Generating Entanglement from Frustration-Free Dissipation

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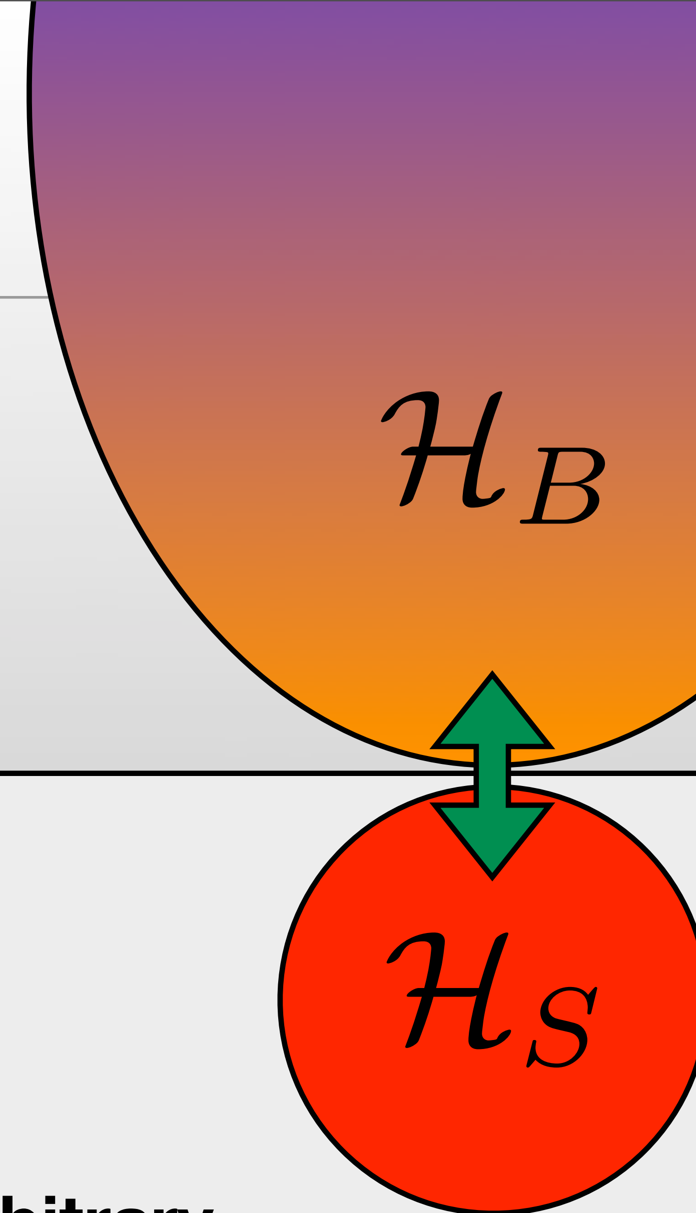
Key Reference: arXiv:1506.07756

Before we start, a couple of things on...

Attainability of Quantum Cooling, Third Law, and all that...

[T. - Viola Sci.Rep. 2014, arXiv:1403.8143]

Open System Dynamics



Bipartition:

S: system of interest (*finite dimensional*);

B: environment/bath

Unitary *joint* dynamics:

$$\rho_{SB}(t) = U(t)\rho_S(0) \otimes \rho_B(0)U(t)^\dagger$$

Assume the *joint* system is controllable/ U is **arbitrary.**

How well can we cool (or purify) the system? Are there intrinsic limits?

Note: with controllability, purification and ground-state cooling are equivalent.

Def: By ε- purification at time *t* we mean that exists U and a pure state such that:

$$\rho_S' = \mathrm{Tr}_B(\rho_{SB}(t)) \quad \text{satisfies} \quad \|\rho_S', |\psi\rangle\langle\psi|\|_1 \leq \varepsilon,\ \forall \rho_S$$

Subsystem Principle for Purification

✓ **Results in** [T-Viola, Sci.Rep. 2014]
Most general subsystem: associated to a tensor factor of a subspace,

$$\mathcal{H}_B = (\mathcal{H}_{S'} \otimes \mathcal{H}_F) \oplus \mathcal{H}_R$$

✓ **[Thm]** If the joint system is completely controllable and initially factorized:

(1) $\mathcal{E}$ **- purification can be achieved if** $\|\rho_B - \tilde{\rho}_B\| \leq \varepsilon$
for some:

$$\tilde{\rho}_B = (|\psi\rangle\langle\psi| \otimes \rho_F) \oplus 0_R$$

(2) Exact ($\varepsilon = 0$ **) purification if and only if** $\rho_B = (|\psi\rangle\langle\psi| \otimes \rho_F) \oplus 0_R$

(3) $\mathcal{E}$ **- purification is possible if**

$$\varepsilon \geq \tilde{\varepsilon}(\rho_B) \equiv \tilde{\varepsilon} = 1 - \sum_{j=1}^{d_F} \lambda_j(\rho_B) \geq 0$$

Strategy: Swap the state of the system with the subsystem one.
Claim: (1) is actually “if and only if”, i.e. either swap works or nothing does.

Example: Thermal Bath States

- ✓ In [Wu, Segal & Brumer, No-go theorem for ground state cooling given initial system-thermal bath factorization. Sci.Rep. 2012], it is claimed that a no-go theorem for cooling holds, under similar (actually weaker) hypothesis.

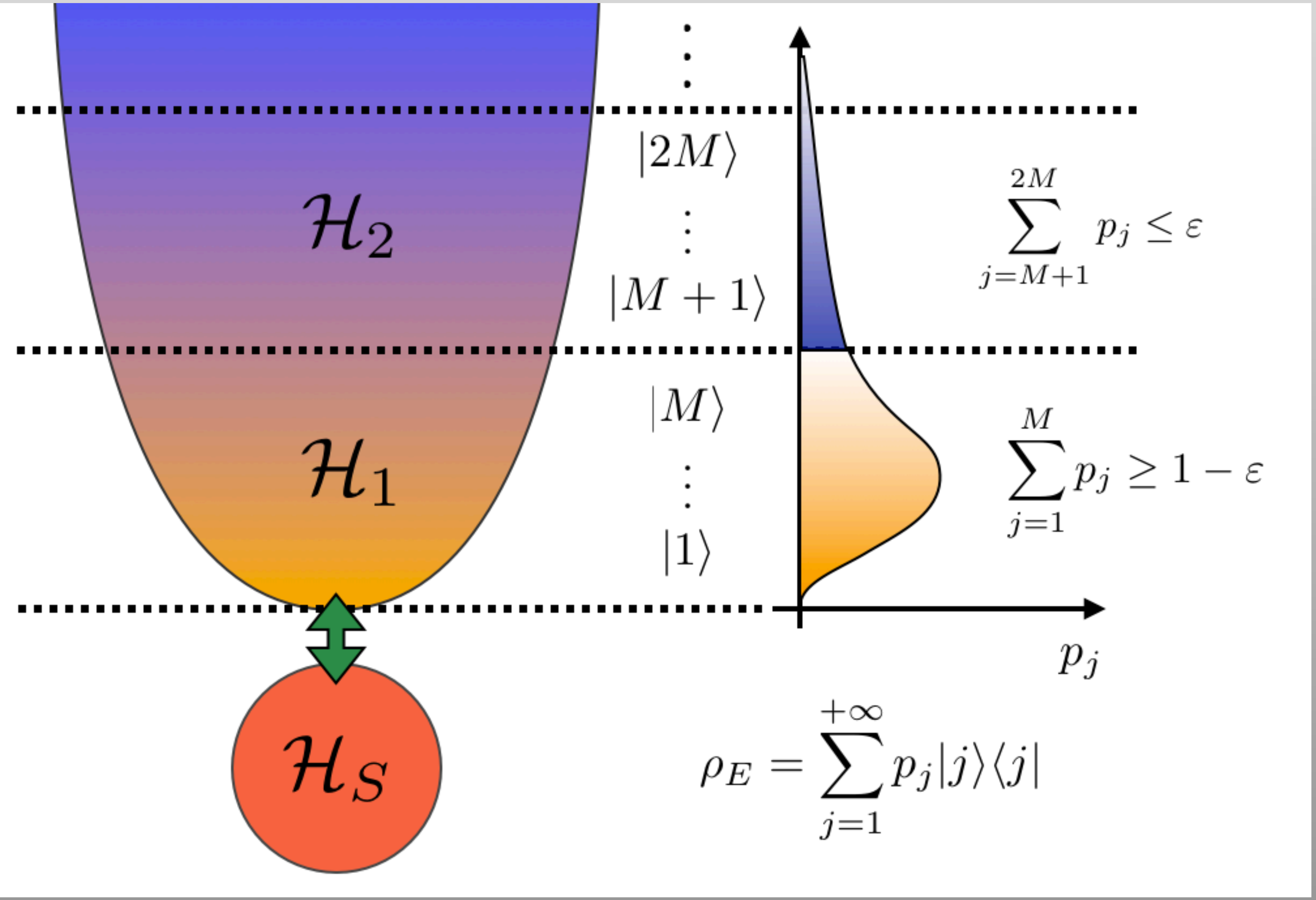
  Ok, for perfect cooling, but arbitrarily good cooling is possible!

