A one-dimensional model of plasma expansion

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Abstract —We consider a system consisting of a vacuum gap separated by two electrodes. A plasma is injected at the cathode and expands. From the sharp plasma boundary, electrons are accelerated towards the anode, forming an electron beam. In this paper, from a two-fluid isentropic Euler system for each species of particles coupled with the Poisson equation, we perform a formal asymptotic analysis, yielding a quasineutral model for the plasma region and a Child-Langmuir law for the beam region. These two models are connected through a transmission layer problem. Finally, a numerical validation of this asymptotic model against the original two-fluid model is given.

Keywords —Plasma-vacuum interface, quasineutral limit, Child-Langmuir asymptotics, transmission layer problem

1. INTRODUCTION

In this paper, we consider a system consisting of a vacuum gap separated by two electrodes. A plasma is injected at the cathode and expands. From the sharp plasma boundary, electrons are accelerated towards the anode, forming an electron beam. Such a device has been proposed and analyzed in relation with the technology of electron beam accelerators [1].

The goal of the present work is to propose a suitable model for this system. Our starting point is a two-fluid isentropic Euler system for each species (electrons and ions) coupled with the Poisson equation. However, in the plasma region, due to the large particle density, the space scale associated with the Poisson equation (the so-called Debye length [2]) is very small and makes numerical simulations extremely delicate. Therefore, for numerical purposes, it is highly desirable to derive a quasineutral model consisting of a single Euler equation for an averaged fluid. A first point of our work is to show that, in the present case, the 'correct' quasineutral model is not the standard one-fluid Euler equations (see e.g. [3]) but an extension of it accounting for a non-zero plasma current. It differs from the standard one by an additional flux term in the momentum conservation equation accounting for the counter-pressure induced by electron acceleration. This model is shown to be hyperbolic only if the plasma density is larger than a threshold value (depending on the value of the current).

However, quasineutrality breaks down at the plasma-vacuum interface. Beyond the interface, in the beam region, only electrons are present and require the use of a coupled single electronic fluid Euler system with Poisson's equation. Scaling considerations however allow again to reduce the system to a simpler one, namely the Child-Langmuir model (see [4], [5]).

The key modeling problem now consists in connecting the quasineutral model in the plasma region with the Child-Langmuir model in the beam region. To some extent, the problem reduces to the evaluation of the pressure exerted by the beam onto the plasma interface. This problem is solved by analysing a transmission layer problem, deduced from the original two-fluid model by stretching the position variable about the plasma-vacuum interface. We show that, for the interface problem to have a well-behaved solution, the plasma density on the left of the interface must lie in a certain interval $[n_H, n_P]$. Unfortunately, the layer analysis does not provide a definite value and we need to rely on numerical simulations to specify this value. We experimentally see that the value n_p should be imposed to the plasma density on the left of the interface.

The present work is a shorter version of [6]. Quasineutral limits are investigated in [3] for the Euler equations and in [7] for the drift-diffusion model. The result of [3] is concerned with smooth solution with a lower bound on the species density. We also refer to [8] for an earlier version of the present work. In [8], it was postulated that the plasma current was zero (thus reducing the quasineutral model in the plasma to the standard one) and that the electron counter-pressure was concentrated at the interface. However, numerical simulations showed a significant discrepancy with the two-fluid Euler-Poisson model for the plasma density at the interface.

In the present model, we show that a non-zero plasma current gives a better approximation of the original model. However, we are unable so far to provide a mathematically rigorous justification of this fact.

2. THE ASYMPTOTIC MODEL

Our starting point is the same as in [8]: we consider the two-fluid Euler-Poisson model in scaled form (see [8] for details about the scaling):

$$(n_i)_t + (n_i u_i)_x = 0, \qquad (n_i u_i)_t + \left((n_i u_i^2) + p_i(n_i)\right)_x = -n_i \phi_x/\eta, \tag{1}$$

$$(n_e)_t + (n_e u_e)_x = 0, \qquad \varepsilon\left((n_e u_e)_t + (n_e u_e^2)_x\right) + (p_e(n_e))_x = n_e \phi_x/\eta, \tag{2}$$

$$-\lambda \phi_{xx} = (n_i - n_e), \tag{3}$$

with the following boundary conditions $n_i(x=0) = n_e(0) = n_0$, $u_i(0) = u_e(0) = u_0$, $\phi(0) = 0$ and $\phi(1) = 1$. Here, the indices i (resp. e) stand for ions (resp. electrons). n, u and p are respectively the fluid density, mean velocity and pressure for each species, while ϕ is the electrostatic potential. We postulate pressure laws of the form $p(n) = Cn^\gamma$ where $C > 0$ and $\gamma > 1$ can be different for electrons and ions. The cathode is supposed located at $x = 0$ and the anode at $x = 1$. The dimensionless parameters ε, η and λ respectively measure the electron to ion mass ratio, the ratio of the thermal energy of the plasma to the applied bias and the ratio of the applied potential to the typical self-consistent potential. It should be noted that the product $\eta\lambda$ is the square of the dimensionless Debye length. At time $t = 0$, it is assumed that the domain is devoid of particles. The boundary conditions express the injection of a quasineutral plasma of scaled density n_0 and scaled velocity u_0, both of order unity in the present scaling.

