent to there exists one of these three defined strategies.
For instance, look at $2 \Rightarrow 3$, if $\langle \theta_0, S_0 \rangle = a < 0$, a new strategy is defined, satisfying the last property:

$$\theta'^n = \theta^n, n = 1, \cdots, N \; ; \; \theta'^0(t) = \theta^0(t) - ae^{-rt}, \forall t \in [0, T].$$

Then,

$$\langle \theta'_0, S_0 \rangle = \theta'^0_0, S^0_0 + \sum_1^N \langle \theta^n_0, S^n_0 \rangle = \langle \theta_0, S_0 \rangle - a = 0$$

and $\langle \theta'_T, S_T \rangle = \langle \theta_T, S_T \rangle - ae^{-rT} e^{rT} > \langle \theta_T, S_T \rangle \geq 0$. Thus $\langle \theta'_T, S_T \rangle$ is positive non null.

•

6.4 Viable market

Definition 6.7 *A market without arbitrage strategy is said to be* **viable** *(cf. AOA hypothesis definition, Chapter 3).*

Below, some sufficient conditions to make a market S viable are given.

Theorem 6.8 *(cf. [6], 12.2 et sq.) If the set $\mathcal{Q}$ is non empty then market is viable.*

Preuve as an exercise, with the following steps, let $Q \in \mathcal{Q}_S$:
1. If for any self-financing strategy $\tilde{V}_t(\theta)$ is a $Q-$supermartingale, then the market is viable.
Since $\tilde{V}_t(\theta)$ is a $Q-$supermartingale:

$$\forall s \leq t, \;\; E_Q[\tilde{V}_t(\theta)/\mathcal{F}_s] \leq \tilde{V}_s(\theta).$$

So for $s = 0$, since $\mathcal{F}_0$ is the trivial σ-algebra,

$$E_Q[\tilde{V}_T(\theta)] \leq \tilde{V}_0(\theta) \text{ i.e. } \langle \theta_0, S_0 \rangle$$

Suppose that θ is an arbitrage strategy: $\langle \theta_0, S_0 \rangle = 0, \langle \theta_T, S_T \rangle \geq 0$.
So $E_Q[\tilde{V}_T(\theta)] \leq 0$ and since $\tilde{V}_T(\theta) = e^{-rT}\langle \theta_T, S_T \rangle \geq 0, \;\; \tilde{V}_T(\theta) = 0$ the strategy θ cannot be an arbitrage strategy.
2. If any self-financing strategy of $\mathcal{P}(\tilde{S})$ is such that $\tilde{V}_t(\theta) \geq 0$, then the market is viable.
Since the strategy θ is self-financing,

$$\tilde{V}_t(\theta) = \langle \theta_0, S_0 \rangle + \int_0^t \langle \theta_s, d\tilde{S}_s \rangle.$$

Corollary 6.5 shows that $\tilde{V}_t(\theta)$ then is a local $Q-$martingale. Since it is positive, it is a supermartingale (cf. Lemma 5.6 proof) and we go to 1. to conclude. •

As a conclusion, to avoid arbitrage, we add a condition in admissibility definition of a strategy θ: the obligation to satisfy

$$V_t(\theta) \geq 0, \;\; dt \otimes d\mathbb{P} \text{ almost surely }.$$

Remark 6.9 *Let us stress the implication sequence: $\mathcal{Q}_S$ is non empty $\Rightarrow$ arbitrage opportunity absence $\Rightarrow$ the price processes are semi-martingales,*
BUT the converse aren't necessarily satisfied!! cf. once again Delbaen-Schachermayer, $\mathcal{Q}_S \neq \emptyset \Leftrightarrow NFLVR$.

6.5 Complete market

Here we use the tools which are defined in Subsection 5.3 (representation property). Let $Q \in \mathcal{Q}_S$.

Definition 6.10 *A contingent claim $X \in L^1(\Omega, \mathcal{F}_T, Q)$ is* **attainable** *under probability Q if there exists a self-financing admissible strategy θ and $x \in \mathbb{R}$ such that*

$$X = \langle \theta_T, S_T \rangle = x + \int_0^T \theta_s . dS_s.$$

A market is said to be **complete** *under the probability Q for the price system S if any $X \in L^1(\Omega, \mathcal{F}_T, Q)$ is attainable.*

Here we look for a characterization of complete markets, at least we try to exhibit some completeness sufficient conditions.