- ✓ **E.g. Qubit target:**

  1) Choose a good subspace;

  2) Construct a 2D subsystem;

  3) Swap the state with the qubit of interest;



Comments

- **What is this useful for? Why did I speak about this?**
  First steps towards a general/systematic construction that achieve optimal purity/ground state cooling for the target system.
  Other connections to thermodynamics...

- **It is reminiscent of the third law:** attaining perfect cooling would imply using infinitely many degrees of freedom, and (likely) infinite energy.

  Usual problem: finding a formulation of the third law with clear hypothesis.

- **It is connected to Landauer's principle [David's lectures]:**

  - Exact purification is erasure.

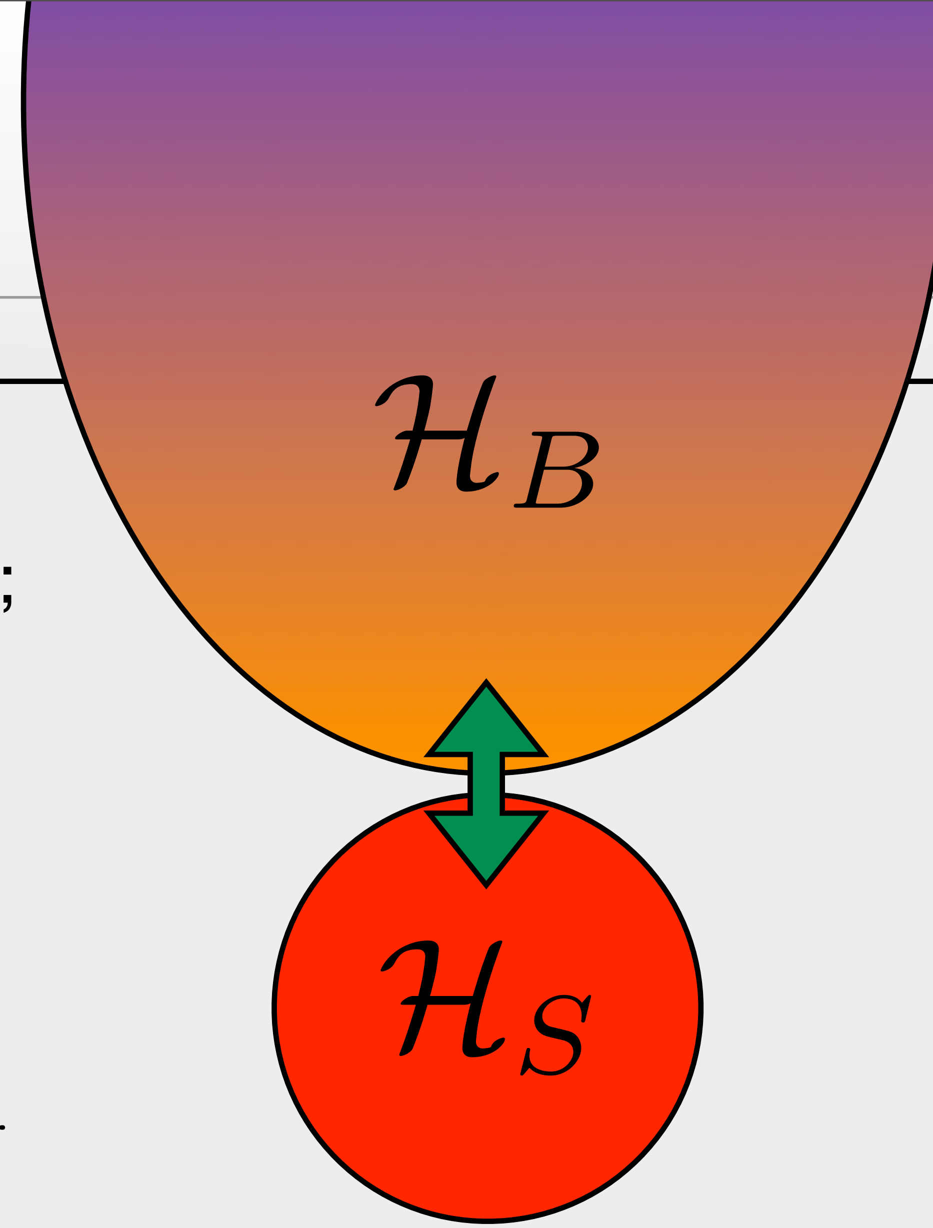
  - Swap operations seem to be the key.

INTRODUCTION (to the main talk)

Open quantum systems,
quantum dynamical semigroups
and long-time behavior.

Dissipative state preparation.

Open System Dynamics



Bipartition:

S: system of interest (*finite dimensional*);

B: uncontrollable environment

Full description via joint Hamiltonian:

$$H = H_S \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B + H_{SB}$$

Unitary *joint* dynamics:

$$\rho_{SB}(t) = U(t)\rho_S(0) \otimes \rho_B(0)U(t)^\dagger$$

Under suitable Markovian approximation (weak coupling, singular), generating an effective memoryless, time-invariant bath, we can obtain convenient *reduced* dynamics:

$$\rho_S(t) = \mathcal{E}_t(\rho_S(0)), \quad \{\mathcal{E}_t = e^{\mathcal{L}t}\}_{t\geq 0}$$

Forward composition law: Continuous Semigroup of CPTP linear maps

Quantum Dynamical Semigroups

- Assume the dynamics to be a semigroup (i.e. the environment to be memoryless). The general form of the Markovian **generator** is:

[Gorini-Kossakovski-Sudarshan/Lindblad, 1974]

$$\dot{\rho}_t = \mathcal{L}(\rho) = -i[H, \rho_t] + \sum_{k=1}^{p} L_k \rho_t L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k,\ \rho_t\}$$

Hamiltonian part

$$H = H^\dagger, \quad L_k \in \mathbb{C}^{n\times n}.$$

Dissipative, "noisy" part

H may contain environment induced terms.