We are now interested in the limit $\eta \to 0$ with $\lambda = O(1)$, corresponding to values observed in high current diodes [1]. The forthcoming analysis is independent of whether ε is assumed to be small or not, and we shall consider it as an $O(1)$ quantity for simplicity.

The formal limit η goes to 0 gives rise to $n_e, n_i \to n$, $u_i \to u$, $u_e \to u - j/n$, $\phi \to 0$, where

$$\begin{cases} (1+\varepsilon)\,\left((nu)_t + (nu^2)_x\right) + (p_i(n) + p_e(n))_x + \varepsilon\left(-2uj + \dfrac{j^2}{n}\right)_x = \varepsilon j_t, \\ n_t + (nu)_x = 0\,, \quad j_x = 0. \end{cases} \tag{4}$$

with boundary conditions $n(x=0) = n_0$, $u(x=0) = u_0$.

Remark 1 *i) If $j \neq 0$, the system (4) is hyperbolic if and only if $(1+\varepsilon)\,n^2\,p'(n) > \varepsilon j$ where $p(n) = p_i(n) + p_e(n)$, or equivalently, iff $n > n_H(j)$, thereby defining $n_H(j)$.*
ii) It is assumed that in the limit, we lose the boundary conditions for the electrons due to some boundary layer effects. Therefore, the value of $j(t)$ is left undefined at this stage and will be determined from the beam region.
iii) The term $\varepsilon\,(-2uj(t) + j(t)^2/n)$ expresses the electron counter-pressure (and maintains total momentum balance).

The quasineutral model (4) is valid up to the plasma-vacuum interface, denoted by $X(t)$. Beyond $X(t)$, i.e. in the beam, only electrons remain and the quasineutral model breaks down. Subject to the anode potential, the electrons are accelerated to large velocities. Electron velocity is better analyzed after rescaling according to $u_e = (\varepsilon\eta)^{-\frac{1}{2}}\bar{u}_e$ in equation (2). In the limit $\eta \to 0$, $(\bar{n}_e, \bar{u}_e)$ converges to a solution of the well-known Child-Langmuir model (see [8], [4], [5] for details). This model has a one-parameter family of solutions parametrized by the electron flux $\bar{j}_e = \bar{n}_e\bar{u}_e$ in the interval $[0, \bar{j}_{CL}]$.

To pursue the modeling further, we need to specify the plasma current j for the quasineutral model (4), the electron current $\bar{j}_e$ for the Child-Langmuir beam model. We also need an interface condition in order to determine the dynamic of the plasma-beam interface $X(t)$. The formal asymptotics performed so far does not provide such informations, essentially because different scaling hypotheses were made in the plasma and in the beam.

Therefore, we shall assume that the electron current $\bar{j}_e$ for the Child-Langmuir beam model coincides with the maximal current $\bar{j}_{CL}$, or after scaling the variables back to those of system (4), is given by $j_e = j_{CL} = (4/9)\sqrt{2}\lambda(1 - X(t))^{-2}(\varepsilon\eta)^{-1/2}$. Secondly, by current conservation at the interface, we assume that the plasma and beam currents coincide, i.e. $j = -j_{CL}$.

Now, in order to determine the extra interface condition, we look for a travelling-wave solution of system (1)-(3), connecting electron and ion quantities to the quasineutral model to the left of $X(t)$ and to the Child-Langmuir model to the right of $X(t)$. Let $U(x,t) = \tilde{U}(\xi,t)$ be a travelling wave profile (where U denotes the vector of unknowns of system (1)-(3)), with $\xi = (x - X(t))/\sqrt{\eta}$ and $\phi = \eta\tilde{\phi}$. In the formal limit $\eta \to 0$, the system (1)-(3) reduces to the following travelling wave problem (defining $\sigma = \dot{X}(t)$):

