# Convergence of Finite Volumes schemes for an elliptic-hyperbolic system with boundary conditions

**Marie Hélène Vignal**
*UMPA, E.N.S. Lyon*
*46 Allée d'Italie*
*69364 Lyon, Cedex 07, France*

ABSTRACT. *We are interested in the convergence of the approximate solution given by a finite volumes scheme for the problem* $\Delta P = 0$, $u_t - \operatorname{div}(\nabla P\, f(u)) = 0$ *on an open bounded set, where* $f$ *is a non decreasing function. On the elliptic equation, a "four points" finite volumes scheme is used. An error estimate, in a discrete* $H^1$ *norm, of order* $h$ *is proved (*$h$ *defines the size of the mesh). On the hyperbolic equation, one uses an upstream scheme with respect to the flow. The convergence of the approximate solution toward the entropy solution of the problem is shown. Assuming the initial condition in* $BV(\Omega)$*, one establishes an error estimate of order* $h^{1/4}$ *in* $L^1$ *norm.*

KEY WORDS : *elliptic, hyperbolic, boundary conditions, finite volumes, measure valued solution, entropy process solution, error estimate.*

## 1. Introduction

One considers a non linear elliptic-hyperbolic system defined on an open bounded and convex set $\Omega$ of $\mathbb{R}^2$, one denotes by $\Gamma$ the boundary of $\Omega$. Then the problem is the following :

$$\Delta P(x) = 0, \quad x \in \Omega \tag{1}$$

$$u_t(x,t) - \operatorname{div}\Big(\nabla P(x)\, f\big(u(x,t)\big)\Big) = 0, \quad x \in \Omega,\ t \in \mathbb{R}^+ \tag{2}$$

with the following boundary and initial conditions :

$$\nabla P(\tau).n(\tau) = g(\tau), \quad \tau \in \Gamma \tag{3}$$

$$u(\tau,t) = \overline{u}(\tau,t), \quad (\tau,t) \in \Gamma^+ \times \mathbb{R}^+ \tag{4}$$

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{5}$$

where $\Gamma^+ = \{\tau \in \Gamma \ ;\ g(\tau) \geq 0\}$, and $n$ is the unit normal to $\Gamma$ outward to $\Omega$.

One supposes that

$$\begin{cases} \bullet \text{ each connected component of } \Gamma^+ \text{ is a line segment} \\ \bullet\ u_0 \in L^\infty(\Omega) \text{ and } \overline{u} \in C^1 \cap L^\infty(\overline{\Gamma}^+ \times \mathbb{R}_+) \\ \bullet\ f \in C^1(\mathbb{R},\mathbb{R}) \ \text{ and f is a nondecreasing function} \\ \bullet\ g \in L^\infty(\Gamma), \text{ such that } P \text{ is in } C^2(\overline{\Omega}), \\ \qquad \displaystyle\int_\Gamma g(\gamma)\, d\gamma = 0 \text{ and } \int_{\Gamma^+} \frac{1}{g(\tau)}\, d\tau < +\infty \end{cases} \tag{6}$$

One denotes by $a\top b = \max(a,b)$ and $a\bot b = \min(a,b)$ for all $a$, $b$ in $\mathbb{R}$, then for all $\kappa \in \mathbb{R}$, $|. - \kappa|$ and $f(.\top\kappa) - f(.\bot\kappa)$ are the Kruzkov entropy pairs.

More precisely, we search $u \in L^\infty(\Omega \times \mathbb{R}_+)$ entropy solution of (2), (4), (5), i.e. which verifies the following inequality :

$$\begin{aligned} &\iint_{\Omega\times\mathbb{R}_+} |u(x,t) - \kappa|\, \varphi_t(x,t)\, dx\, dt + \int_\Omega |u_0(x) - \kappa|\, \varphi(x,0)\, dx \\ &- \iint_{\Omega\times\mathbb{R}_+} \Big( f\Big(u(x,t)\top\kappa\Big) - f\Big(u(x,t)\bot\kappa\Big)\Big)\, \nabla P(x).\nabla\varphi(x,t)\, dx\, dt \\ &+ \iint_{\Gamma^+\times\mathbb{R}_+} \Big( f\Big(\overline{u}(\tau,t)\top\kappa\Big) - f\Big(\overline{u}(\tau,t)\bot\kappa\Big)\Big)\, g(\tau)\, \varphi(\tau,t)\, d\tau\, dt \geq 0 \end{aligned} \tag{7}$$

for all $\kappa \in \mathbb{R}$ and all $\varphi \in C_c^\infty(\overline{\Omega} \times \mathbb{R}_+, \mathbb{R}_+)$.