- Linear CPTP system: exponential convergence, well-known theory;
- Uniqueness of the equilibrium implies it is attracting.

Question:

Where does, or can the state asymptotically converge?

Physical Answer

[Davies Generator, 1976]

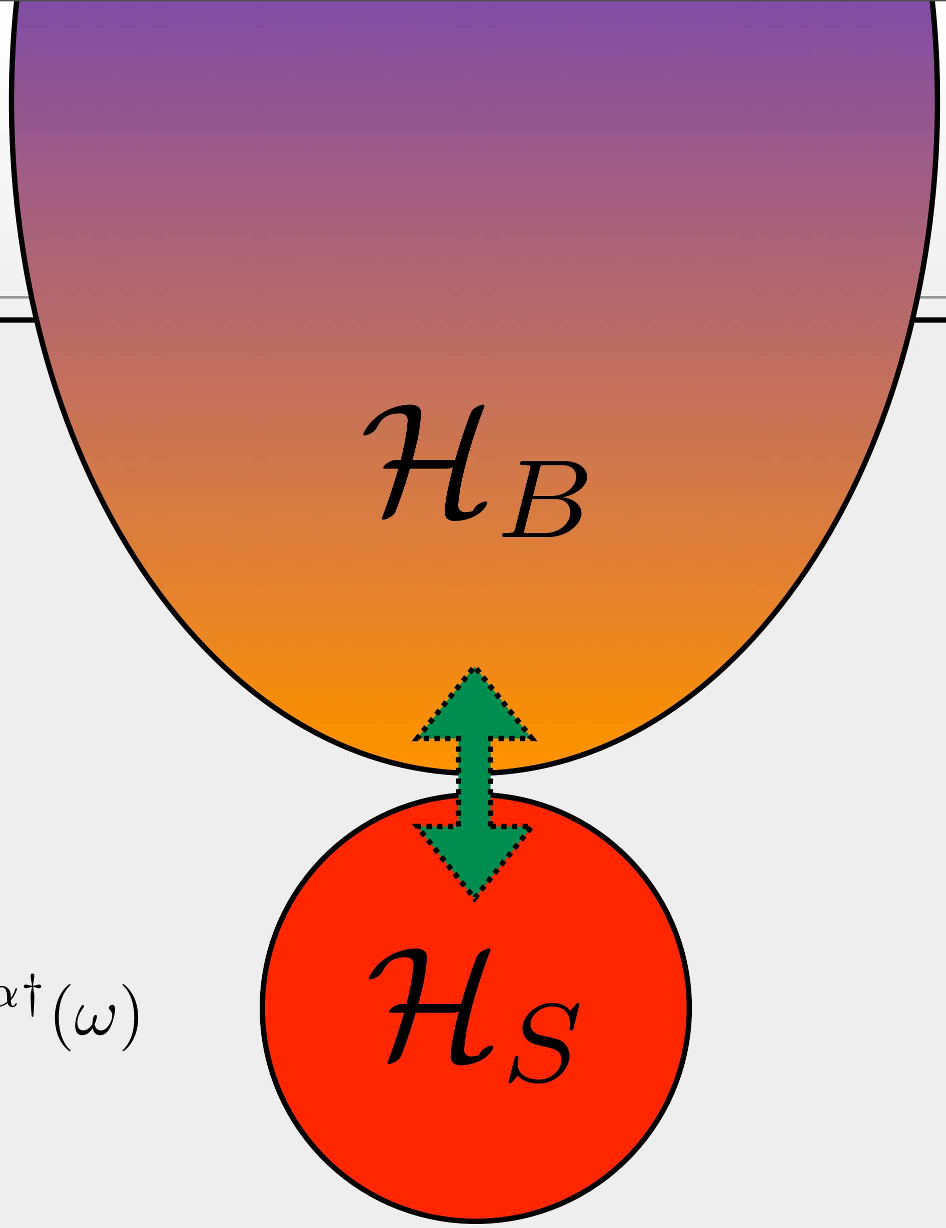
Under weak-coupling limit, consider:

$$H_{SB} = \sum_{\alpha} S^{\alpha} \otimes B^{\alpha}$$

$$e^{iH_S t} S^{\alpha} e^{-iH_S t} = \sum_{\omega} S^{\alpha}(\omega) e^{i\omega t}$$

we get:

$$\begin{aligned} \mathcal{L}(\rho) &= -i[H_S, \rho] + \sum_{\omega,\alpha} g^{\alpha}(\omega)(S^{\alpha}(\omega)\rho S^{\alpha\dagger}(\omega) \\ &\quad -\frac{1}{2}\{S^{\alpha\dagger}(\omega)S^{\alpha}(\omega), \rho\}) \end{aligned}$$