$$
\left\{
\begin{array}{c}
\tilde{n}_i = 0 \quad \text{for } \xi > 0, \quad -\lambda\tilde{\phi}_{\xi\xi} = \tilde{n}_i - \tilde{n}_e, \quad \text{for } \xi \in \mathbb{R}, \\
(\tilde{n}_e(\tilde{u}_e - \sigma))_\xi = 0, \quad \varepsilon(\tilde{n}_e(\tilde{u}_e - \sigma)\tilde{u}_e)_\xi + (p_e)_\xi = \tilde{n}_e\tilde{\phi}_\xi, \quad \text{for } \xi \in \mathbb{R}, \\
(\tilde{n}_i(\tilde{u}_i - \sigma))_\xi = 0, \quad (\tilde{n}_i(\tilde{u}_i - \sigma)\tilde{u}_i)_\xi + (p_i)_\xi = -\tilde{n}_i\tilde{\phi}_\xi, \quad \text{for } \xi < 0.
\end{array}
\right. \tag{5}
$$

with the following boundary conditions expressing the reconnection with the asymptotic models at infinity: when $\xi \to -\infty$, $\tilde{n}_i, \tilde{n}_e \to n_-$, $\tilde{u}_i \to u_-$, $\tilde{u}_e \to u_- - j/n_-$, $\tilde{\phi} \to 0$, where n_- and u_- denote the limits of the quasineutral quantities to the left of the interface, and when $\xi \to +\infty$, $\tilde{n}_i \to 0$, $\tilde{n}_e(\tilde{u}_e - \sigma) \to j_{CL}$.

The traveling wave problem (5) can be reduced to a nonlinear Poisson equation as follows: First, from the mass conservation equations and the boundary conditions, we deduce that $\sigma = \tilde{u}_i = u_-$ and $\tilde{n}_e(\tilde{u}_e - \sigma) = j_{CL}$, which yields $j = -j_{CL}$ as already noted. Then, let us define $h_{e,i}(n)$ such that $h'_{e,i}(n) = p'_{e,i}(n)/n$ with $h_{e,i}(0) = 0$ and $k_e(n) = \frac{\varepsilon j^2}{2\,n^2} + h_e(n)$. The momentum conservation relations and the boundary conditions yield

$$
h_i(\tilde{n}_i) + \tilde{\phi} = h_i(n_-) \quad \text{and} \quad k_e(\tilde{n}_e) - \tilde{\phi} = k_e(n_-). \tag{6}
$$

Note that, since h_i is increasing, $\tilde{n}_i = h_i^{-1}(h_i(n_-) - \tilde{\phi})$ is decreasing with respect to $\tilde{\phi}$. In contrast with h_i, k_e is non monotonous, decreasing on $[0, n_{min}]$ and increasing on $[n_{min}, \infty)$ with $n_{min} > n_H$ (see remark 1 for the definition of n_H). A choice of a branch of k_e^{-1} has to be made in accordance with the position of n_- with respect to n_{min}. With either choices, after inserting (6) into the Poisson equation, we are led to a nonlinear Poisson system.

Next, We analyze the nonlinear Poisson problem by means of a phase-plane analysis. This analysis shows that no profile with the correct behavior at $+\infty$ can be obtained in the case $n_- > n_{min}$. In the case, $n_- < n_{min}$, such a profile does exist provided that $n_- < n_P$, where n_P is a limit value greater than the limit for hyperbolicity n_H. Therefore, any choice of $n_- \in [n_H, n_P]$ leads to an admissible travelling wave profile, as announced in the introduction.

We have numerically determined that $n_- = n_P$ gives the best approximation. Therefore, the quasineutral model (4) in the plasma region, the Child-Langmuir model in the beam region, connected with the conditions $-j = j_e = j_{CL}$ and $n_- = n_P$ constitute our asymptotic model. We shall validate it in the forthcoming section by comparing it with the solutions of (1)-(3)

3. NUMERICAL SIMULATIONS

The quasi-neutral system (4) is approximated using a finite volume method [10], [11], [9]. The treatment of the interface follows [8]. We choose $\Delta x = 2.10^{-4}$, $\gamma = 2$, $\varepsilon = 5.10^{-1}$, $\lambda = 10^{-3}$ and $\eta = 10^{-4}$. Figure 1 and 2 compare the quasineutral model in the plasma region with electron and ion quantities of the two-fluid model. The agreement seems fairly satisfactory. We note that, contrary to a standard fluid expansion in vacuum, the plasma forms a sharp interface with the vacuum.
Figure 3 compare the child-Langmuir electron model with the two-fluid model in the beam region. Again, the agreement looks rather good.

