Estimation of the instantaneous volatility and detection of volatility jumps

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Abstract

Concerning price processes, the fact that the volatility is not constant has been observed for a long time. So we deal with models as $dX_t = \mu_t dt + \sigma_t dW_t$ where σ is a stochastic process. Recent works on volatility modeling suggest that we should incorporate jumps in the volatility process. Empirical observations suggest that simultaneous jumps on the price and the volatility [8, 9] exist. The hypothesis that jumps occur simultaneously makes the problem of volatility jump detection reduced to the prices jump detection. But in case of this hypothesis failure, we try to work in this direction. Among others, we use Jacod and Ait-Sahalia' recent work [3] giving estimates of cumulated volatility $\int_0^t |\sigma_s|^p ds$ for any $p \geq 2$. This tool allows us to deliver an estimate of instantaneous volatility. Moreover we prove a central limit theorem for it. Obviously, such a theorem provides a confidence interval for the instantaneous volatility and leads us to a test of the jump existence hypothesis. For instance, we consider a simplest model having volatility jumps, when volatility is piecewise constant: $\sigma_t = \sum_{i=0}^{N_t-1} \sigma_i \mathbf{1}_{[\tau_i,\tau_{i+1}[}(t)$. The jump times are $\tau_i, i \geq 1$, and σ_i is a $\mathcal{F}_{\tau_i}$-measurable random variable. Another example is studied: $\sigma_t = |Y_t|$ where (Y_t) is a solution to a Lévy driven SDE, with suitable coefficients. Finally, the volatility estimator is tested on some simulations.

1 Introduction

The financial market objects offer a great complexity of modeling. Moreover, recent improvements in high-frequency data processing make continuous models relevant. A natural question arises then: the introduction of jumps into these continuous models. The importance of jumps in Finance is now widely recognized (cf. [9] for a review and [10] for a list of recent studies on this topic).

The key parameter of these models is the volatility influence on price process, which is of paramount importance. Concerning price processes, the fact that the volatility is not constant has been observed for a long time. More recently stochastic volatility appears in models (cf. [20, 21] and references therein) mainly because it is able to fit skews and smiles and relevant to the problem of options pricing. So we deal here with models such as

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

where W is a Brownian motion and σ is a stochastic process.

Recent works on volatility modeling suggest that jumps occur in the volatility process. Empirical observations suggest that there exists simultaneous jumps on the price and the volatility [7, 8, 9, 11]. The hypothesis that jumps occur simultaneously makes the problem of volatility jump detection reduced to the prices jump detection. Even if we know that the price and the volatility processes are not independent processes, the hypothesis that their jumps occured simultaneously appears mathematically very strong, so it will be nice if it could be relaxed. It is the way we chose to follow. Our aim is to detect when jumps in the volatility process occur.

To do this, we have to evaluate the volatility. A classic tool is to give estimates of integrated volatility $\int_0^t |\sigma_s|^p ds$ for any $p \geq 2$ which appears naturally as the limit of some power variations. The use of power variation comes from the link between the quadratic variation and the integrated volatility (cf. [7, 8, 22] for the continuous setting, [2, 4, 18, 19, 23, 22] for the discontinuous setting and the recent paper [16] for a more general setting). Our work is based on Theorem 2 which comes from Aït Sahalia and Jacod's recent works [3, 15]. In a sense, we propose in this paper to approximate the instantaneous volatility using some finite differences of power variations.

A likelihood-based estimation of jumps parameter is difficult to reach [2, 1]. But using this kind of results on volatility, we can derive results on jump detections (cf. [5, 10]). Here we have a Central Limit Theorem and we can go further and get an expression of the confidence interval on the instantaneous volatility and thus a test of jump occurence.

The paper is organized as follows. In Section 2, we introduce the model we deal with, we present the different assumptions for this study and we state our main theorem, a Central Limit Theorem for the instantaneous volatility. Section 3 is the proof of this theorem. In Section 4, we present two examples of processes σ satisfying the assumptions: a pure jump process and the absolute value of the solution of a Lévy process driven Stochastic Differential Equation. From Theorem 5, we easily deduce a confidence interval for the instantaneous volatility, this is shown in Section 5. In Section 6, still from Theorem 5, we construct a test of jump occurence in the first example case. Finally Section 7 is devoted to some simulations.

2 Model

We consider a stochastic process $(X_t)_{t\geq 0}$ defined on a filtered probability space by:

(1) $$dX_t = \mu_t dt + \sigma_t dW_t, \quad t \geq 0,$$

where W is a $(\mathcal{F}_t)$-adapted Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, $\mu : \mathbb{R}_+ \to \mathbb{R}$ and σ are some càdlàg $(\mathcal{F}_t)$-adapted processes. We also assume that σ is non-negative.

In this paper, we want to estimate $(\sigma_t)_{t\geq 0}$ using the asymptotic properties of the observed discrete increments of X: let T be a positive number and assume that X is observed at times $i\Delta_n$ for all $i = 0, 1, \ldots, [\frac{T}{\Delta_n}]$. In the sequel, we will assume that

$$\Delta_n \xrightarrow[n\to+\infty]{} 0.$$

Then, for $p > 0$, we denote by $\hat{B}(p, \Delta_n)$ the stochastic process defined by

$$\widehat{B}(p, \Delta_n)_t := \sum_{i=1}^{[\frac{t}{\Delta_n}]} |\Delta_i^n X|^p, \quad t \geq 0,$$

where $\Delta_i^n X := X_{i\Delta_n} - X_{(i-1)\Delta_n}$.

Before going further, we recall two results about the asymptotic properties of the observe discrete increments of X, from Lépingle [17] and Aït Sahalia and Jacod [3, Theorem 2] respectively.

Proposition 1. *Let p be a positive number and set $m_p := \mathbb{E}[|U|^p]$ where $U \sim \mathcal{N}(0,1)$. Then, locally uniformly in t,*

$$\Delta_n^{1-\frac{p}{2}} \hat{B}(p, \Delta_n)_t \xrightarrow[n\to+\infty]{\mathbb{P}} m_p A(p)_t \quad \textit{with} \quad A(p)_t = \int_0^t \sigma_s^p ds.$$

Proposition 2. *Let $p \geq 2$ and assume technical assumptions as in [3]. Then, the sequence of continuous processes $(Y(n,p))_{n\in\mathbb{N}}$ defined for any $n \in \mathbb{N}$ by*

$$Y(n,p)_t := \frac{1}{\sqrt{\Delta_n}} \Big(\Delta_n^{1-\frac{p}{2}} \hat{B}(p, \Delta_n)_t - m_p A(p)_t \Big), \quad t \geq 0,$$

converges stably (in particular in law) to a random variable $Y(p)$ on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}})$ of the original filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ such that, for any $t \geq 0$, conditionally on $\mathcal{F}$, $Y(p)_t$ is a centered Gaussian variable with variance $\tilde{\mathbb{E}}[Y(p)_t^2 | \mathcal{F}] = (m_{2p} - m_p^2) A(2p)_t$.

Looking at these results, it is natural to try to estimate σ_{t-}^p by the following statistic: $(\Sigma(p,\Delta_n,h_n)_t)$ defined for every $t \geq h_n$ by:

$$\Sigma(p,\Delta_n,h_n)_t := \frac{\Delta_n^{1-\frac{p}{2}}\left(\hat{B}(p,\Delta_n)_t - \hat{B}(p,\Delta_n)_{t-h_n}\right)}{m_p h_n}.$$

Actually, this estimate is the mean of p-variations in a window of length h_n where (h_n) is a sequence of positive numbers such that

$$h_n \xrightarrow[n\to+\infty]{} 0.$$

Remark 3. *Note that we chose to build $\Sigma(p,\Delta_n,h_n)$ with the observations before time t. Actually, the estimation of σ_t is really of interest in finance if it is based on the prices that we know at time t.*

Before stating our main result, we introduce the following assumptions :

$(\mathbf{C^1})$:

$$\forall r>0, \quad \sup_{t\in[0,T]} \mathbb{E}\left[|\sigma_t|^r\right] < +\infty \quad \text{and} \quad \sup_{t\in[0,T]} \mathbb{E}\left[|\mu_t|^r\right] < +\infty.$$

$(\mathbf{C^2_\alpha})$: There exists $0<\alpha\leq 1$, such that for all $r \geq \frac{1}{\alpha}$ there exists a constant C_r such that

$$\mathbb{E}\left[|\sigma_t-\sigma_s|^r\right] < C_r|t-s| \quad \forall (s,t)\in[0,T]^2.$$

Remark 4. *Since $\alpha \in]0,1]$, Assumption $(\mathbf{C^2_\alpha})$ and Jensen's inequality yield*

$$\mathbb{E}\left[|\sigma_t-\sigma_s|^r\right] \leq C|t-s|^{r\alpha\wedge 1} \quad \forall r>0,\ \forall (s,t)\in[0,T]^2. \tag{2}$$

Note that Assumptions $(\mathbf{C^1})$ and $(\mathbf{C^2_\alpha})$ are convenient for Lévy driven Stochastic Differential Equations when all the moments are finite (see subsection 4.2 for details). Assuming that all the moments of σ_t are finite is not fundamental but alleviating these assumptions would complicate the conditions on Δ_n and h_n in the main result.

We are now able to state our main result.

Theorem 5. *Let $p \geq 2$ and let (X_t) be a stochastic process solution to (1). Assume $(\mathbf{C^1})$and $(\mathbf{C^2_\alpha})$. Assume that $\frac{\Delta_n}{h_n} \xrightarrow[n\to+\infty]{} 0$ and that*

$$h_n = \begin{cases} o\left(\Delta_n^{\frac{1}{2\alpha+1}}\right) & \text{if } p=2 \text{ or } p\geq 3,\\ o\left(\Delta_n^{\frac{1}{2\alpha+1}\vee(3-p)}\right) & \text{if } p\in]2,3]. \end{cases}$$

Then, for every $t>0$,

$$\sqrt{\frac{h_n}{\Delta_n}}\left(\Sigma(p,\Delta_n,h_n)_t - \sigma_{t-}^p\right) \xrightarrow[n\to+\infty]{\mathcal{L}} \frac{\sqrt{m_{2p}-m_p^2}}{m_p}\sigma_{t-}^p U,$$

where $U \sim \mathcal{N}(0,1)$ and U is independent of $\mathcal{F}_{t-}$.