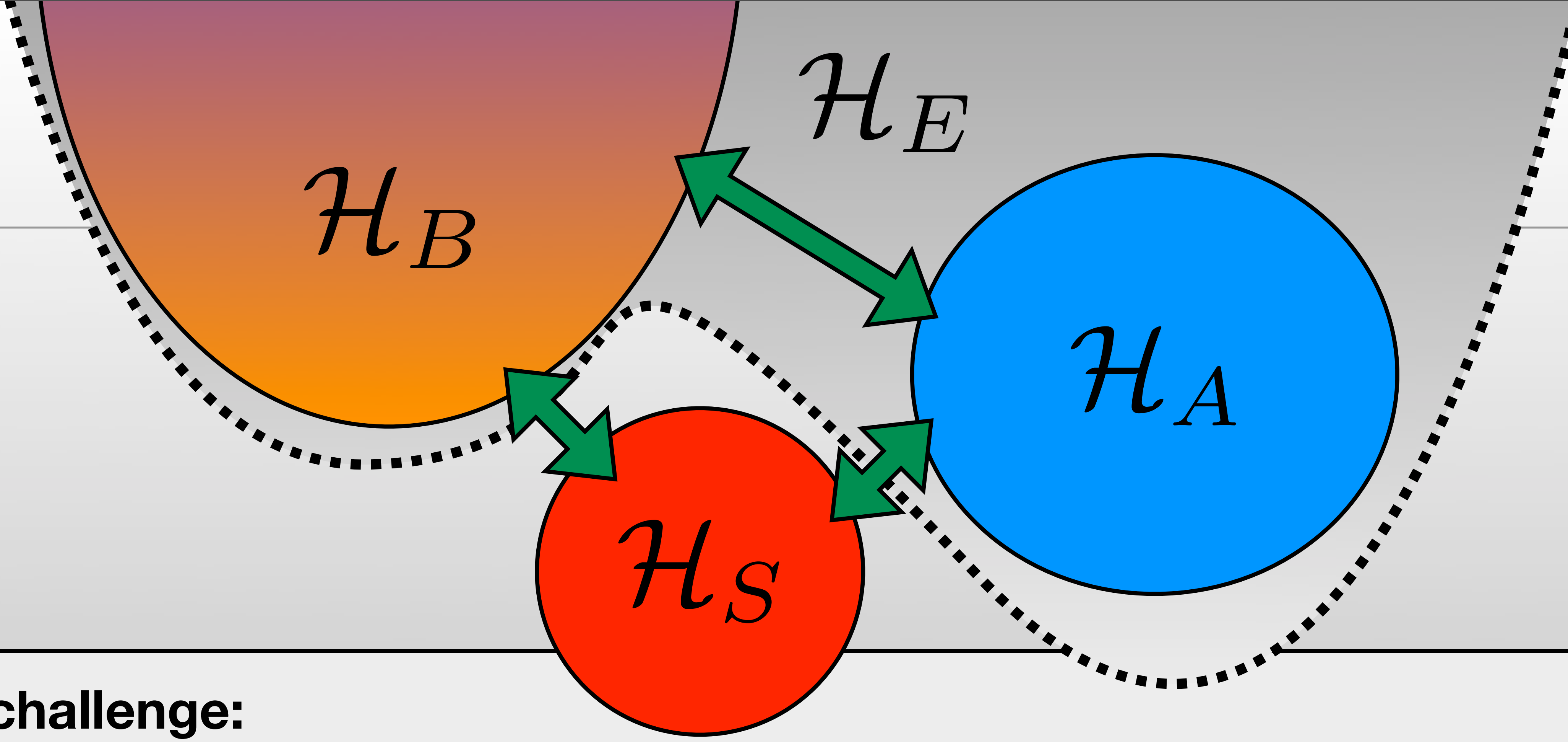


Let B be a bath at inverse temperature β. Under some additional condition (irreducibility of algebra), it is possible to show that it admits the **Gibbs state** as unique equilibrium:

$$\rho_\beta = \frac{e^{-\beta H_S}}{\mathrm{Tr}(e^{-\beta H_S})}$$

Physically consistent, expected result.

Why keep looking into it?



New challenge:

Engineering of open quantum dynamics

Key Applications:
Control &
Quantum Simulation

S: system of interest;

E: environment, including possibly:

B: uncontrollable environment

A: auxiliary, engineered system (quantum and/or classical controller)

Full description via Joint Hamiltonian:

$$H = (H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + H_{SE}) + H_c(t)$$

Reduced description via controlled generator (not just weak coupling!):

$$\mathcal{L}_t(\rho) = -i[H_S + H_C(t), \rho] + \sum_k \lambda_k(t)(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\})$$

Design of Open Quantum Dynamics

- **Two Prevailing & Complementary Approaches:**

I. **Environment as Enemy: we want to "remove" the coupling.**
Noise suppression methods, active and passive, including *hardware engineering, noiseless subsystems, quantum error correction, dynamical decoupling*;

II. **Environment as Resource: we want to "engineer" the coupling.**
Needed for state preparation, open-system simulation, and much more...

Dissipation for Information Engineering

- **Dissipation allows for:**

✓ **Entanglement Generation**

✓ **Computing**

✓ **Open System Simulator**

nature physics
LETTERS
PUBLISHED ONLINE: 20 JULY 2009 | DOI: 10.1038/NPHYS1342

Quantum computation and quantum-state engineering driven by dissipation

Frank Verstraete[1]*, Michael M. Wolf[2] and J. Ignacio Cirac[3]*

ARTICLE

An open-system quantum simulator with trapped ions

Julio T. Barreiro[1]*, Markus Müller[2,3]*, Philipp Schindler[1], Daniel Nigg[1], Thomas Monz[1], Michael Chwalla[1,2], Markus Hennrich[1], Christian F. Roos[1,2], Peter Zoller[2,3] & Rainer Blatt[1,2]

LETTER

doi:10.1038/nature12802

Autonomously stabilized entanglement between two superconducting quantum bits

S. Shankar[1], M. Hatridge[1], Z. Leghtas[1], K. M. Sliwa[1], A. Narla[1], U. Vool[1], S. M. Girvin[1], L. Frunzio[1], M. Mirrahimi[1,2] & M. H. Devoret[1]

PRL 107, 080503 (2011) PHYSICAL REVIEW LETTERS

Entanglement Generated by Dissipation and Steady State Entanglement of Two Macroscopic Objects

Hanna Krauter,[1] Christine A. Muschik,[2] Kasper Jensen,[1] Wojciech Wasilewski,[1,*] Jonas M. Petersen,[1] J. Ignacio Cirac,[2] and Eugene S. Polzik[1,†]

LETTER

Deterministic entanglement of supercondu… qubits by parity measurement and feedbac…

D. Ristè[1], M. Dukalski[1], C. A. Watson[1], G. de Lange[1], M. J. Tiggelman[1], Ya. M. Blanter[1], K. W. Lehnert[2], R. N. Schouten[1] & L. DiCarlo[1]

Scalable dissipative preparation of many-body entanglement

Florentin Reiter[1], David Reeb[2], and Anders S. Sørensen[1]
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(Dated: January 28, 2015)

Focus: Dissipative State Preparation

- **Can we design an environment that "prepares" a desired state?**
  Naive Answer: YES!
  mathematically easy: $\dot{\rho} = \mathcal{L}(\rho) = \mathcal{E}(\rho) - \rho, \quad \mathcal{E}(\rho) = \rho_{\text{target}} \text{trace}(\rho)$

- Choice is non-unique: "simple" Markov evolutions that do the job always exist:
  - Pure state: generator with single L is enough, with ladder-type operator;
    [T-Viola, IEEE T.A.C., 2008, Automatica 2009]
  - Mixed state: generator with H and a single L (tri-diagonal matrices);
    [T-Schirmer-Wang, IEEE T.A.C., 2010]

- However...
  Can we do it with experimentally-available controls? Typically NOT.
  We need to take into account:
  - The control method [open-loop, switching, feedback, coherent feedback,...]
  - Limits on speed and strength of the control actions;
  - Faulty controls;
  - **Locality constraints.**

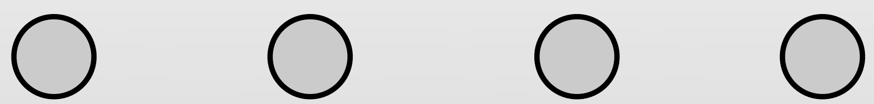
Physical relevance;
Key limitation for large-scale
entanglement generation

Main Task

Understanding the role of locality constraints and providing general design rules for dissipative state preparation

Multipartite Systems and Locality

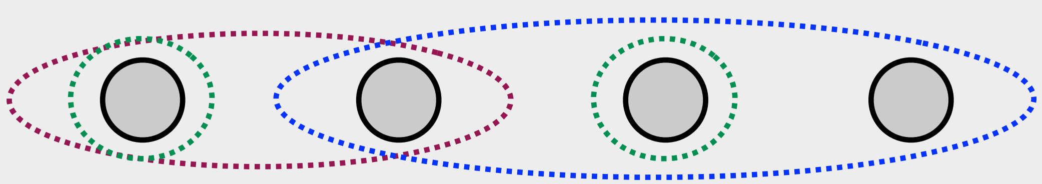
- Consider *n* finite-dimensional systems, indexed:



$a = 1 \quad 2 \quad 3 \quad \cdots$

$$\mathcal{H}_{\mathcal{Q}} = \bigotimes_{a=1}^{n} \mathcal{H}_a$$

- **Locality notion:** from the start, we specify *subsets of indexes,* or ***neighborhoods,*** corresponding to group of subsystems:



$$\mathcal{N}_1 = \{1,2\}$$
$$\mathcal{N}_2 = \{1,3\}$$
$$\mathcal{N}_3 = \{2,3,4\}$$

*...*on which "**we can act simultaneously**": how?