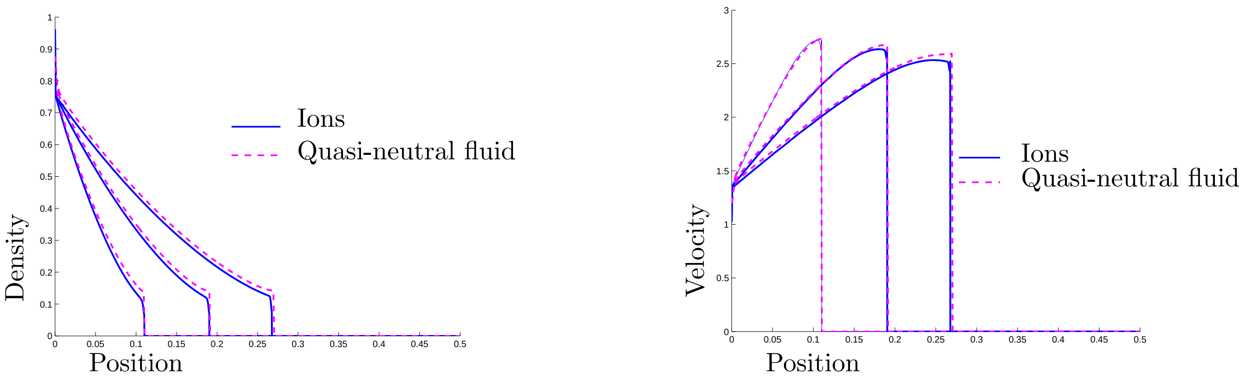

Figure 1: Ion density and mean velocity from the 2-fluid model compared with the quasi-neutral model : values between the cathode and the interface $X(t)$ at time $t = 0.04, t = 0.07, t = 0.1$ (in dimensionless units)

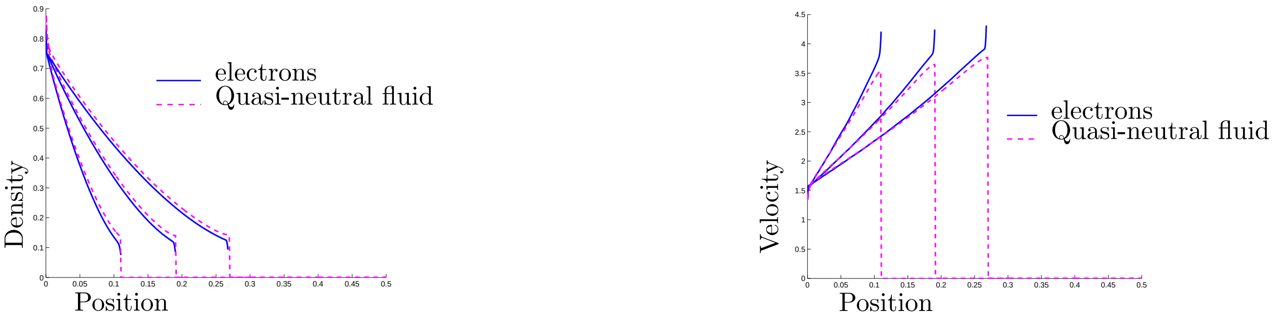

Figure 2: Electron density and mean velocity from the 2-fluid model compared with the quasi-neutral model : values between the cathode and the interface $X(t)$ at time $t = 0.04, t = 0.07, t = 0.1$ (in dimensionless units)

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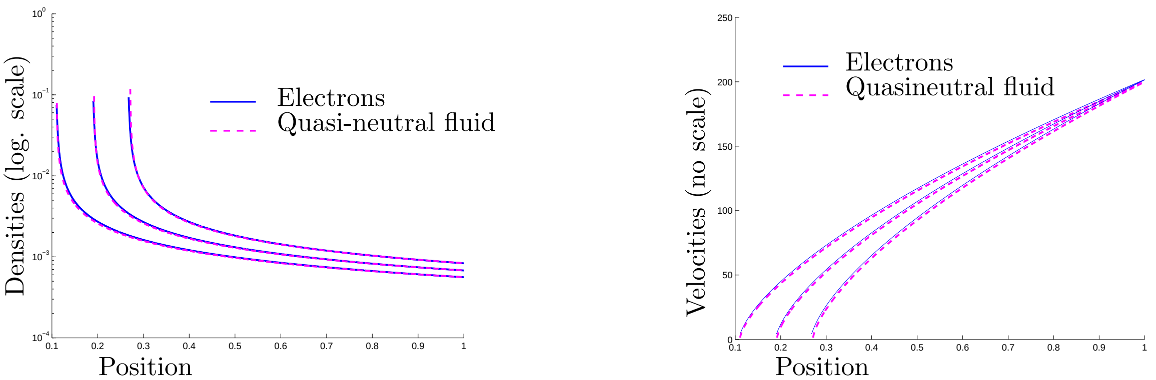

Figure 3: Electron density and mean velocity from the 2-fluid model compared with the Child-Langmuir model : values between the interface $X(t)$ and the anode at time $t = 0.04, t = 0.07, t = 0.1$ (in dimensionless units)

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