Corollary 6. *Let $p \geq 2$. Assume $(\mathbf{C^1})$ and $(\mathbf{C^2_\alpha})$ with $\alpha \in]0,1]$. Set $r_n := \frac{h_n}{\Delta_n}$ which corresponds to the number of observations on the interval $[t-h_n, t]$. Suppose $\Delta_n = \frac{1}{n}$ and $r_n := n^\rho$ with*

$$\begin{cases} \rho \in (0, \frac{2\alpha}{2\alpha+1}) & \textit{when } p \in \{2\} \cup \left[\frac{6\alpha+2}{2\alpha+1}, \infty\right[, \\ \rho \in (0, p-2) & \textit{when } p \in \left]2, \frac{6\alpha+2}{2\alpha+1}\right]. \end{cases}$$

Then, for every $t > 0$,

$$n^{\frac{\rho}{2}} \left(\Sigma(p, \Delta_n, h_n)_t - \sigma^p_{t-}\right) \xrightarrow[n\to+\infty]{\mathcal{L}} \frac{\sqrt{m_{2p} - m_p^2}}{m_p} \sigma^p_{t^-} U,$$

where $U \sim \mathcal{N}(0,1)$ and U is independent of $\mathcal{F}_{t^-}$.

These results are both enough to obtain an estimation of σ_t^p and to obtain a confidence interval for it, together the convergence speed.

Remark 7. *Considering the window width h_n, the conditions of Theorem 5 could be written as follows: $r_n \longrightarrow +\infty$ and*

$$\begin{cases} r_n^{-1} = o\left(\Delta_n^{\frac{2\alpha}{2\alpha+1}}\right) & \textit{if } p = 2, \\ r_n^{-1} = o\left(\Delta_n^{\frac{2\alpha}{2\alpha+1}\wedge(p-2)}\right) & \textit{if } p > 2. \end{cases}$$

Given the Hölder exponent α, the algorithm could be to choose the size discretization Δ_n, then the number of observations in the window r_n to get all the errors less than the main one for relevant p, according to Corollary 6.

Let us focus on case $\alpha = 1/2$ which contains a large class of SDE's (see Section 4.2). Then, if $\Delta_n = 1/n$, the previous corollary implies that when $p \in \{2\} \cap (5/2, +\infty)$, taking a window width $r_n = n^\rho$ with $\rho \in (0, 1/2)$, one obtains a rate of order $n^{\rho/2}$. Actually the rate is optimised in the limit case $\rho = 1/2$ for which the order is $n^{1/4}$.

On the other hand, if one assumes moreover that there is no Brownian part and that the jump component has bounded variations, thus $\alpha = 1$, we can choose $r_n = n^\rho$, $\rho < 2/3$, for instance $\rho = 0.59$, and the order of the optimised rate is $n^{1/3}$ (see simulations in the last section). As an example in such a case, let us choose $r_n = n^{0.59} \sim 300$. It means 300 data which can be the daily observations and globally $n \sim 16\ 000$.

3 Proof

In every proofs C or C_p are constants which can change from a line to another.

3.1 Proof of Theorem 5

Following Jacod [15] scheme, we first decompose $\Sigma(p,\Delta_n,h_n)_t - \sigma^p_{t-}$ as follows:

$$\Sigma(p,\Delta_n,h_n)_t - \sigma^p_{t-} = \frac{Z^{(n,p)}_t - Z^{(n,p)}_{t-h_n}}{m_p h_n} + \Big(\frac{1}{h_n}\int_{t-h_n}^t \sigma^p_u du - \sigma^p_{t-}\Big), \tag{3}$$

where

$$Z^{(n,p)}_t := \Delta_n^{1-\frac{p}{2}} \hat{B}(p,\Delta_n)_t - m_p \int_0^t |\sigma_s|^p ds.$$

In the following lemma, we study the second part of the right-hand member of (3).

Lemma 8. *Let $p \geq 2$ and assume $(\mathbf{C^1})$and $(\mathbf{C^2_\alpha})$. Then,*

$$|t-s| \leq 1 \quad \Rightarrow \quad \mathbb{E}\left[|\sigma^p_{t-} - \sigma^p_{s-}|\right] \leq C|t-s|^\alpha. \tag{4}$$

thus, there exists C_p such that for all t,

$$\mathbb{E}\left[\left|\frac{1}{h_n}\int_{t-h_n}^t |\sigma_u|^p du - \sigma^p_{t-}\right|\right] \leq C_p h_n^\alpha \quad \textit{as soon as} \quad h_n \leq 1. \tag{5}$$

Proof. First of all, let us show relation (4). Denote, for any $t \geq 0$, $\tau_t := \sigma_{t-}$. Using Fatou's lemma, it follows from Assumption $(\mathbf{C^2_\alpha})$ that

$$\mathbb{E}\left[|\sigma_{t-} - \sigma_{s-}|^r\right] < C_r|t-s| \quad \forall (s,t) \in [0,T]^2, \quad \forall r \in [1/\alpha, \infty). \tag{6}$$

Now, using the following elementary inequalities

$$\forall (u,v) \in (\mathbb{R}^+)^2, \qquad |u^q - v^q| \leq \begin{cases} |u-v|^q & \text{if } q \leq 1 \\ C_q\left(|u-v|u^{q-1} + |u-v|^q\right) & \text{if } q > 1, \end{cases} \tag{7}$$

we manage with

$$|\tau^p_t - \tau^p_s| \leq C_p\left(|\tau_t - \tau_s|\tau_s^{p-1} + |\tau_t - \tau_s|^p\right).$$

First, since $\alpha \leq 1$, we can use

$$\mathbb{E}\left[|\tau_t - \tau_s|^p\right] \leq C_p|t-s| \leq C_p|t-s|^\alpha. \tag{8}$$

Second, Hölder's inequality with $(\frac{1}{\alpha}, \frac{1}{1-\alpha})$ and Hypothesis $(\mathbf{C^1})$ yield

$$\mathbb{E}\left[|\tau_t - \tau_s|\tau_s^{p-1}\right] \leq (\mathbb{E}\left[|\tau_t - \tau_s|^{\frac{1}{\alpha}}\right])^\alpha . \left(\mathbb{E}\left[\tau_s^{\frac{p-1}{1-\alpha}}\right]\right)^{1-\alpha} \leq C|t-s|^\alpha. \tag{9}$$

Thus, (8) and (9) lead to (4) which concludes the proof of the first part.
Now, let us show relation (5). Obviously, σ being non-negative, we have :

$$\int_{t-h_n}^t |\sigma_u|^p\, du = \int_{t-h_n}^t \sigma^p_u\, du = \int_{t-h_n}^t \tau^p_u\, du,$$

thus

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{1}{h_n}\int_{t-h_n}^{t}|\sigma_u|^p du-\sigma_{t-}^p\right|\right] &\leq \mathbb{E}\left[\left|\frac{1}{h_n}\int_{t-h_n}^{t}\tau_u^p du-\frac{1}{h_n}\int_{t-h_n}^{t}\tau_t^p du\right|\right], \\
&\leq \frac{1}{h_n}\int_{t-h_n}^{t}|\tau_u^p-\tau_t^p|\,du, \\
&\leq \frac{C_p}{h_n}\int_{t-h_n}^{t}|u-t|^{\alpha}\,du, \\
&\leq C_p h_n^{\alpha}
\end{aligned}$$

which ends the proof of the lemma. □

Hence, when

$$h_n^{\alpha}=o\left(\sqrt{\Delta_n/h_n}\right)\ \text{ as } n\to+\infty,\ \text{ i.e. when } h_n=o\left(\Delta_n^{\frac{1}{2\alpha+1}}\right), \tag{10}$$

it follows from (3) that proving Theorem 5 comes to showing that

$$\sqrt{\frac{h_n}{\Delta_n}}\left(\frac{Z_t^{(n,p)}-Z_{t-h_n}^{(n,p)}}{h_n}\right)\xrightarrow[n\to+\infty]{\mathcal{L}}\left(\sqrt{m_{2p}-m_p^2}\right)\sigma_{t-}^p U, \tag{11}$$

where $U\sim\mathcal{N}(0,1)$ and U is independent of $\mathcal{F}_{t^-}$.

In order to prove (11), we first decompose $Z_t^{(n,p)}$ as follows:

$$Z_t^{(n,p)}=\Delta_n\sum_{i=1}^{[t/\Delta_n]}\left(A_i^n-B_i^n-\left(\mathbb{E}_{i-1}^n\left[A_i^n\right]-\mathbb{E}_{i-1}^n\left[B_i^n\right]\right)\right) \tag{12}$$

$$+\Delta_n\sum_{i=1}^{[t/\Delta_n]}\left(B_i^n-\mathbb{E}_{i-1}^n\left[B_i^n\right]\right) \tag{13}$$

$$+\Delta_n\sum_{i=1}^{[t/\Delta_n]}\mathbb{E}_{i-1}^n\left[A_i^n\right]-m_p\int_0^t|\sigma_u|^p du. \tag{14}$$

where $\mathbb{E}_{i-1}^n[*]$ denote the conditional expectation with respect to $\mathcal{F}_{(i-1)\Delta_n}$ and

$$A_i^n:=\left|\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right|^p \quad \textit{and} \quad B_i^n:=\left|\sigma_{(i-1)\Delta_n}\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\right|^p.$$

Hence,

$$\frac{Z_t^{(n,p)}-Z_{t-h_n}^{(n,p)}}{h_n}=\Lambda_1^n+\Lambda_2^n+\Lambda_3^n,$$

with

$$\Lambda_1^n(t) := \frac{\Delta_n}{h_n} \sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} (A_i^n - B_i^n - (\mathbb{E}_{i-1}^n [A_i^n] - \mathbb{E}_{i-1}^n [B_i^n])),$$

$$\Lambda_2^n(t) := \frac{\Delta_n}{h_n} \sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} (B_i^n - \mathbb{E}_{i-1}^n [B_i^n]),$$

$$\Lambda_3^n(t) := \frac{\Delta_n}{h_n} \left[\sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n [A_i^n] - \frac{m_p}{\Delta_n} \int_{t-h_n}^{t} |\sigma_u|^p du \right].$$

Let us now study $\Lambda_1^n(t)$, $\Lambda_2^n(t)$ and $\Lambda_3^n(t)$.