- **Neighborhood operator:** $M_k = M_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$
- A Hamiltonian is said **Quasi-Local (QL)** if:

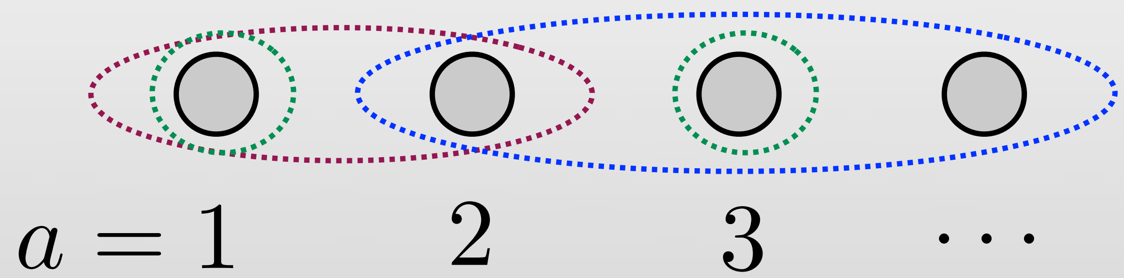
$$H = \sum_k H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$$

This framework encompasses different notions: graph-induced locality, N-body locality, etc...

Neighborhood operators will model the allowed interactions.

Constraints: Frustration-Freeness & Locality

- Consider *n* finite-dimensional systems, and a *fixed* locality notion.



$$\mathcal{N}_1 = \{1,2\} \; \mathcal{N}_2 = \{1,3\}$$
$$\mathcal{N}_3 = \{2,3,4\} \quad \cdots$$

- A dynamical generator $\mathcal{L}(\rho)$ is:
- **Quasi-Local (QL)** if

Sum of neighborhood components!

$$\mathcal{L} = \sum_k \mathcal{L}_{\mathcal{N}_k} \otimes \mathcal{I}_{\bar{\mathcal{N}}_k}$$

or, explicitly:

$$H = \sum_k H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k} \quad L_{k,j} = L_{\mathcal{N}_k(j)} \otimes I_{\bar{\mathcal{N}}_k}$$

- **Frustration-Free (FF)** [Kastoryano,Brandao, 2014; Johnson-T-Viola, 2015] if it is QL and

$$\mathcal{L}(\rho) = 0 \quad \Longrightarrow \quad \mathcal{L}_{\mathcal{N}_k} \otimes \mathcal{I}_{\bar{\mathcal{N}}_k}(\rho) = 0$$

- A state is a global equilibrium *if and only if* it is so for the local generators.

Frustration-Freeness as “Robustness”

- **Inspired by:** Let $\rho = |\psi\rangle\langle\psi|$ be a ground state of a QL Hamiltonian:

$$H = \sum_k H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$$

Def: If all $|\psi\rangle$ are eigenvectors of minimal energy for *both the global and neighborhood Hamiltonians,* namely:

$$\langle\psi|H|\psi\rangle = \min \sigma(H) \quad \Longrightarrow \quad \langle\psi|H_k|\psi\rangle = \min \sigma(H_k), \quad \forall k.$$

such an ***H*** **is said Frustration-Free (FF).**

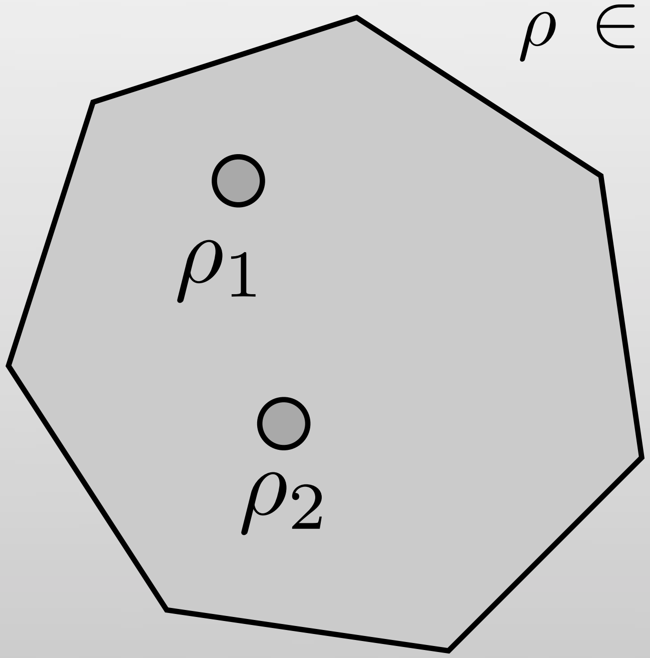
- If the global ground state is unique, we can obtain it by simultaneously “cooling” the system on each neighborhood, **and it does not change if we scale the neighborhood terms:** $$H = \sum_k \alpha_k H_k, \ \alpha_1, \ldots, \alpha_k \in \mathbb{R},$$

No fine tuning!

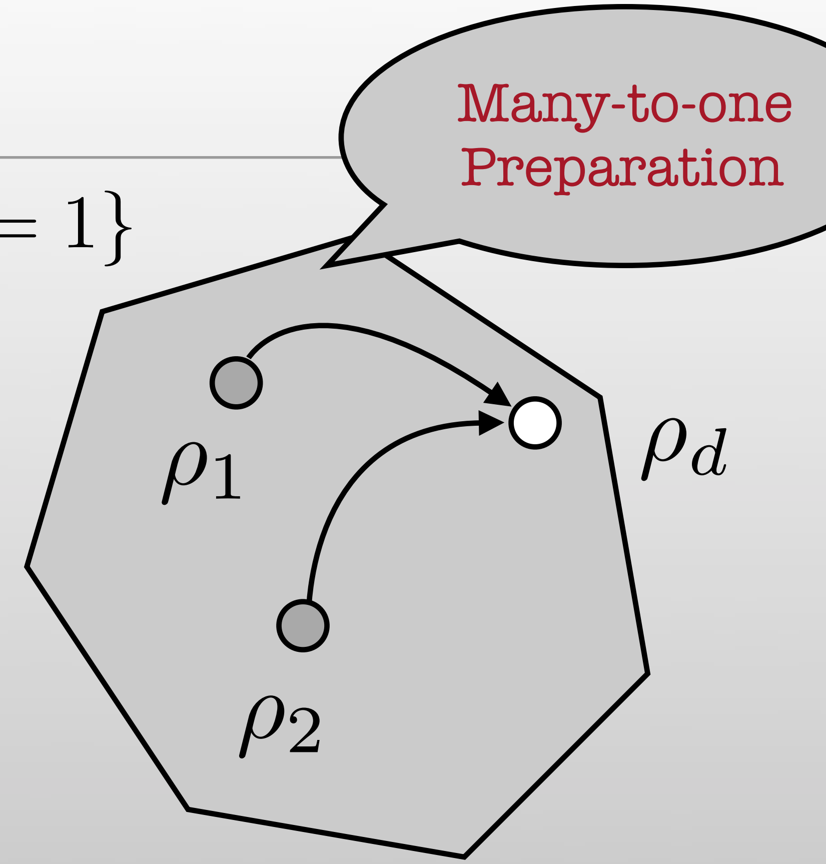
- Same robustness holds for a FF generator and its equilibria.
Key Property:
Summing neighborhood terms in FF generators does not add equilibria.