Theorem 6.11 *A contingent claim X is attainable if and only if there exists a vector process $\alpha \in \mathcal{P}(\tilde{S})$ taking its values in $\mathbb{R}^N$ such that:*

$$E_Q[X/\mathcal{F}_t] = e^{-rT} E_Q[X] + \int_0^t \langle \alpha_s, d\tilde{S}_s \rangle.$$

Proof:

If X is attainable, this means that there exists a self-financing admissible strategy θ and $x \in \mathbb{R}$ such that $X = V_T(\theta) = \langle \theta_T, S_T \rangle = x + \int_0^T \langle \theta_s, dS_s \rangle$.
Since θ is admissible, by definition, it is stochastiquely integrable with respect to S so to $\tilde{S}$; it is self-financing means that (cf. Theorem 6.4) $d\tilde{V}_t(\theta) = \langle \theta_t, d\tilde{S}_t \rangle$. But
$X = \langle \theta_T, S_T \rangle$ i.e. $\tilde{V}_T(\theta) = e^{-rT} X$, finally process $\tilde{V}_.(\theta)$ is a martingale:

$$\tilde{V}_t(\theta) = E_Q[\tilde{V}_T(\theta)/\mathcal{F}_t] = E_Q[\tilde{V}_T(\theta)] + \int_0^t \langle \theta_s, d\tilde{S}_s \rangle.$$

The first term actually is $e^{-rT} E_Q[X]$, we identify the process α, which we are looking for, as the strategy θ for the coordinates 1 to N (θ^0 is out since $d\tilde{S}_t^0 = 0$).

Reciprocally, if α exists, the strategy is defined as:

$$\theta^n = \alpha^n, \ n = 1, \cdots, N \ ; \ \theta_t^0 = \int_0^t \langle \alpha_s, d\tilde{S}_s \rangle - \sum_1^N \langle \alpha_t^n, \tilde{S}_t^n \rangle,$$

and $x = E_Q[X]$. We verify that the pair (strategy, initial value) actually attains the contingent claim X, then that the proposed θ is actually self-financing. •

We admit the theorem :

Theorem 6.12 *Let Q be a risk neutral probability measure. If $\mathcal{F}_0 = \{\Omega, \emptyset)$, the following are equivalent:*

(i) The market concerning the price system $\{S\}$ is complete.

(ii) $\mathcal{Q}_S = \{Q\}$

Exercise: prove (i) implies (ii) assuming there exists two equivalent risk neutral probability measures, then $\forall i,\ \tilde{S}^i \in \mathcal{M}_{loc}(Q_1) \cap \mathcal{M}_{loc}(Q_2)$.

6.6 Exercises

We assume that the prices system is:

$$S_t^n = \mathcal{E}_t(X^n), t \in [0,T],$$

where:

$$dX_t^n = \sum_{j=1}^{d} \sigma_j^n(t) dW_t^j + b^n(t)dt, n = 1, \cdots, N; dx_t^0 = rdt.$$

The matrix σ rank is N $dt \otimes d\mathbb{P}$ almost surely and $\exists \alpha > 0$ such that $\sigma\sigma^* \geq \alpha I$. The coefficients b, σ, r are bounded on $[0,T] \times \Omega$.

6.6.1 Is the market viable?

Meaning is there no arbitrage opportunity?
Recall that an admissible strategy is a process θ taking its values in $\mathbb{R}^{N+1}$ belonging to $\mathcal{P}(S.\sigma.W)$ and satisfying the condition

$$\langle \theta, S \rangle \geq 0, \; dt \otimes d\mathbb{P} \text{ almost surely.}$$

Prove:
The market is viable as soon as there exists a risk neutral probability measure.
Give some sufficient condition on the coefficients σ, b, r such that the market would be viable.

6.6.2 Is the market complete?

Indication: describe the set $\mathcal{Q}_S$ of risk neutral probability measures, depending on $d < N, d = N, d > N$.

6.6.3 Admissibility necessary and sufficient conditions

Prove that:
the discounted consumption $\int_0^T e^{-rs} c_s ds$ being fixed

$$E_Q[\int_0^T e^{-rs} c_s ds] \leq x$$

is equivalent to the existence of an admissible strategy π which can simulate X_T starting from initial wealth x.

6.6.4 Optimal strategies

Let a small trader optimizing his strategy using the so called utility functions U_i as follows:

$$(c, X_T) \to E_{\mathbb{P}}[\int_0^T U_1(c_s)ds + U_2(X_T)]$$

where U_i are class C^1 functions, positive, concave, strictly increasing. It is a constrained optimisation problem:

$$\sup_{(c,X_T)} \{E_{\mathbb{P}}[\int_0^T U_1(c_s)ds + U_2(X_T)]/E_Q[\int_0^T e^{-rs}c_s ds] \leq x\}. \tag{25}$$

Solve this problem using Lagrange method. Let $\mathcal{L}$ be the Lagrangian function:

$$\mathcal{L} : C_1 \times C_2 \times \mathbb{R}^+ \to \mathbb{R}$$

$$\mathcal{L}(c, X, \lambda) = E_{\mathbb{P}}[\int_0^T U_1(c_s)ds + U_2(X_T)] - \lambda(E_Q[\int_0^T e^{-rs}c_s ds] - x),$$

$C_1 = \{c \in L^1([0,T] \times \Omega, dt \times dQ)$, $C_2 = L^1(\mathbb{P}) \cap L^1(Q)$.
Indication: think of Kuhn and Tucker Theorem (saddle point).