Lemma 9. *Let $p \geq 2$ and assume $(\mathbf{C^1})$ and $(\mathbf{C^2_\alpha})$. Then, there exists $C_p > 0$ such that for all t,*

$$\sup_{t\in[0,T]} \mathbb{E}\left[(\Lambda_1^n(t))^2\right] \leq C_p \frac{\Delta_n^{1+\frac{1}{p}}}{h_n}.$$

As a consequence, for every $t \in [0,T]$,

$$\sqrt{\frac{h_n}{\Delta_n}} \Lambda_1^n(t) \xrightarrow[n\to+\infty]{\mathbb{L}^1} 0.$$

Proof. By a martingale argument, we have,

$$\begin{aligned}
\mathbb{E}\left[(\Lambda_1^n(t))^2\right] &= \mathbb{E}\left[\left(\frac{\Delta_n}{h_n} \sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} (A_i^n - B_i^n - (\mathbb{E}_{i-1}^n [A_i^n] - \mathbb{E}_{i-1}^n [B_i^n]))\right)^2\right], \\
&= \frac{\Delta_n^2}{h_n^2} \left(\sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} \mathbb{E}\left[(A_i^n - B_i^n)^2\right] - \sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} \mathbb{E}\left[(\mathbb{E}_{i-1}^n [A_i^n - B_i^n])^2\right] \right), \\
&\leq \frac{\Delta_n^2}{h_n^2} \sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} \mathbb{E}\left[(A_i^n - B_i^n)^2\right], \\
&\leq \frac{\Delta_n^{2-p}}{h_n^2} \sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} \mathbb{E}\left[\left(|\Delta_i^n X|^p - \left|\sigma_{(i-1)\Delta_n} \Delta_i^n W\right|^p\right)^2\right].
\end{aligned}$$

But

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

Thus we have

$$\Delta_i^n X = \sigma_{(i-1)\Delta_n} \Delta_i^n W + \chi_i^n,$$

with

$$\chi_i^n = \int_{(i-1)\Delta_n}^{i\Delta_n} (\sigma_s - \sigma_{(i-1)\Delta_n}) dW_s + \int_{(i-1)\Delta_n}^{i\Delta_n} \mu_s ds.$$

Using a Taylor expansion of $g(x) = |x|^p$ on the interval

$$\mathcal{D}_i^n = [\sigma_{(i-1)\Delta_n} \Delta_i^n W ; \Delta_i^n X],$$

we have :

$$\big| |\Delta_i^n X|^p - |\sigma_{(i-1)\Delta_n} \Delta_i^n W|^p \big| \leq \sup_{x \in \mathcal{D}_i^n} |g'(x)| \; |\chi_i^n|.$$

But $|g'(x)| = O(|x|^{p-1})$ thus using the relation $|x+y|^p \leq C_p(|x|^p + |y|^p)$ with C_p a constant, we have

$$\begin{aligned} \sup_{x \in \mathcal{D}_i^n} |g'(x)| \;&\leq\; C_p(|\sigma_{(i-1)\Delta_n} \Delta_i^n W|^{p-1} \;+\; |\chi_i^n|^{p-1}), \\ \big| |\Delta_i^n X|^p - |\sigma_{(i-1)\Delta_n} \Delta_i^n W|^p \big| &\leq C_p(|\sigma_{(i-1)\Delta_n} \Delta_i^n W|^{p-1} |\chi_i^n| \;+\; |\chi_i^n|^p). \end{aligned}$$

Finally there is a constant C_p such that, for all $t \geq 0$:

$$\begin{aligned} \mathbb{E}\left[(\Lambda_1^n(t))^2 \right] &\leq C_p \frac{\Delta_n^{2-p}}{h_n^2} \sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} \mathbb{E}\left[|\chi_i^n|^2 |\sigma_{(i-1)\Delta_n} \Delta_i^n W|^{2p-2} \;+\; |\chi_i^n|^{2p} \right] \\ \mathbb{E}\left[(\Lambda_1^n(t))^2 \right] &\leq C_p \frac{\Delta_n^{2-p}}{h_n^2} \sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} \Big(\mathbb{E}\left[|\chi_i^n|^{2p} \right]^{\frac{1}{p}} \mathbb{E}\left[|\sigma_{(i-1)\Delta_n} \Delta_i^n W|^{2p} \right]^{\frac{p-1}{p}} \\ &\qquad + \mathbb{E}\left[|\chi_i^n|^{2p} \right] \Big) \end{aligned} \tag{15}$$

First of all, we have by the independence between $\sigma_{(i-1)\Delta_n}$ and $\Delta_i^n W$ and the Assumption ($\mathbf{C^1}$) on σ :

$$\mathbb{E}\left[|\sigma_{(i-1)\Delta_n} \Delta_i^n W|^{2p} \right] = \Delta_n^p \, m_{2p} \, \mathbb{E}\left[|\sigma_{(i-1)\Delta_n}|^{2p} \right] \leq C_p \Delta_n^p.$$

So it remains to give a majoration of $\mathbb{E}\left[|\chi_i^n|^{2p} \right]$.

Using Assumption ($\mathbf{C^1}$), we have for every $s \in [(i-1)\Delta_n, T], \; \forall q > 0,$

$$\begin{aligned} \mathbb{E}\left[\left| \int_{(i-1)\Delta_n}^{s} \mu_u du \right|^q \right] &\leq (s-(i-1)\Delta_n)^q \sup_{t \in [0,T]} \mathbb{E}\left[|\mu_t|^q \right], \\ &\leq C(s-(i-1)\Delta_n)^q. \end{aligned} \tag{16}$$

Now, an application of Burkholder-Davies-Gundy inequality [14, Inequality (2.34) p. 38] and of Jensen's inequality at the fourth line lead us to:

$$
\begin{aligned}
\mathbb{E}\left[|\chi_i^n|^{2p}\right] &\leq C.\left(\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{i\Delta_n}(\sigma_s-\sigma_{(i-1)\Delta_n})dW_s\right|^{2p}\right]\right.\\
&\qquad\left.+\,\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{i\Delta_n}\mu_s ds\right|^{2p}\right]\right),\\
&\leq C.\left(\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{i\Delta_n}(\sigma_s-\sigma_{(i-1)\Delta_n})^2 ds\right|^{p}\right]+\Delta_n^{2p}\right),\\
&\leq C.\left(\Delta_n^p\,\mathbb{E}\left[\left|\frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n}(\sigma_s-\sigma_{(i-1)\Delta_n})^2 ds\right|^{p}\right]+\Delta_n^{2p}\right),\\
&\leq C.\left(\Delta_n^p\,\mathbb{E}\left[\frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n}\left|\sigma_s-\sigma_{(i-1)\Delta_n}\right|^{2p} ds\right]+\Delta_n^{2p}\right),\\
&\leq C.\left(\Delta_n^p\,\mathbb{E}\left[\int_0^1\left|\sigma_{(i-1+u)\Delta_n}-\sigma_{(i-1)\Delta_n}\right|^{2p} du\right]+\Delta_n^{2p}\right),\\
&\leq C.\left(\Delta_n^p\int_0^1\mathbb{E}\left[\left|\sigma_{(i-1+u)\Delta_n}-\sigma_{(i-1)\Delta_n}\right|^{2p}\right] du+\Delta_n^{2p}\right).
\end{aligned}
$$

Now, by the Assumption ($\mathbf{C}_\alpha^\mathbf{2}$), we have, since $p\geq 1$,

$$
\mathbb{E}\left[|\chi_i^n|^{2p}\right]\leq C.\left(\Delta_n^p\int_0^1 u\Delta_n du+\Delta_n^{2p}\right)\leq C.\left(\Delta_n^{p+1}+\Delta_n^{2p}\right)\leq C.\Delta_n^{p+1}
$$

Thus (15) becomes:

$$
\begin{aligned}
\mathbb{E}\left[(\Lambda_1^n(t))^2\right] &\leq C.\frac{\Delta_n^{2-p}}{h_n^2}\sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]}\left[(\Delta_n^{p+1})^{\frac{1}{p}}\Delta_n^{p-1}+\Delta_n^{p+1}\right],\\
&\leq C.\frac{\Delta_n^{2-p}}{h_n^2}\frac{h_n}{\Delta_n}\left[\Delta_n^{p+\frac{1}{p}}+\Delta_n^{p+1}\right],\\
&\leq C.\frac{\Delta_n^{1-p}}{h_n}\left[\Delta_n^{p+\frac{1}{p}}+\Delta_n^{p+1}\right],\\
&\leq \frac{C}{h_n}\left[\Delta_n^{1+\frac{1}{p}}+\Delta_n^{2}\right],
\end{aligned}
$$

the constant C does not depend on t and as $p\geq 2$, we have,

$$
\sup_{t\in[0,T]}\mathbb{E}\left[(\Lambda_1^n(t))^2\right]\leq C_p\frac{\Delta_n^{1+\frac{1}{p}}}{h_n},
$$

which ends the proofs. $\square$

Lemma 10. *If $\Delta_n = o(h_n)$ as $n \to +\infty$, then, for every $t \in [0,T]$,*

$$\sqrt{\frac{h_n}{\Delta_n}}\Lambda_2^n(t) \xrightarrow[n\to+\infty]{\mathcal{L}} \sqrt{m_{2p} - m_p^2}(\sigma_{t-})^p U,$$

where $U \sim \mathcal{N}(0,1)$ and U is independent of $\mathcal{F}_{t^-}$.

Proof. Let t be a positive number. Let

$$\{(\xi_i^n), i = [(t-h_n)/\Delta_n], \ldots, [t/\Delta_n], n \geq 1\}$$

be the sequence of triangular arrays of square-integrable martingale increments (with respect to the filtration $(\mathcal{F}_{(i-1)\Delta_n})_{n\geq 0}$) defined by:

$$\xi_i^n := \sqrt{\frac{\Delta_n}{h_n}}\left(\left|\sigma_{(i-1)\Delta_n}\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\right|^p - \mathbb{E}_{i-1}^n\left[\left|\sigma_{(i-1)\Delta_n}\frac{\Delta_i^n W}{\sqrt{\Delta_n}}\right|^p\right]\right), \tag{17}$$
$$\sim \sqrt{\frac{\Delta_n}{h_n}}\sigma_{(i-1)\Delta_n}^p(|U|^p - m_p).$$

First,

$$\mathbb{E}_{i-1}^n\left[(\xi_i^n)^2\right] = \frac{\Delta_n}{h_n}(m_{2p} - m_p^2)(\sigma_{(i-1)\Delta_n})^{2p}. \tag{18}$$

Hence,

$$\left|\sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\left[(\xi_i^n)^2\right] - (m_{2p} - m_p^2)(\sigma_{t-})^{2p}\right| \leq C \sup_{u\in[t-h_n,t]}\left|(\sigma_u)^{2p} - (\sigma_{t-})^{2p}\right|,$$

and the right continuity of (σ_{t-}) implies that,

$$\sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\left[(\xi_i^n)^2\right] \xrightarrow[n\to+\infty]{} (m_{2p} - m_p^2)|\sigma_{t-}|^{2p} \quad a.s.$$

Then, the lemma will follow from the fact that,

$$\sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} \xi_i^n = \sqrt{\frac{h_n}{\Delta_n}}\Lambda_2^n(t),$$

and from the central limit theorem for arrays of square-integrable martingale increments (see $e.g.$ [13]) provided the Lindeberg's condition holds, $i.e.$ if,

$$\sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]} \mathbb{E}_{i-1}^n\left[(\xi_i^n)^2 1_{|\xi_i^n|^2\geq\varepsilon}\right] \xrightarrow[n\to+\infty]{} 0 \quad a.s. \quad \forall\varepsilon > 0. \tag{19}$$