Asymptotic State Stabilization

$$\rho \in \mathfrak{D}(\mathcal{H}) := \{\rho = \rho^\dagger > 0,\ \mathrm{trace}(\rho) = 1\}$$



Task: Prepare a target state *irrespective* of the initial one.



When is it possible with FF dynamics?

Define: ρ_d is **Frustration-Free Stabilizable [FFS]** if it is

1) **Invariant:** $\mathcal{L}(\rho_d) = 0$

2) **Attracting:** $\forall \rho \in \mathfrak{D}(\mathcal{H}), \quad \lim_{t \to +\infty} e^{\mathcal{L}t}(\rho) = \rho_d$

for **some** quasi-local FF dynamics.

Relevance: Basic task of QIP; Cooling to ground state; *Entanglement generation and preservation;* One-way computing; Metropolis-type sampling

General Fact in Dissipative Design:

Making a state invariant is the hard part;
After that, making everything else converge to it is (relatively) easy.

Invariance-ensuring generators are a zero-measure set.
In there, stabilizing ones are generic.
[T. et al, IEEE TAC 2012]
[T.,Viola, QIC 2014]

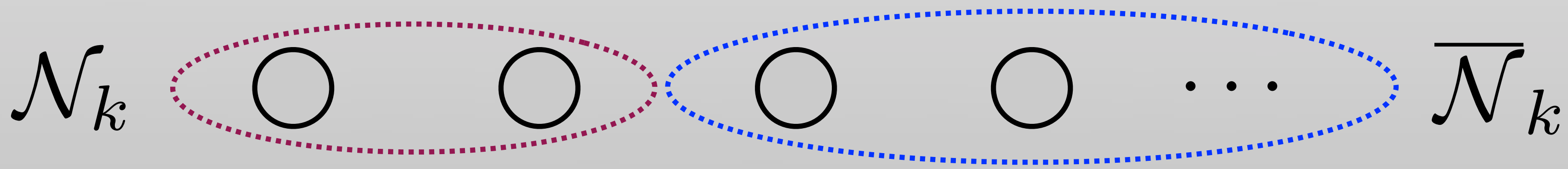
Characterizing Invariance: Schmidt Span

- **When is a state invariant for a FF generator?**

FF hypothesis: we have an equilibrium *if and only if*

$$\mathcal{L}_{\mathcal{N}_k} \otimes \mathcal{I}_{\bar{\mathcal{N}}_k}(\rho_d) = 0, \ \forall k$$

Consider one neighborhood and its complement:



- Write the **operator Schmidt decomposition** with respect to the partition $\mathcal{H}_{\mathcal{N}_k} \otimes \mathcal{H}_{\overline{\mathcal{N}}_k}$:

$$\rho_d = \sum_j A_j \otimes B_j$$

- Define the *Schmidt Span:* $\Sigma_k(\rho_d) = \text{span}\{A_j\}$

Operator subspace!

- **Lemma:** ρ_d is invariant if and only if $\Sigma_k(\rho_d) \subset \ker(\mathcal{L}_{\mathcal{N}_k}), \quad \forall k$

- This implies invariance of the reduced state: $\rho_{\mathcal{N}_k} = \text{trace}_{\bar{\mathcal{N}}_k}(\rho_d)$

Stabilizing Dynamics?

Invariance is characterized!
Now we have a good idea of what the stabilizing QL generators have to do!

I. **Locally preserve the *Schmidt spans*;**

II. **Perturb and destabilize everything else;**

However....

Towards Stabilization: Distorted Algebras

- $\mathcal{L}_{\mathcal{N}_k}$ is the generator of a CPTP semigroup. The structure of the *fixed points* is well known [Ng,Blume-Kohut,Viola; Wolf], they form a distorted *algebra*:

$$\ker(\mathcal{L}_{\mathcal{N}_k}) = \left(\bigoplus_{\ell} \mathfrak{B}(\mathcal{H}_\ell^A) \otimes \tau_\ell\right) \oplus \mathbb{O}$$

- **Why is this important?** We (may) need to enlarge the *set of invariant operators* with respect to (just) the Schmidt span **(~no pancake theorem).**

- Let ρ be a maximum rank fixed state for $\mathcal{L}_{\mathcal{N}_k}$. Given the Schmidt span, we can construct the ***minimal*** **distorted algebra** $\mathcal{A}_k$ so that $\Sigma_k(\rho_d) \subseteq \mathcal{A}_k$, by making it **closed** with respect to:
  (i) Linear combinations and adjoint;
  (ii) Modified product:

$$X \times_\rho Y = X\,\rho^{-1} Y$$

  with: $\rho = \rho_{\mathcal{N}_k}$.

Lemma: ρ_d is invariant if and only if $\mathcal{A}_k \subseteq \ker(\mathcal{L}_{\mathcal{N}_k}), \quad \forall k$

- As we hoped for, for generic states, the condition turns out to be not only necessary, but also sufficient....

Main Result: Full-rank States

- For each neighborhood, we can construct the *enlarged distorted algebra:* $$\mathcal{A}_k^g = \mathcal{A}_k \otimes \mathfrak{B}(\overline{\mathcal{N}}_k)$$

Provides a test with only two inputs: <u>the state and the neighborhoods</u>

Theorem: *Assume* ρ_d is full rank. Then it is **FFS** if and only if

$$\bigcap_k \mathcal{A}_k^g = \mathrm{span}(\rho_d)$$

- *Proof idea:* **Necessity** follows from Lemmas. Proving **sufficiency,** we consider an *explicit choice of generators:* $$\mathcal{L}_{\mathcal{N}_k}(\rho) = \mathcal{E}_{\mathcal{N}_k}(\rho) - \rho;$$

  with **CPTP non-orthogonal projections onto the minimal distorted algebras** (dual of conditional expectations):

$$\mathcal{E}_{\mathcal{N}_k}(\rho) \in \mathcal{A}_k; \quad \mathcal{E}^2_{\mathcal{N}_k}(\rho) = \mathcal{E}_{\mathcal{N}_k}(\rho).$$

Key technical point: proving the dynamics is *frustration free.*
Then the shared equilibrium is unique, and there cannot be any other one.

Key Result

- Assume that for all *k*, $\mathcal{L}_k = \mathcal{L}_{\mathcal{N}_k(j)} \otimes \mathcal{I}_{\bar{\mathcal{N}}_k}$:

$$\mathrm{alg}(\mathcal{L}_k) \subseteq \mathrm{alg}(\mathcal{L})$$

- **Note:** This is true if there is no Hamiltonian;
- Then we have the following chain of equality/inclusions (with full rank states):

$$\begin{array}{ccccc}\ker(\mathcal{L}) & = \rho^{\frac{1}{2}}\ker(\mathcal{L}^\dagger)\rho^{\frac{1}{2}} & = & \rho^{\frac{1}{2}}\mathrm{alg}\{\mathcal{L}\}'\rho^{\frac{1}{2}} \\ & & & \mathbin{\rotatebox[origin=c]{90}{$\subseteq$}} \\ \ker(\mathcal{L}_k) & = \rho^{\frac{1}{2}}\ker(\mathcal{L}_k^\dagger)\rho^{\frac{1}{2}} & = & \rho^{\frac{1}{2}}\mathrm{alg}\{\mathcal{L}_k\}'\rho^{\frac{1}{2}},\end{array}$$

- This proves that the chosen generator is FF (does not have Hamiltonian).