GLOSSARY

admissible strategy: Definition 6.2, pages 50, 52, 54.

arbitrage strategy: Definition 6.6, pages 1, 23, 51, 54.

assets: pages 1, 15, 33, 34, 39, 45, 48, 49, 50.

bond: pages 46, 48.

complete market: pages 45, 52.

contingent claim: pages 1,34, 39, 52, Definition 5.19.

attainable (contingent claim): Definition 6.10, Theorem 6.11.

fair price: Definition 5.19, pages 34, 45, 46.

maturity: Definition 5.18.

stocks=assets.

trend: page 34.

viable market: pages 34, 51, 54.

volatility: page 34.

References

[1] L. BACHELIER: “Théorie de la spéculation”, Annales scientifiques de l’école normale supérieure,17, 21-88, Paris, 1900.

[2] F. BLACK and M. SCHOLES, “The pricing of options and corporate liabilities”, Journal of Political Economy, 3, 637-654,1973.

[3] J.C. COX and S.A. ROSS : “ The valuation of options for alternative stochastics processes”, Journal of Financial Economics, 7, 229-263, 1979.

[4] Rose Ann DANA et Monique JEANBLANC, “Marchés financiers en temps continu, valorisation et équilibre”, Economica, deuxième édition, Paris, 1998.

[5] F. DELBAEN and W. SCHACHERMAYER, “A general version of the fundamental theorem of asset pricing”, Math. Ann. 300, 463-520, 1994

[6] M.U. DOTHAN, “Prices in financial Markets”, Oxford University Press, Oxford, 1990.

[7] A. FRIEDMAN : Stochastic Differential Equations and Applications, I , Academic Press, New-York ,1975.

[8] J.M. HARRISON and D.M.KREPS, “Martingales and arbitrage in multiperiod securities markets”, Journal of Economic Theory, 20, 381-408, 1979.

[9] J.M. HARRISON and S. PLISKA : “Martingales and stochastic integrals in the theory of continuous trading”,Stochastics Processes and their Applications, 11,215-260, 1981.

[10] C. HUANG, “Information structure and equilibrium asset prices”, Journal of Economic Theory, 35, 33-71, 1985.

[11] K. ITO and H.P. Mc KEAN, “Diffusion Processes and their sample paths”, Springer-Verlag, New York, 1974.

[12] J. JACOD, “Calcul stochastique et problèmes de martingales”, Lecture Notes 714, Springer-Verlag, New York, 1979.

[13] I. KARATZAS and S.E. SHREVE, “Brownian Motion and Stochastic calculus”, Springer-Verlag, New York, 1988.

[14] I. KARATZAS and S.E. SHREVE, “Methods of Mathematical Finance”, Springer-Verlag, New York, 1998.

[15] H. KUNITA and S. WATANABE, “On square integrable martingales” Nagoya Mathematics Journal, 30, 209-245, 1967.

[16] D. LAMBERTON et B. LAPEYRE, “Introduction au calcul stochastique appliqué à la finance”, Ellipses, Paris,1991.

[17] D. LEPINGLE et J. MEMIN, “Sur l’intégrabilité uniforme des martingales exponentielles”, Z. Wahrscheinlichkeitstheorie, 42, p175-203, 1978.

[18] R.S. LIPTSER and A.N. SHIRYAEV, “Statistics of Random Processes”, Springer-Verlag, New York, 1977.

[19] H.P. Mc KEAN : “Stochastic Integrals”, Academic Press, New-York, 1969.

[20] M. MUSIELA and M. RUTKOWSKI, “Martingale Methods in Financial Modelling”,Springer-Verlag, New-York, 1997.

[21] J. NEVEU, “Martingales à temps discrets”, Masson, Paris, 1972.

[22] S. R. PLISKA, “Introduction to Mathematical Finance”, Blackwell, Oxford, 1998.

[23] P. PROTTER, “Stochastic Integration and Differential Equations”, Springer-Verlag, New York, 1990.

[24] F. QUITTARD-PINON, “Marchés de capitaux et théorie financière”, Economica, Paris, 1993.