Let us prove (19). We derive from the Cauchy-Schwarz and Chebyshev inequalities that,

$$\begin{aligned}\mathbb{E}^n_{i-1}\left[(\xi^n_i)^2\,1_{|\xi^n_i|^2\geq\varepsilon}\right] &\leq \mathbb{E}^n_{i-1}\left[(\xi^n_i)^4\right]^{\frac{1}{2}}\left[\mathbb{P}\left[\{|\xi^n_i|^2\geq\varepsilon\}|\mathcal{F}_{(i-1)\Delta_n}\right]\right]^{\frac{1}{2}},\\ &\leq \frac{1}{\varepsilon}\mathbb{E}^n_{i-1}\left[(\xi^n_i)^4\right].\end{aligned}$$

Therefore, using (17),

$$\mathbb{E}^n_{i-1}\left[(\xi^n_i)^4\right]=\frac{\Delta_n^2}{h_n^2}\sigma^{4p}_{(i-1)\Delta_n}\mathbb{E}\left[(|U|^p-m_p)^4\right],$$

and since σ is locally bounded, we obtain that there exists $C(\omega)$ such that for all $t\geq 0$,

$$\sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]}\mathbb{E}^n_{i-1}\left[(\xi^n_i)^2 1_{|\xi^n_i|^2\geq\varepsilon}\right]\leq\frac{C(\omega)}{\varepsilon}\sum_{i=[(t-h_n)/\Delta_n]}^{[t/\Delta_n]}\frac{\Delta_n^2}{h_n^2}=\frac{C(\omega)}{\varepsilon}\frac{\Delta_n}{h_n}.$$

Assertion (19) follows from the fact that $\Delta_n=o(h_n)$ as $n\to+\infty$. □

Proposition 11. *Let $p\geq 2$ and assume $(\mathbf{C^1})$ and $(\mathbf{C^2_\alpha})$. Then,*

(20)
$$\mathbb{E}\left[\left|\mathbb{E}^n_{i-1}\left[\left|\frac{\Delta^n_i X}{\sqrt{\Delta_n}}\right|^p\right]-\frac{m_p}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n}|\sigma_u|^p du\right|\right]\leq\begin{cases}C\Delta_n^{\frac{1}{2}} & \text{if } p=2\\ C\Delta_n^{\frac{p-2}{2}\wedge\alpha\wedge\frac{1}{2}} & \text{if } p>2.\end{cases}$$

As a consequence,

$$\sqrt{\frac{h_n}{\Delta_n}}\Lambda^n_3(t)\xrightarrow[n\to+\infty]{\mathbb{L}^1}0\quad \text{if}\quad\begin{cases}h_n\xrightarrow[n\to+\infty]{}0 & \text{when } p=2,\\ h_n=o\left(\Delta_n^{(3-p)\vee(1-2\alpha)\vee 0}\right) & \text{when } p>2.\end{cases}$$

Proof. We begin the proof by the following remark. Using scaling and independance properties of the Brownian motion and the Ito's formula yield

$$\begin{aligned}m_p&=\mathbb{E}^n_{i-1}\left[\left|\frac{W_{i\Delta_n}-W_{(i-1)\Delta_n}}{\sqrt{\Delta_n}}\right|^p\right],\\ &=\frac{p(p-1)}{2\Delta_n^{\frac{p}{2}}}\int_{(i-1)\Delta_n}^{i\Delta_n}\mathbb{E}^n_{i-1}\left[|W_s-W_{(i-1)\Delta_n}|^{p-2}\right]ds.\end{aligned}$$

Keeping in mind this representation of m_p, we decompose the integrand of (20) as follows:

$$\mathbb{E}^n_{i-1}\left[\left|\frac{\Delta^n_i X}{\sqrt{\Delta_n}}\right|^p\right]-\frac{m_p}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n}(\sigma_u)^p du=(21)+(22)+(23)$$

where

$$\mathbb{E}_{i-1}^n\left[\left|\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right|^p\right] - \frac{p(p-1)}{2\Delta_n^{\frac{p}{2}}}\int_{(i-1)\Delta_n}^{i\Delta_n}\mathbb{E}_{i-1}^n\left[\left|\int_{(i-1)\Delta_n}^s \sigma_u dW_u\right|^{p-2}\sigma_s^2\right]ds \tag{21}$$

$$\frac{p(p-1)}{2\Delta_n^{\frac{p}{2}}}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\mathbb{E}_{i-1}^n\left[\left|\int_{(i-1)\Delta_n}^s \sigma_u dW_u\right|^{p-2}\sigma_s^2 - (\sigma_{(i-1)\Delta_n})^p\left|\int_{(i-1)\Delta_n}^s dW_u\right|^{p-2}\right]\right)ds \tag{22}$$

$$-\frac{m_p}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n}\left((\sigma_s)^p - (\sigma_{(i-1)\Delta_n})^p\right)ds. \tag{23}$$

In case $p = 2$, one can check that (22) and (23) cancel themselves. Then, it only appears the term (21). So, except for this term, we suppose $p > 2$.

We first manage with (23). By (4) we deduce that, for every $p > 2$,

$$\mathbb{E}\left[\left|(\sigma_s)^p - (\sigma_{(i-1)\Delta_n})^p\right|\right] \le C_p(s-(i-1)\Delta_n)^\alpha.$$

Hence,

$$\mathbb{E}\left[\frac{1}{\Delta_n}\left|\int_{(i-1)\Delta_n}^{i\Delta_n}\left((\sigma_s)^p - (\sigma_{(i-1)\Delta_n})^p\right)ds\right|\right] \le C_p\Delta_n^\alpha. \tag{24}$$

Then, the sequel of the proof is based on Lemmas 12 and 13 corresponding to (21) and (22) respectively.

Lemma 12. *Let $p \ge 2$ and assume $(\mathbf{C^1})$ and $(\mathbf{C^2_\alpha})$. Then,*

$$\mathbb{E}\left[|(21)|\right] \le \begin{cases} C.\Delta_n^{\frac{1}{2}} & \text{if } p = 2, \\ C.\Delta_n^{(\frac{p}{2}-1)\wedge\frac{1}{2}} & \text{if } p > 2. \end{cases} \tag{25}$$

Proof. First, we use Itô's formula to develop A_i^n:

$$A_i^n = \left|\frac{\Delta_i^n X}{\sqrt{\Delta_n}}\right|^p = \int_{(i-1)\Delta_n}^{i\Delta_n} \pm p\frac{|X_s - X_{(i-1)\Delta_n}|^{p-1}}{\Delta_n^{\frac{p}{2}}}\mu_s ds$$
$$+\frac{1}{2}p(p-1)\int_{(i-1)\Delta_n}^{i\Delta_n}\frac{|X_s - X_{(i-1)\Delta_n}|^{p-2}}{\Delta_n^{\frac{p}{2}}}\sigma_s^2 ds + M_i^n,$$

with $\mathbb{E}_{i-1}^n[M_i^n] = 0$. It follows that:

$$(21) = \mathbb{E}_{i-1}^n\left[\int_{(i-1)\Delta_n}^{i\Delta_n} \pm p\frac{|X_s - X_{(i-1)\Delta_n}|^{p-1}}{\Delta_n^{\frac{p}{2}}}\mu_s ds\right]$$
$$+\frac{1}{2}p(p-1)\mathbb{E}_{i-1}^n\left[\int_{(i-1)\Delta_n}^{i\Delta_n} R_i^n(s)\sigma_s^2 ds\right],$$
$$\text{with}\quad R_i^n(s) := \frac{|X_s - X_{(i-1)\Delta_n}|^{p-2}}{\Delta_n^{\frac{p}{2}}} - \frac{|\int_{(i-1)\Delta_n}^s \sigma_u dW_u|^{p-2}}{\Delta_n^{\frac{p}{2}}}.$$

Now,

$$\mathbb{E}\left[|X_s - X_{(i-1)\Delta_n}|^{p-1}|\mu_s|\right]$$
$$\leq C.\mathbb{E}\left[\left(\left|\int_{(i-1)\Delta_n}^{s}\mu_u du\right|^{p-1} + \left|\int_{(i-1)\Delta_n}^{s}\sigma_u dW_u\right|^{p-1}\right)|\mu_s|\right].$$

On the one hand, using for the first term the Cauchy-Schwarz inequality, Assumption ($\mathbf{C^1}$) and (16), we obtain that

$$\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s}\mu_u du\right|^{p-1}|\mu_s|\right] \leq C(s-(i-1)\Delta_n)^{p-1}.$$

On the other hand, using Cauchy-Schwarz and Burkholder-Davies-Gundy inequalities,

$$\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s}\sigma_u dW_u\right|^{p-1}|\mu_s|\right] \leq \sqrt{\mathbb{E}\left[\mu_s^2\right]}\sqrt{\mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^{s}\sigma_u^2 du\right)^{p-1}\right]}$$

Thus Jensen's inequality yields:

$$\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s}\sigma_u dW_u\right|^{p-1}|\mu_s|\right]$$
$$\leq \sqrt{\mathbb{E}\left[\mu_s^2\right]}\sqrt{(s-(i-1)\Delta_n)^{p-2}\int_{(i-1)\Delta_n}^{s}\mathbb{E}\left[\sigma_u^{2(p-1)}\right]du},$$

thus finally ($\mathbf{C^1}$) yields

$$\mathbb{E}\left[|X_s - X_{(i-1)\Delta_n}|^{p-1}|\mu_s|\right] \leq C(s-(i-1)\Delta_n)^{p-1} + C(s-(i-1)\Delta_n)^{\frac{p-1}{2}}.$$

Hence, for every $p \geq 2$,

$$\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{i\Delta_n} \pm p\frac{|X_s - X_{(i-1)\Delta_n}|^{p-1}}{\Delta_n^{\frac{p}{2}}}\mu_s ds\right|\right] \leq C\Delta_n^{\frac{1}{2}}.$$

Now, we observe that $R_i^n(s) = 0$ when $p = 2$ so the proof is ended in this case.