Main Result: Comments and Extensions

- **What is this useful for?**
  Allows for checking if a target state is **in principle** stabilizable under **given** (and strict) locality constraints, with frustration-free dynamics.
  The checking procedure can be automated.

- **If full quasi-local control/simulation is available,** we give a recipe for stabilization of desired state, where possible.
  More constraints can be included later, e.g. via suitable numerical methods. Our result gives a preliminary check.

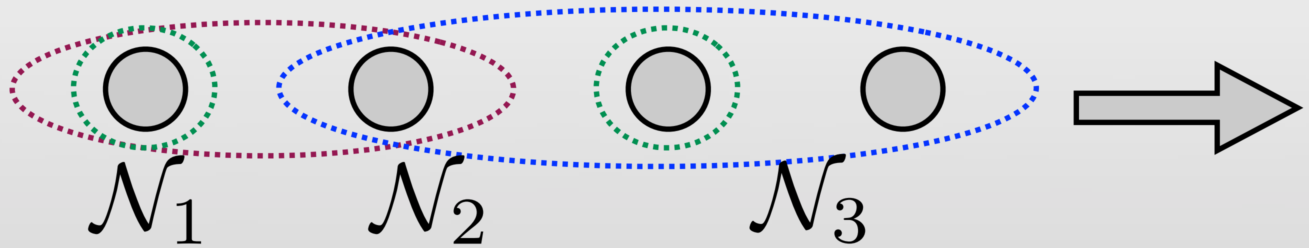
- **It can be seen as a way to construct quantum "sampler"** [Kastoryano,Brandao, 2014] **- a way to obtain a density we do not have.**
  Complements to other work by Temme, Cubitt, Wolf, and co-workers where focus is on studying the scalability/speed, when convergence is already guaranteed.

- **For general states,** the same necessary condition holds. However, we do not have a full proof for sufficiency.
  An additional condition is used, but we conjecture is not needed.

- Full and simpler characterization **for pure states**.

Specialization for Pure States

- For each *neighborhood* compute the reduced states;



$\rho_{\mathcal{N}_1},\ \rho_{\mathcal{N}_2},\ \rho_{\mathcal{N}_3}$

- **Being** ρ_d **pure, it can be shown that:** $\mathcal{A}_k = \Sigma_k(\rho) = \mathfrak{B}(\mathrm{supp}(\rho_{\mathcal{N}_k}))$
- Instead of intersecting distorted algebras, I can just look at heir supports.

- For each neighborhood calculate the *support* of the reduced state times the identity on the rest: $\mathcal{H}_{\mathcal{N}_k} = \mathrm{supp}(\rho_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k})$

- ***Theorem*** [T.-Viola, 2012]***:***

$$\mathcal{H}_0 := \bigcap_k \mathcal{H}_{\mathcal{N}_k} = \mathrm{supp}(\rho)$$

if and only if ρ **is FFS;**

IDEA: the support is “where the probability is”;
Locally I only see the reduced state, and I try to prepare it.

FFS, Or Not? Physical Interpretation

- **Equivalent characterization:** $\rho = |\psi\rangle\langle\psi|$ is FFS **if and only if it is the unique ground state of a Frustration-Free QL Hamiltonian**, that is:
  - There exists a QL Hamiltonian for which $|\psi\rangle$ is the unique ground state and

$$H = \sum_k H_k, \quad H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$$

such that $\langle\psi|H_k|\psi\rangle = \min \sigma(H_k), \quad \forall k.$

Proof: It suffices to choose $H_k = \Pi^{\perp}_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}$, $\Pi^{\perp}_{\mathcal{N}_k}$ projects on $\mathrm{supp}(\rho_{\mathcal{N}_k})^{\perp}$.

- **We retrieve the FF Hamiltonian - the analogy with FF generators fully works!**
- Interesting connection to physically-relevant cases, and previous work by Verstraete, Perez-Garcia, Cirac, Wolf, B. Kraus, Zoller and co-workers.
- **Differences:**
  In their setting, the proper locality notion is induced by the target state itself.
  In our setting, *the locality is fixed a priori. We also prove necessity of the condition.*

Applications

Generating entanglement from quasi-local dissipation.

Is Frustration-Free Enough for Pure States?

- **Which states are FFS? Using our test, it turns out that...**
  - All product states are FFS.
  - ***GHZ states (maximally entangled) and W states are not FFS***
    Unless we have neighborhoods that cover the whole network/nonlocal interactions;
    $$\rho_{\rm GHZ} = |\Psi\rangle\langle\Psi|, \quad |\Psi\rangle \equiv |\Psi_{\rm GHZ}\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}.$$
  - ***Any graph state is FFS*** **with respect to the locality induced by the graph;**
    To each node is assigned a neighborhood, which contain all the nodes connected by edges.
    $$U_G|00\ldots0\rangle = |\varphi_{\rm graph,0}\rangle$$
  - ***Generic (injective) MPS/PEPS are FFS*** for **some** locality definition...
    Neighborhood size may be big! [see work by Peres-Garcia, Wolf, Cirac and co-workers]
  - Some **Dicke states** *that are not graph* can be stabilized!
    E.g. on linear graph with NN interaction:
    $$\frac{1}{\sqrt{6}}(|1100\rangle + |1010\rangle + |0110\rangle + |0101\rangle + |0011\rangle + |1001\rangle)$$

Is Frustration-Free Enough for Mixed States?

- **Which states are FFS? Using our test, it turns out that...**

- There are **non-entangled states that are not FFS!**

$$\rho_{\text{sep}} = \frac{1}{2}(00^{\otimes n} + 11^{\otimes n}).$$

- **Product graph states are FFS,** with locality induced by the graph.

U_G : prepares the graph basis.

$$\rho_G = U_G\Big(\bigotimes_{j=1}^{n} \rho_j\Big){U_G}^\dagger,$$

- **Commuting Gibbs states are FFS,** with locality generated by the Hamiltonian (NNN).

$$\rho_\beta = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$$

with:

$$H = \sum_k H_k,\ H_k = H_{\mathcal{N}_k} \otimes I_{\bar{\mathcal{N}}_k}, \boxed{[H_k, H_j] = 0,\ \forall j, k}$$

- **Some non-commuting Gibbs states** are FFS!

e.g. zero-temperature states as certain Dicke states,

and their mixtures with e.g. GHZ states!

Summary and Outlook

- **Locality constraints are key for state preparation.**
- **We obtain a way to check if a target state is "compatible" with *given constraints***
- **If it is, we provide intuition on what the stabilizing dynamics should do, as well as *one that works*.**
- **We show that there are new *(non commuting)* states that are genuinely FFS.**
- **It is possible to relax invariance constraints for preparation of GHZ and W.** Two steps: first initialization and then **conditional stabilization.**

➡**Next:**
Relation to Encoders and Memories; Numerical approaches;
When is FFS generic? More general constraints.

➡**Open problems:** The above mentioned conjecture and...
Better *classification* of FFS states; Scalable non-commuting Gibbs;
Stabilization beyond Frustration-Free; Discrete-time models;
Speed of convergence (when the system size grows - scalability).