When $p > 2$, (7) applied with $q = p-2$ yields

$$|R_i^n(s)| \leq \begin{cases} \frac{1}{\Delta_n^{\frac{p}{2}}}\left|\int_{(i-1)\Delta_n}^{s}\mu_u du\right|^{p-2} & \text{if } p \leq 3 \\ C.\frac{1}{\Delta_n^{\frac{p}{2}}}\left(\left|\int_{(i-1)\Delta_n}^{s}\mu_u du\right| . \left|\int_{(i-1)\Delta_n}^{s}\sigma_u dW_u\right|^{p-3}\right. & \\ \left. + \left|\int_{(i-1)\Delta_n}^{s}\mu_u du\right|^{p-2}\right) & \text{if } p > 3. \end{cases} \tag{26}$$

First, let $p \in (2,3]$. We derive from (16) that

$$\|R_i^n(s)\|_2 \leq C\Delta_n^{-\frac{p}{2}}(s-(i-1)\Delta_n)^{p-2} \ .$$

Then, Cauchy-Schwarz and $(\mathbf{C^1})$ yield

$$\mathbb{E}\left[\int_{(i-1)\Delta_n}^{i\Delta_n} |R_i^n(s)\sigma_s^2|ds\right] \leq C. \sup_{s\in[0,T]} \sqrt{\mathbb{E}\left[\sigma_s^4\right]}\Delta_n^{\frac{p}{2}-1} \leq C\Delta_n^{\frac{p}{2}-1}.$$

Assume now that $p>3$. First, for all $s \in [(i-1)\Delta_n, i\Delta_n]$, we derive from the Hölder's inequality applied with $\bar{p} = \frac{p}{p-3}$ and $\bar{q} = \frac{p}{3}$ that

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s} \mu_u du\right| \left|\int_{(i-1)\Delta_n}^{s} \sigma_u dW_u\right|^{p-3} \sigma_s^2\right] \\
&\quad \leq \left(\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s} \sigma_u dW_u\right|^{p}\right]\right)^{\frac{p-3}{p}} \left(\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s} \mu_u du\right|^{\frac{p}{3}} |\sigma_s|^{\frac{2p}{3}}\right]\right)^{\frac{3}{p}}.
\end{aligned}$$

Therefore, Burkholder-Davis-Gundy inequalities and Assumption $(\mathbf{C^1})$ yield

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s} \mu_u du\right| . \left|\int_{(i-1)\Delta_n}^{s} \sigma_u dW_u\right|^{p-3} .\sigma_s^2\right] \\
&\quad \leq C\left(\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s} \sigma_u^2 du\right|^{\frac{p}{2}}\right]\right)^{\frac{p-3}{p}} . \left(\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s} \mu_u du\right|^{\frac{p}{3}} |\sigma_s|^{\frac{2p}{3}}\right]\right)^{\frac{3}{p}}, \\
&\quad \leq C(s-(i-1)\Delta_n)^{\frac{p}{2}.\frac{p-3}{p}}.\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s} \mu_u du\right|^{\frac{2p}{3}}\right]^{\frac{3}{2p}} \mathbb{E}\left[|\sigma_s|^{\frac{4p}{3}}\right]^{\frac{3}{2p}},
\end{aligned}$$

and finally using (16) yields

$$\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s} \sigma_u dW_u\right|^{p-3} \left|\int_{(i-1)\Delta_n}^{s} \mu_u du\right| \sigma_s^2\right] \leq C.(s-(i-1)\Delta_n)^{\frac{p-3}{2}+1}.$$

Thus, we derive from the preceeding inequality and from (26) that when $p>3$,

$$\begin{aligned}
\mathbb{E}\left[\int_{(i-1)\Delta_n}^{i\Delta_n} |R_i^n(s)\sigma_s^2|ds\right] &\leq \frac{C}{\Delta_n^{\frac{p}{2}}} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big[(s-(i-1)\Delta_n)^{p-2} \\
&\qquad +(s-(i-1)\Delta_n)^{\frac{p-1}{2}}\Big] ds, \\
&\leq C.\Delta_n^{\frac{1}{2}}.
\end{aligned}$$

□

We now focus on (22).

Lemma 13. *Let $p > 2$ and assume $(\mathbf{C^1})$ and $(\mathbf{C^2_\alpha})$. Then,*

$$\mathbb{E}\left[|(22)|\right] \leq C.\Delta_n^{(\frac{p}{2}-1)\wedge\alpha\wedge\frac{1}{2}}. \tag{27}$$

Proof. First,

$$\frac{1}{\Delta_n^{\frac{p}{2}}}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\mathbb{E}_{i-1}^n\left[\left|\int_{(i-1)\Delta_n}^{s}\sigma_u dW_u\right|^{p-2}\sigma_s^2 - (\sigma_{(i-1)\Delta_n})^p\left|\int_{(i-1)\Delta_n}^{s}dW_u\right|^{p-2}\right]\right)ds$$

$$= \frac{1}{\Delta_n^{\frac{p}{2}}}\int_{(i-1)\Delta_n}^{i\Delta_n}\mathbb{E}_{i-1}^n\left[(\sigma_s^2 - \sigma_{(i-1)\Delta_n}^2)\left|\int_{(i-1)\Delta_n}^{s}\sigma_u dW_u\right|^{p-2}\right]ds \tag{28}$$

$$+\frac{\sigma_{(i-1)\Delta_n}^2}{\Delta_n^{\frac{p}{2}}}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\mathbb{E}_{i-1}^n\left[\left|\int_{(i-1)\Delta_n}^{s}\sigma_u dW_u\right|^{p-2} - \left|\int_{(i-1)\Delta_n}^{s}\sigma_{(i-1)\Delta_n} dW_u\right|^{p-2}\right]\right)ds. \tag{29}$$

Let us focus on (28) and let $q > 1$ and $r > 1$ satisfying $\frac{1}{q} + \frac{1}{r} = 1$ and $r \geq 2 \vee \frac{2}{p-2}$. Using Hölder inequality, we have

$$\begin{aligned}&\mathbb{E}\left[\left|\sigma_s^2 - \sigma_{(i-1)\Delta_n}^2\right|.\left|\int_{(i-1)\Delta_n}^{s}\sigma_u dW_u\right|^{p-2}\right]\\ &\qquad \leq \left(\mathbb{E}\left[\left|\sigma_s^2 - \sigma_{(i-1)\Delta_n}^2\right|^q\right]\right)^{\frac{1}{q}}.\left(\mathbb{E}\left[\left|\int_{(i-1)\Delta_n}^{s}\sigma_u dW_u\right|^{r(p-2)}\right]\right)^{\frac{1}{r}}.\end{aligned}$$

Then, we use Burkholder-Davis-Gundy and Assumption $(\mathbf{C^1})$ to obtain the majoration

$$\leq C.\left(\mathbb{E}\left[|\sigma_s^2 - \sigma_{(i-1)\Delta_n}^2|^q\right]\right)^{\frac{1}{q}}.\left(\sup_{t\in[0,T]}\mathbb{E}\left[|\sigma_t|^{r(p-2)}\right]\right)^{\frac{1}{r}}(s-(i-1)\Delta_n)^{\frac{p-2}{2}}.$$

Using Hölder again, Assumptions $(\mathbf{C^1})$ and $(\mathbf{C^2_\alpha})$ yield for every $h > 1$ and $h' > 1$ with $\frac{1}{h} + \frac{1}{h}' = 1$:

$$\begin{aligned}\mathbb{E}\left[|\sigma_s^2 - \sigma_{(i-1)\Delta_n}^2|^q\right] &\leq \mathbb{E}\left[|\sigma_s - \sigma_{(i-1)\Delta_n}|^{hq}\right]^{\frac{1}{h}}\mathbb{E}\left[|\sigma_s + \sigma_{(i-1)\Delta_n}|^{h'q}\right]^{\frac{1}{h'}},\\ &\leq C(s-(i-1)\Delta_n)^{(q\alpha)\wedge\frac{1}{h}},\end{aligned}$$

where in the last inequality we used (2). Thus,

$$\begin{aligned}&\mathbb{E}\left[\left|\sigma_s^2 - \sigma_{(i-1)\Delta_n}^2\right|.\left|\int_{(i-1)\Delta_n}^{s}\sigma_u dW_u\right|^{p-2}\right]\\ &\qquad \leq C.(s-(i-1)\Delta_n)^{\frac{p-2}{2}}\left(\mathbb{E}\left[\sigma_s^2 - \sigma_{(i-1)\Delta_n}^2\right]^q\right)^{\frac{1}{q}},\\ &\qquad \leq C.(s-(i-1)\Delta_n)^{\frac{p-2}{2}+(\alpha\wedge\frac{1}{qh})}.\end{aligned} \tag{30}$$

Hence, for every $q \in (1,2]$ $(p \geq 3)$ or $q \in (1, \frac{2}{4-p})$ $(2 \leq p \leq 3)$ (in order that $r = \frac{q}{q-1} \geq 2 \vee \frac{2}{p-2}$), for every $h > 1$, we have

$$\mathbb{E}\left[|(28)|\right] \leq \frac{C}{\Delta_n^{\frac{p}{2}}} \int_{(i-1)\Delta_n}^{i\Delta_n} (s-(i-1)\Delta_n)^{\frac{p-2}{2}+\alpha\wedge\frac{1}{qh}} ds \leq C\Delta_n^{\alpha\wedge\frac{1}{qh}}.$$

Taking q and h sufficiently small and large respectively, one obtains that for every $\varepsilon > 0$,

$$\mathbb{E}\left[|(28)|\right] \leq C\Delta_n^{\alpha\wedge(1-\varepsilon)}. \tag{31}$$

We now study (29). Set $M_s^n = \int_{(i-1)\Delta_n}^s (\sigma_u - \sigma_{(i-1)\Delta_n}) dW_u$. By (7),

$$\begin{aligned}
&\left|\left|\int_{(i-1)\Delta_n}^s \sigma_u dW_u\right|^{p-2} - \left|\int_{(i-1)\Delta_n}^s \sigma_{(i-1)\Delta_n} dW_u\right|^{p-2}\right| \\
&\leq \begin{cases} |M_s^n|^{p-2} & \text{if } p \leq 3 \\ C.\left(|M_s^n|.\left|\int_{(i-1)\Delta_n}^s \sigma_{(i-1)\Delta_n} dW_u\right|^{p-3} + |M_s^n|^{p-2}\right) & \text{if } p > 3. \end{cases}
\end{aligned} \tag{32}$$

Using Burkholder-Davies-Gundy's inequality, we derive from $(\mathbf{C}_\alpha^2)$ that for every $r \geq 2$,

$$\begin{aligned}
\mathbb{E}\left[|M_s^n|^r\right] &\leq C.\mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^s (\sigma_u - \sigma_{(i-1)\Delta_n})^2 du\right)^{\frac{r}{2}}\right], \\
&\leq C.(s-(i-1)\Delta_n)^{\frac{r}{2}-1} \int_{(i-1)\Delta_n}^s \mathbb{E}\left[|\sigma_u - \sigma_{(i-1)\Delta_n}|^r\right] du, \\
&\leq C.(s-(i-1)\Delta_n)^{\frac{r}{2}+1}. \qquad (33)
\end{aligned}$$

Hence, if $p \leq 3$, it follows from (32), Holder's inequality (applied with $\bar{p} = \frac{2}{p-2}$ and $\bar{q} = \frac{2}{4-p}$) and $(\mathbf{C^1})$ that

$$\begin{aligned}
\mathbb{E}\left[|(29)|\right] &\leq \frac{C}{\Delta_n^{\frac{p}{2}}} \int_{(i-1)\Delta_n}^{i\Delta_n} \mathbb{E}\left[|M_s^n|^2\right]^{\frac{p-2}{2}} .\mathbb{E}\left[|\sigma_{(i-1)\Delta_n}|^{\bar{q}}\right]^{\frac{1}{\bar{q}}} ds, \qquad (34) \\
&\leq \frac{C}{\Delta_n^{\frac{p}{2}}} \int_{(i-1)\Delta_n}^{i\Delta_n} \left[(s-(i-1)\Delta_n)^2\right]^{\frac{p-2}{2}} ds, \\
&\leq C\Delta_n^{\frac{p}{2}-1}. \qquad (35)
\end{aligned}$$

Assume now that $p > 3$. According to (32), we have two terms to manage with. On the one hand, by Cauchy-Schwarz, (33) and $(\mathbf{C^1})$, we have

$$\begin{aligned}
&\mathbb{E}\left[\sigma_{(i-1)\Delta_n}^2 |M_s^n|.\left|\int_{(i-1)\Delta_n}^s \sigma_{(i-1)\Delta_n} dW_u\right|^{p-3}\right] \\
&\leq \left(\mathbb{E}\left[|M_s^n|^2\right]\right)^{\frac{1}{2}} (s-(i-1)\Delta_n)^{\frac{p-3}{2}} \mathbb{E}\left[|\sigma_{(i-1)\Delta_n}|^{2(p-1)}\right]^{\frac{1}{2}}, \\
&\leq C(s-(i-1)\Delta_n)^{\frac{p-1}{2}}.
\end{aligned}$$

On the other hand, Holder's inequality $(\frac{1}{2},\frac{1}{2})$ and relation (33) yield

$$\begin{aligned}\mathbb{E}\left[\sigma^2_{(i-1)\Delta_n}|M^n_s|^{p-2}\right] &\le \left(\mathbb{E}\left[\sigma^2_{(i-1)\Delta_n}\right]\right)^{\frac{1}{2}}.\left(\mathbb{E}\left[|M^n_s|^{2(p-2)}\right]\right)^{\frac{1}{2}},\\ &\le C(s-(i-1)\Delta_n)^{\frac{p-1}{2}}.\end{aligned}$$

Thus, it follows that when $p>3$,

$$\mathbb{E}\left[|(29)|\right] \le C.\Delta_n^{\frac{1}{2}}. \tag{36}$$

Finally, we derive the lemma from (31), (35), (36). □

We conclude the proof of Proposition 11 with the combination of (24), (25), (27). □

3.2 Comparison of sufficient conditions and the error sizes

Finally, we gather all the constraints on (h_n,Δ_n) related to α and p, sufficient conditions to apply Lemmas 8, 9, 10 and Proposition 11:

$$\frac{h_n}{\Delta_n^{\frac{1}{2\alpha+1}}} \to 0, \qquad \textit{cf. } (10), \tag{37}$$

$$\frac{\Delta_n}{h_n} \to 0, \qquad \textit{cf. Lemmas 9 and 10}, \tag{38}$$

$$h_n \to 0, \qquad \textit{for } p=2, \textit{ Proposition 11}, \tag{39}$$

$$\frac{h_n}{\Delta_n^{(3-p)\vee 0\vee(1-2\alpha)}} \to 0, \qquad \textit{for } 2<p, \textit{ Proposition 11}, \tag{40}$$

Proposition 14. *Let $p\ge 2$ and assume $(\mathbf{C^1})$and $(\mathbf{C^2_\alpha})$. Then, the following conditions are sufficient to get Theorem 5:*

$$\Delta_n/h_n \xrightarrow{n\to+\infty} 0,$$

and

$$h_n = \begin{cases} o\left(\Delta_n^{1/(2\alpha+1)}\right) & \textit{if } p=2\ ,\\ o\left(\Delta_n^{\frac{1}{2\alpha+1}\vee(3-p)}\right) & \textit{if } p>2.\end{cases}$$

Proof. Let $t>0$. Firstly, hypothesis $h_n = o\left(\Delta_n^{1/(2\alpha+1)}\right)$ is (37) and implies (39). So in case $p=2$, every conditions are satisfied to conclude Theorem 5.

Second, in case $p>2$, using that $1-2\alpha<\frac{1}{1+2\alpha}$ for every α, $(3-p)\vee 0\vee(1-2\alpha)\le \frac{1}{1+2\alpha}\vee(3-p)$ and the hypotheses imply (40) and once again every conditions are satisfied to conclude Theorem 5.

□

Remark 15. *Concerning the L^1-errors, we remark in Proposition 11 that in case $p > 2$ $\|\Lambda_3^n(t)\|_1 \le C\Delta_n^{\frac{p-2}{2}\wedge\alpha\wedge\frac{1}{2}}$, and this error is always bigger that the one in case $p = 2$, $\sqrt{\Delta_n}$. So we now fix $p = 2$ and we estimate σ_{t-}^2. Moreover, equation (18) in Lemma 10 proof implies that $\sup_n \|\Lambda_2^n(t)\|_2^2 < \infty$, hence the family is uniformly integrable and $\|\sqrt{\frac{h_n}{\Delta_n}}\Lambda_2^n(t)\|_1$ goes to a constant, thus the order of $\|\Lambda_2^n(t)\|_1$ is $\sqrt{\frac{\Delta_n}{h_n}}$. Hence, using (3), $\Lambda_i^n(t), i = 1,3$ definitions, Lemmas 8, 9 and Proposition 11, the error*

$$\|\Sigma(2, h_n, \Delta_n)_t - \sigma_{t-}^2\|_1 \le C\sup\{h_n^\alpha, \sqrt{\frac{\Delta_n^{1+1/p}}{h_n}}, \sqrt{\frac{\Delta_n}{h_n}}, \sqrt{\Delta_n}\} = C\sup\{h_n^\alpha, \sqrt{\frac{\Delta_n}{h_n}}\}$$

since $p > 0$ and $h_n \to 0$. We keep in mind that we need $\frac{\Delta_n}{h_n} \to 0$ and $\frac{h_n}{\Delta_n^{\frac{1}{2\alpha+1}}} \to 0$ to get Theorem 5. The error could be the least as soon as $\sqrt{\frac{\Delta_n}{h_n}} \sim h_n^\alpha$. For care of simplicity and without loss of generality, we put $\Delta_n = 1/n$; h_n has to satisfy

$$nh_n \to \infty \ ; \ h_n n^{\frac{1}{2\alpha+1}} \to 0 \ ; \ h_n \sim n^{-\frac{1}{2\alpha+1}}.$$

For instance we put $h_n = n^\rho$, then the best choice could be $\rho = -\frac{1}{2\alpha+1} - \varepsilon$, ε as small as possible.

4 Examples

4.1 Pure jump process

Using A. Alvarez's thesis [5], we consider the simplest model having volatility jumps, when volatility is piecewise constant:

$$\sigma_t = \sum_{i=0}^{N_t - 1} \sigma_i \mathbf{1}_{[\tau_i, \tau_{i+1}[}(t), \tag{41}$$

where N is a Poisson process with intensity λ. The N jump times are $\tau_i, i \ge 1$, and σ_i are positive $\mathcal{F}_{\tau_i}$-measurable random variables satisfying for all q: there exists $K_q > 0$ such that the conditional expectations satisfy:

$$\sup_i \mathbb{E}\,[\sigma_i^q / N] \le K_q < +\infty. \tag{42}$$

This yields Hypothesis ($\mathbf{C^1}$) and is less than Alvarez's hypothesis (σ bounded). Let us remark that σ is not a Lévy process except if $(\sigma_i - \sigma_{i-1})_{i \ge 1}$ are independant equidistributed random variables.

Below we check Hypothesis ($\mathbf{C}^{\mathbf{2}}_\alpha$) with $\alpha = 1$. Let us remark that actually $\sigma_t = \sigma_{N_t}$ thus when $s < t$,

$$\begin{aligned}
\mathbb{E}\left[|\sigma_t - \sigma_s|^r\right] &= \sum_{i<j} e^{-\lambda t} \frac{(\lambda s)^i}{i!} \frac{(\lambda(t-s))^{j-i}}{(j-i)!} \mathbb{E}\left[|\sigma_j - \sigma_i|^r \mid N_s = i, N_t = j\right], \\
&\leq K_r \sum_{i<j} e^{-\lambda t} \frac{(\lambda s)^i}{i!} \frac{(\lambda(t-s))^{j-i}}{(j-i)!}, \\
&\leq 1 - e^{-\lambda(t-s)}, \\
&\leq t - s.
\end{aligned}$$

4.2 Lévy driven SDE's

Assume that $\sigma_t = |Y_t|$ where (Y_t) is a solution to the following SDE:

$$dY_t = b(Y_{t-})dt + \varsigma(Y_{t-})d\tilde{W}_t + \kappa(Y_{t-})dZ_t, \tag{43}$$

where $b : \mathbb{R} \to \mathbb{R}$, $\varsigma : \mathbb{R} \mapsto \mathbb{R}$ and $\kappa : \mathbb{R} \mapsto \mathbb{R}$ are some continuous functions with sublinear growth, $(\tilde{W}_t)_{t\geq 0}$ is a Brownian motion and $(Z_t)_{t\geq 0}$ is an integrable purely discontinuous Lévy process independent of $(W_t)_{t\geq 0}$ with Lévy measure π and characteristic function given for every $t \geq 0$ by:

$$\mathbb{E}\left[e^{i<u,Z_t>}\right] = \exp\left[t\left(\int e^{i<u,y>} - 1 - i < u, y > \pi(dy)\right)\right].$$

We also assume that $\int_{|y|>1} |y|^r \pi(dy) < +\infty$ for every $r > 0$, *i.e.* that $\mathbb{E}\left[|Z_t|^r\right] < +\infty$ for every $r > 0$. We denote by $\underline{q}$, the Blumenthal-Getoor index defined by:

$$\underline{q} = \inf\left\{q \geq 0, \int_{|y|\leq 1} |y|^q \pi(dy) < +\infty\right\}.$$

We recall that $\underline{q} \leq 2$ since $\int_{|y|\leq 1} |y|^2 \pi(dy) < +\infty$. Then, Assumptions ($\mathbf{C}^{\mathbf{1}}$) and ($\mathbf{C}^{\mathbf{2}}_\alpha$) hold with

- $\alpha = \frac{1}{2}$ in the general case,
- $\alpha = 1$ if $\varsigma = 0$ and $\underline{q} < 1$,
- $\alpha = \frac{1}{q}$ for every $q > \underline{q}$ if $\varsigma = 0$ and $\underline{q} > 1$.