A case study: GHZ States

- **GHZ states are never QLS for non trivial topology:**

$$\rho_{\mathrm{GHZ}} = |\Psi\rangle\langle\Psi|, \quad |\Psi\rangle \equiv |\Psi_{\mathrm{GHZ}}\rangle = (|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}.$$

By symmetry, $\mathcal{H}_0$ **must contain** $|000\ldots0\rangle, |111\ldots1\rangle$ **.**

Hence the following orthogonal states **must remain** stable for the QL dynamics.

$$\begin{aligned} |\Psi_{\mathrm{GHZ}^+}\rangle &= (|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}; \\ |\Psi_{\mathrm{GHZ}^-}\rangle &= (|000\ldots0\rangle - |111\ldots1\rangle)/\sqrt{2}; \end{aligned}$$

We need to "select" the right one How?

$$\sigma_x^{\otimes n}|\Psi_{\mathrm{GHZ}^+}\rangle = |\Psi_{\mathrm{GHZ}^+}\rangle \quad \sigma_x^{\otimes n}|\Psi_{\mathrm{GHZ}^-}\rangle = -|\Psi_{\mathrm{GHZ}^-}\rangle$$

- **Trick:** First prepare the system in the +1-eigenspace of $\sigma_x^{\otimes n}$ (e.g. $|+\rangle^{\otimes n}$). Then we show there exists a QL $\{\mathcal{E}_t\}_{t\geq 0}$ that prepares $\mathcal{H}_0$ leaving the **eigenspace invariant.**

- By our Theorem, ρ_{GHZ} **is Conditionally QLS!** (scalable on the linear graph)

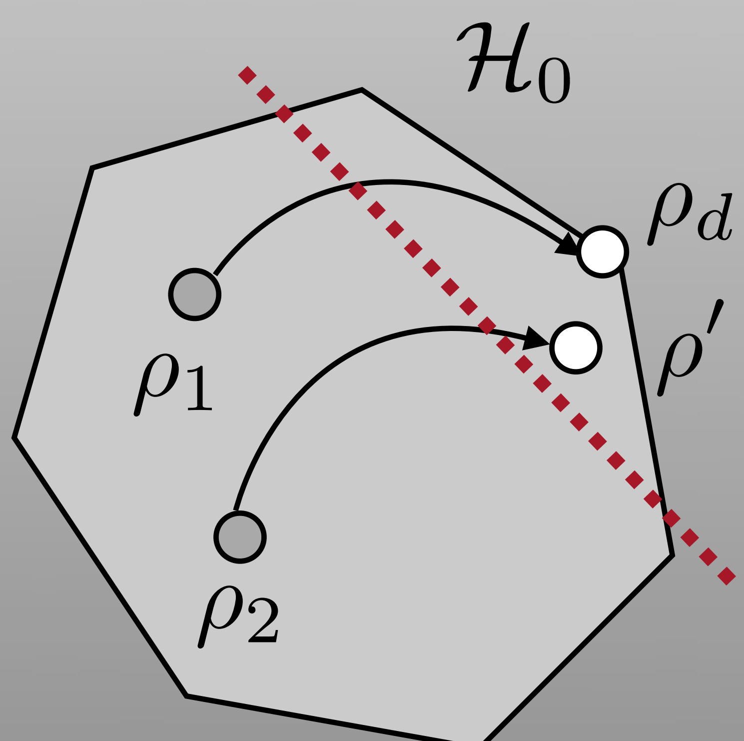
Conditional Preparation: Some Intuition

FFS Problem: unfeasible global stabilization task because I can only prepare (nec. cond.):

$$\mathcal{H}_0 := \bigcap_k \mathcal{H}_{\mathcal{N}_k}$$

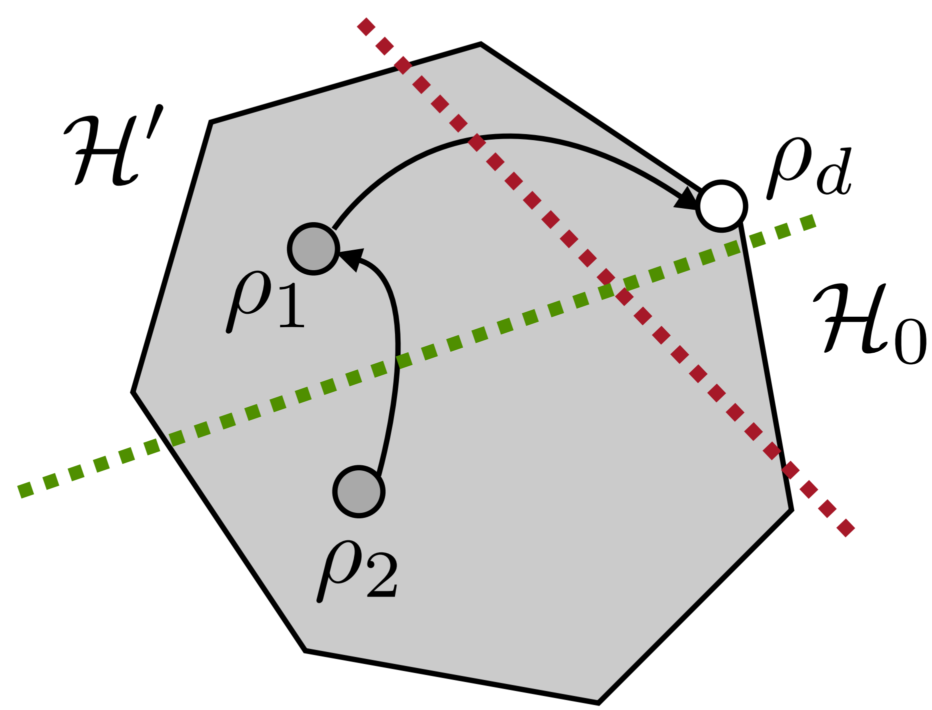
The necessity follows from:

$$\forall\, t \geq 0 \quad \mathcal{E}_t(\rho) = \rho$$



If we relax this assumptions, we can obtain scalable protocols!

$$\forall\, t \geq T > 0 \quad \mathcal{E}_t(\rho) = \rho$$



First I prepare a subspace that (1) is invariant for the QL sequence; (2) is attracted directly to ρ_d

Problem: finding such $\mathcal{H}'$!

Conditional Preparation: Definition & Result

- **Definition:** A state $\rho = |\psi\rangle\langle\psi|$ is **Quasi-Local Stabilizable (QLS) conditional to** $\mathcal{H}'$ if there exist a **dynamical semigroup** $\{\mathcal{E}_t\}_{t\geq 0}$ such that

$$\forall\, t \geq 0 \quad \mathcal{E}_t(\rho) = \rho \qquad \lim_{t\to\infty} \|\rho_t - \rho\| = 0$$

for every ρ_0 with support on $\mathcal{H}'$.

- **Lemma:** It is not restrictive to take $\mathcal{H}'$ invariant.

With some additional hypothesis, the search for the subspace can be automated.

- **Theorem:** If $\mathcal{H}'$

(1) contains $|\Psi\rangle$;

(2) is orthogonal to $\mathcal{H}_0 \ominus \{|\Psi\rangle\}$;

(3) is invariant for $\{\mathcal{E}_t\}_{t\geq 0}$ that stabilizes $\mathcal{H}_0$;

Then $\rho = |\psi\rangle\langle\psi|$ is QLS conditional to $\mathcal{H}'$.