5 Asymptotic confidence interval

According to Theorem 5, with for instance asymptotic probability 0.95,

$$\mathbb{P}\left[\frac{\sqrt{r_n}|\Sigma(p, \Delta_n, h_n)_t - \sigma^p_{t-}| m_p}{\sigma^p_{t-}\sqrt{m_{2p} - m_p^2}} \leq 1.96\right] \xrightarrow[n\to\infty]{} 0.95.$$

Thus, with 0.95 asymptotic confidence,

$$\sigma^p_{t-} \in \left[\frac{m_p\sqrt{r_n}\,\Sigma(p,\Delta_n,h_n)_t}{m_p\sqrt{r_n}+1.96\sqrt{m_{2p}-m_p^2}} \, , \, \frac{m_p\sqrt{r_n}\,\Sigma(p,\Delta_n,h_n)_t}{m_p\sqrt{r_n}-1.96\sqrt{m_{2p}-m_p^2}} \right]. \tag{44}$$

The confidence interval length is about $r_n^{-\frac{1}{2}}$. We denote it as

$$CI(n,t).$$

But actually the most interesting point is that we obtain an asymptotic confidence interval for the relative error:

$$\mathbb{P}\left[\left| \frac{\Sigma(p,\Delta_n,h_n)_t}{\sigma^p_{t-}} - 1 \right| \leq \frac{1.96\sqrt{m_{2p}-m_p^2}}{m_p\sqrt{r_n}} \right] \xrightarrow[n\to\infty]{} 0.95.$$

Remark 16. *Actually, to compare this result with respect to p, we have to compare asymptotic confidence interval of σ_{t-}, namely*

$$\sigma_{t-} \in \left[\left(\frac{m_p\sqrt{r_n}\,\Sigma(p,\Delta_n,h_n)_t}{m_p\sqrt{r_n}+1.96\sqrt{m_{2p}-m_p^2}} \right)^{\frac{1}{p}} , \left(\frac{m_p\sqrt{r_n}\,\Sigma(p,\Delta_n,h_n)_t}{m_p\sqrt{r_n}-1.96\sqrt{m_{2p}-m_p^2}} \right)^{\frac{1}{p}} \right],$$

this interval length is about $r_n^{-\frac{1}{2}}\frac{\sqrt{m_{2p}-m_p^2}}{pm_p}$, and this length order is unhappily increasing with p, so it could be not so good to use $p > 2$.

6 Jump detection, test of jump occurence in model 4.1

Proposition 17. *Let the volatility definedby (41) does, satisfying Hypothesis* (42)*. Let n fixed and the j-th window $V_j = [j\Delta_n, (j+1)\Delta_n[$. We test the hypothesis*

$$\mathcal{H}_0 = \text{“there is no jump } T_k \in V_j\text{”}.$$

The test with critical region:

$$CR = \left\{ \left| \Sigma(p,\Delta_n,h_n)_{(j+1)\Delta_n} - \Sigma(p,\Delta_n,h_n)_{j\Delta_n} \right| \geq \varepsilon^j_n \right\},$$

where

$$\varepsilon^j_n = 2\Sigma(p,\Delta_n,h_n)_{j\Delta_n} \frac{q_{1-\frac{\beta}{16}}\sqrt{m_{2p}-m_p^2}}{\sqrt{r_n}m_p - q_{1-\frac{\beta}{8}}\sqrt{m_{2p}-m_p^2}},$$

q_α being the α-quantile of standard Gaussian law, is a test of level β.

Proof. We can compute under $\mathcal{H}_0$, and conditionally in $\mathcal{F}_{j\Delta_n}$ the probability:

$$\mathbb{P}_{\mathcal{H}_0}\left[CR\right] = \mathbb{P}\left[\left|\Sigma(p,\Delta_n,h_n)_{(j+1)\Delta_n} - \Sigma(p,\Delta_n,h_n)_{j\Delta_n}\right| \geq \varepsilon_n^j\right].$$

Let us remark that under $\mathcal{H}_0$, $\sigma^p_{j\Delta_n} = \sigma^p_{(j+1)\Delta_n}$. Below, we note σ^p this common value. Hence,

$$\left|\Sigma(p,\Delta_n,h_n)_{(j+1)\Delta_n} - \Sigma(p,\Delta_n,h_n)_{j\Delta_n}\right| \leq |A^n_{(j+1)\Delta_n}| + |A^n_{j\Delta_n}|.$$

where $A^n_t = \Sigma(p,\Delta_n,h_n)_t - \sigma^p,\ \forall t \in V_j$.

Lemma 18. *Let a sequence of distribution functions F_n going to a distribution function F and a real sequence x_n going to infinity when n goes to infinity. Then,*

$$\lim_{n\to\infty} |F_n(x_n) - F(x_n)| = 0.$$

Proof. let $\varepsilon > 0$ and A great enough so that $|1-F(A)| < \varepsilon/3$. Let N_1 such that $\forall n \geq N_1,\ x_n > A$.

$$|F_n(x_n) - F(x_n)| \leq |1 - F_n(A)| + |1 - F(A)| + |F_n(A) - F(A)|.$$

The first term in the majoration goes to $|1 - F(A)|$, thus there exists N_2 such that $\forall n \geq N_2,\ |1 - F_n(A)| \leq \varepsilon/3$. Finally there exists N_3 such that $\forall n \geq N_3,\ |F(A) - F_n(A)| \leq \varepsilon/3$. □

Using Theorem 5, t being fixed, given $\mathcal{F}_{t-}$, asymptotically the law of

$$Y_n := \left(\frac{\Sigma(p,\Delta_n,h_n)_t}{\sigma^p} - 1\right).\frac{\sqrt{r_n}m_p}{\sqrt{m_{2p}-m_p^2}},$$

is the standard Gaussian one (with distribution function Φ) and we denote F_n the $\mathcal{F}_{t-}$ conditional distribution function of $\frac{\Sigma(p,\Delta_n,h_n)_t}{\sigma^p} - 1$.

Thus, for every $\beta > 0$, there exists n great enough so that

$$\left|F_n\left(-\alpha\frac{\sqrt{r_n}m_p}{\sqrt{m_{2p}-m_p^2}}\right) - \Phi\left(-\alpha\frac{\sqrt{r_n}m_p}{\sqrt{m_{2p}-m_p^2}}\right)\right| \leq \frac{\beta}{8}. \tag{45}$$

We consider the two events

$$C_n^t(\alpha) = \left\{\frac{\Sigma(p,\Delta_n,h_n)_t}{\sigma^p} \geq 1-\alpha\right\},\quad D_n^t(\alpha) = \left\{\frac{\Sigma(p,\Delta_n,h_n)_t}{\sigma^p} \leq 1-\alpha\right\}.$$

For every $\varepsilon > 0$, we have :

$$\mathbb{P}\left[\left\{|A_t^n| \geq \frac{1}{2}\varepsilon\right\} \mid \mathcal{F}_{t-}\right] \leq \mathbb{P}\left[\left\{|A_t^n| \geq \frac{1}{2}\varepsilon\right\} \cap C_n^t(\alpha) \mid \mathcal{F}_{t-}\right] + \mathbb{P}\left[D_n^t(\alpha) \mid \mathcal{F}_{t-}\right]. \tag{46}$$

On the one hand, using Lemma 18 and (45), we have

$$\mathbb{P}\left[D_n^t(\alpha) \mid \mathcal{F}_{t-}\right] \leq \mathbb{P}\left[U \leq -\alpha\frac{\sqrt{r_n}m_p}{\sqrt{m_{2p}-m_p^2}}\right] + \frac{\beta}{8}$$

We choose α such that

$$\alpha \frac{\sqrt{r_n} m_p}{\sqrt{m_{2p} - m_p^2}} = q_{1-\frac{\beta}{8}}.$$

α is now fixed and verifies

$$\mathbb{P}\left[D_n^t(\alpha) \mid \mathcal{F}_{t-}\right] \leq \frac{\beta}{4}.$$

On the other hand, using the fact that the event $C_n^t(\alpha) \in \mathcal{F}_{t-}$, yields

$$\begin{aligned}
&\mathbb{P}\left[\left\{|A_{t-}^n| \geq \frac{1}{2}\varepsilon\right\} \cap C_n^t(\alpha) \mid \mathcal{F}_{t-}\right] \\
&= \mathbf{1}_{C_n^t} \mathbb{P}\left[\left\{|Y_n| \geq \frac{\varepsilon}{2\Sigma(p,\Delta_n,h_n)_t} . \frac{\sqrt{r_n} m_p}{\sqrt{m_{2p} - m_p^2}} \frac{\Sigma(p,\Delta_n,h_n)_t}{\sigma^p}\right\} \mid \mathcal{F}_{t-}\right] \\
&\leq \mathbb{P}\left[|Y_n| \geq \frac{\varepsilon\sqrt{r_n} m_p}{2\Sigma(p,\Delta_n,h_n)_t\sqrt{m_{2p} - m_p^2}}(1-\alpha) \mid \mathcal{F}_{t-}\right], \\
&\leq \mathbb{P}\left[|U| \geq \frac{\varepsilon\sqrt{r_n} m_p}{2\Sigma(p,\Delta_n,h_n)_t\sqrt{m_{2p} - m_p^2}}(1-\alpha)\right] + \frac{\beta}{8}.
\end{aligned}$$

To get this sum less than $\frac{\beta}{4}$ we finally choose ε such that

$$\frac{\varepsilon\sqrt{r_n} m_p}{2\Sigma(p,\Delta_n,h_n)_t\sqrt{m_{2p} - m_p^2}}(1-\alpha) = q_{1-\frac{\beta}{16}},$$

meaning that

$$\frac{\varepsilon}{2\Sigma(p,\Delta_n,h_n)_t}\left[\frac{\sqrt{r_n} m_p}{\sqrt{m_{2p} - m_p^2}} - q_{1-\frac{\beta}{8}}\right] = q_{1-\frac{\beta}{16}},$$

we conclude the proof applying the previous to $t = j\Delta_n$, so yields ε_n^j. □

Remark 19. *It could be difficult to exactly compute the power of this test, i.e.*

$$\mathbb{P}_{\mathcal{H}_1}[CR] = \mathbb{P}_{\mathcal{H}_1}\left[\left|\Sigma(p,\Delta_n,h_n)_{(i+1)\Delta_n} - \Sigma(p,\Delta_n,h_n)_{i\Delta_n}\right| \geq \varepsilon_n^i\right]$$

where $\mathcal{H}_1$ means that there exists at least one jump in V_i, namely τ_{j+1}. Thus, the volatility is σ_j at time $i\Delta_n$ and σ_k at time $(i+1)\Delta_n$, $k > j$. We now consider $p = 2$ for care of simplicity. A first step is to remark that

$$\mathbb{P}_{\mathcal{H}_1}[CR^c] = \mathbb{E}_{\mathcal{H}_1}[P\{\left|\Sigma(2,\Delta_n,h_n)_{(i+1)\Delta_n} - \Sigma(2,\Delta_n,h_n)_{i\Delta_n}\right| \leq \varepsilon_n^i\}/\mathcal{F}_{((i+1)\Delta_n)-}]/\mathcal{F}_{(i\Delta_n)-}]]$$

and to consider that, asymptotically, the $\mathcal{F}_{((i+1)\Delta_n)_-}$ conditional law of $\Sigma(2,\Delta_n,h_n)_{(i+1)\Delta_n}$ is the Gaussian law $(\sigma_k^2, \frac{1}{r_n}\sigma_k^4)$, and the $\mathcal{F}_{(i\Delta_n)_-}$ conditional law of $\Sigma(2,\Delta_n,h_n)_{i\Delta_n}$ is the Gaussian law $(\sigma_j^2, \frac{1}{r_n}\sigma_j^4)$.

Moreover, we add the hypothesis that σ^2 is a compound Poisson process, so $\sigma_k^2 = \sigma_j^2 + U$, σ_j^2 and U being independant random variable, π denotes the law of U. With such approximations, we can write the power of the test as following:

$$\mathbb{P}_{\mathcal{H}_1}[CR] = 1 - E_{\mathcal{H}_1}\left[\int_{\mathcal{D}} \frac{1}{2\pi}\exp(-\frac{1}{2}(u^2+v^2))dudv\pi(dy)\right]$$

where $\mathcal{D} := \{(u,v,y) \in \mathbb{R}^3$ such that $|\sigma_j^2 + \frac{1}{\sqrt{r}_n}\sigma_j^2 v - \sigma_j^2 - y - \frac{1}{\sqrt{r}_n}(\sigma_k^2+y)u| \leq \varepsilon_i^n\}$. The complement to 1 of this probability is the Gaussian measure of a band between two lines:

$$\sigma_j^2 v - (\sigma_j^2 + y)u = \sqrt{r}_n y \pm \sqrt{r_n}\varepsilon_i^n.$$

But this bandwidth is more or less constant since $\sqrt{r_n}\varepsilon_i^n = 2\Sigma(2,\Delta_n,h_n)_{i\Delta_n}\frac{q_{1-\frac{\beta}{16}}}{1-\frac{1}{\sqrt{r}_n}q_{1-\frac{\beta}{8}}}$ and goes to infinity since $\sqrt{r}_n y$ goes to infinity under the hypothesis that U law support is mainly out of 0. This remark justifies the chosen critical region, since the test power goes to 1 when n goes to infinity.

7 Simulations

In this section, we want to test numerically the volatility estimator. In order to be able to compare the estimations with the true volatility, we do not use some real datas but get our observations from *quasi-exact* simulations of toy models (by quasi-exact, we mean simulations of the process using an Euler scheme with a very small time-discretization step).

7.1 A numerical test in a continuous stochastic volatility model

In this part, we consider the stochastic volatility model proposed in [12] where the volatility is an Ornstein-Uhlenbeck process. Denote the price by (S_t) and by (σ_t) the (non-negative) stochastic volatility. Set $X_t := \log(S_t)$ and $v_t := \sigma_t^2$. The model is defined by:

$$\begin{cases} dX_t = (r - \frac{1}{2}\sigma_t^2)dt + \sigma(t)dW_t^1 \\ dv_t = a(m - v_t)dt + \beta(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2), \end{cases}$$

where r, a, β and m are some positive parameters, $\rho \in [-1,1]$ and the processes W^1 and W^2 are independent one-dimensional Brownian motions. Thus $\alpha = \frac{1}{2}$, cf. Section 4.2.
We set $X_0 = \log(50)$, $v_0 = m$ and simulate *quasi-exactly* (X_t, v_t) at times $0, 1/n, 2/n, \ldots, 1$ with the following parameters:

$$r = 0.05, \quad \rho = 0, \quad a = 1, \quad m = 0.05, \quad \text{and} \quad \beta = 0.05.$$

Using the simulated observations $X_0, X_{1/n}, \ldots, X_1$, we compute the estimator $\Sigma(p, 1/n, h_n)$ on $[h_n, 1]$ and compare its value with the true volatility. In Figures 1 and 2, we represent the corresponding graphics for $n = 1000$ and $n = 10000$ and $h_n = n^{-1/2}$. In all the figures, we choose $p = 2$ since as shown in the computation of the confidence interval length in Remark 16, to increase p is not a good choice. The process (σ_t) is plotted as continuous line whereas the estimator $\Sigma(2, 1/n, h_n)$ is plotted as discontinuous line.

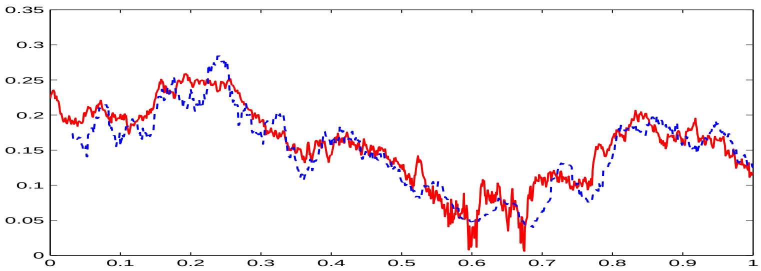

Figure 1: $n = 1000$, $h_n = n^{-1/2}$.

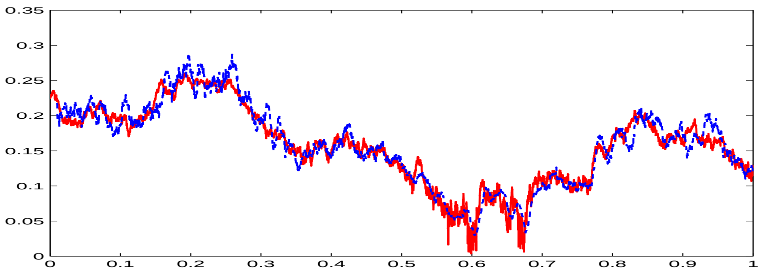

Figure 2: $n = 10000$, $h_n = n^{-1/2}$.

By Corollary (6), taking $r_n = n^\rho$ with $\rho \in (0, 1/2)$ and $p \in \{2\} \cap (5/2, +\infty)$ (or equivalently $h_n = n^{\rho-1}$), we obtain a rate of order $n^{\rho/2}$. In particular, we can derive that the best rate is obtained in the limit case $\rho = 1/2$. This theoretical result is confirmed in the following computation. Denote by $E_n(p, h_n)$ the mean relative error defined by:

$$E_n(p, h_n) := \frac{1}{n} \sum_{k=1}^{n} \frac{\left| \Sigma(p, n^{-1}, h_n)_{k/n}^{1/p} - \sigma(\frac{k}{n}) \right|}{\sigma(\frac{k}{n})}. \tag{47}$$

We obtain the following results:

	$E_n(2, n^{-0.4})$	$E_n(2, n^{-0.5})$	$E_n(2, n^{-0.6})$	$E_n(4, n^{-0.4})$	$E_n(4, n^{-0.5})$	$E_n(4, n^{-0.6})$
$n = 10^3$	18,9%	**16,6%**	18,6%	20,3%	**17,5%**	19,2%
$n = 10^4$	12,2%	**11,0%**	12,3%	13,0%	**11,9%**	12,9%

This phenomena can be explained as follows. In this problem, we recall that there are two conflicting errors that correspond to the two right-hand member terms of (3). For the first term, smaller is h_n, stronger is the error, but, for the second term, it is the exactly the opposite. In a sense, case $h_n = n^{-1/2}$ corresponds to the (best) equilibrium between the two types of errors (cf. Remark 15).

7.2 A numerical test in a jump model

In this last part, we assume that the volatility is a jump process solution to a SDE driven by a tempered stable subordinator $(Z_t^{(\lambda,\beta)})$ with Lévy measure $\pi(dy) = 1_{y>0} \exp(-\lambda y)/y^{1+\beta} dy$. This model can be viewed as a particular case of the Barndorff-Nielsen and Shephard model (see [6]):

$$\begin{cases} dX_t = (r - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW_t^1 \\ dv_t = -\mu v_t dt + dZ_t^{(\lambda,\beta)} \end{cases}$$

with the following choice of parameters:

$$r = 0.05, \quad \mu = 1, \quad \lambda = 1 \quad \text{and} \quad \beta = 1/2,$$

thus $\alpha = 1$ according to Section 4.2.
As in the preceding example, we simulate (X_t, v_t) on the interval $[0, 1]$ with $X_0 = \log(50)$ and $v_0 = 0.05$. In order to compare the two types of models, we chose some similar parameters. The main difference between these two models comes from the variations which are stronger in the first case. We obtain a quasi-exact sequel $(X_{k/n}, v_{k/n})$ with $k \in \{0, \dots, n\}$. In Figures 3 and 4, we represent the estimated and true volatilities for some different choices of $h_n = n^{-1/2}$, $n = 10^3$ and $n = 10^4$.

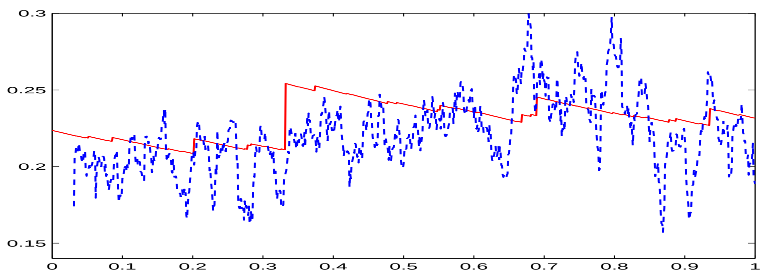

Figure 3: $n = 1000$, $h_n = n^{-1/2}$.

For these computations, we obtain the following mean relative errors:

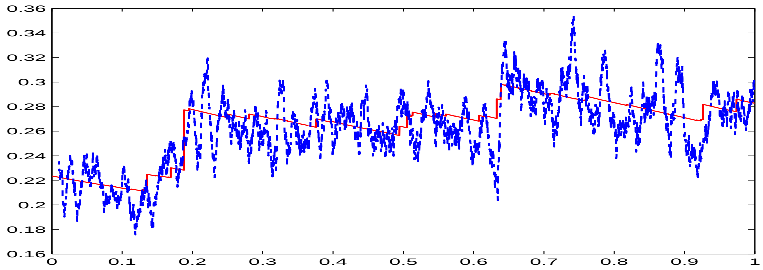

Figure 4: $n = 10000$, $h_n = n^{-1/2}$.

	$E_n(2, n^{-0.6})$	$E_n(2, n^{-0.5})$	$E_n(2, n^{-0.4})$	$E_n(4, n^{-0.6})$	$E_n(4, n^{-0.5})$	$E_n(4, n^{-0.4})$
$n = 10^3$	13,2%	8,3%	**6,3%**	15,5%	11,0%	**8,8%**
$n = 10^4$	9,1%	5,5%	**3,2%**	10,1%	6,6%	**3,9%**

Here the best result is obtained with $h_n = n^\rho$, $\rho = -0.4$, according to Remark 15. Here $\alpha = 1$ so in this case, the error is mainly $\sqrt{\frac{1}{nh_n}} \geq h_n$. This error decreases when ρ increases up to $\rho = -\frac{1}{3} - \varepsilon$, ε as small as possible.

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