## 2. Discretization

Let $\mathcal{T}$ be a mesh of $\Omega$. For an element $p$ of $\mathcal{T}$, one denotes by $N(p)$ the set of the neighbours of $p$ and by $\sigma_{pq}$ the interface between $p$ and $q$, for all $q$ in $N(p)$. One assumes that $\mathcal{T}$ satisfies the four following regularity hypotheses :

1- The intersection between two elements of $\mathcal{T}$ is a line segment or a point.

2- There exist $\alpha > 0$ and $h > 0$ such that for all $p \in \mathcal{T}$ :

$$\alpha\, h^2 \leq m(p), \qquad m(\partial p) \leq \frac{1}{\alpha}\, h \qquad \text{and} \qquad \delta(p) \leq h$$

where $\partial p$ is the boundary of $p$, $m(.)$ is the Lebesgue measure in two space dimensions for $p$ and one space dimension for $\partial p$ and $\delta(p)$ is the diameter of $p$.

3- There exists a point family $(x_p)_{p\in\mathcal{T}}$ such that for all $p \in \mathcal{T}$, $x_p \in p$ and such that for all $q \in N(p)$ the line segment $[x_p, x_q]$ is orthogonal to $\sigma_{pq}$. The intersection between $[x_p, x_q]$ and $\sigma_{pq}$ will be denoted by $x_{pq}$.

4- If one denotes by $d_{pq}$ the length of the line segment $[x_p, x_q]$ then there exists $\beta > 0$ such that :

$$d_{pq} \geq \beta\, h$$

Several meshes, like Voronoï mesh or convenient triangulations, satisfy these hypotheses. So we discretize the elliptic equation generalizing the four points finite volumes scheme described by R. Herbin in [He93]. The unknowns are the $(P_p)_{p\in\mathcal{T}}$, and the approximate solution is given by $P_{\mathcal{T}}(x) = P_p$ for all $x \in p$ $(p \in \mathcal{T})$. The discretized equation is the following :

$$\sum_{q\in N(p)} \frac{P_q - P_p}{d_{pq}}\, m(\sigma_{pq}) + \sum_{a\in\mathcal{A}_{ext}(p)} g_a = 0 \qquad \text{for all } p \in \mathcal{T} \tag{8}$$

where $\mathcal{A}_{ext}(p)$ is the set of the edges of $p$ which are on $\Gamma$ and $g_a = \displaystyle\int_a g(\tau)\, d\tau$.

To discretize the hyperbolic equation one defines a time step $k > 0$ satisfying the following stability condition :

$$\text{for all } p \in \mathcal{T} \quad \frac{k\ M}{m(p)} \left( \sum_{\substack{q\in N(p)\\ P_p<P_q}} \frac{P_q - P_p}{d_{pq}}\, m(\sigma_{pq}) + \sum_{a\in\mathcal{A}_{ext}(p)} g_a^+ \right) \leq (1-\eta) \tag{9}$$

where $\eta \in ]0,1[$ is given, $M = \sup_{s\in[-U,U]} f'(s)$, $U = \max\Big(\|u_0\|_\infty, \|\overline{u}\|_\infty\Big)$, and where $g_a^+ = \displaystyle\int_a \Big(g(\tau)\top 0\Big)\, d\tau$ ; Then one sets $t^n = n\,k$.

The unknowns are the $u_p^n$, $p \in \mathcal{T}$, $n \in \mathbb{N}$, and the approximate solution is given by : $u_{\mathcal{T},k}(x,t) = u_p^n$ if $x \in p$ $(p \in \mathcal{T})$ and $t \in [t^n, t^{n+1}[$ $(n \in \mathbb{N})$. One sets, for all $p \in \mathcal{T}$, $u_p^0$ (resp. $\overline{u}_a^n$, for all $a \in \mathcal{A}_{ext}(p)$) the mean value of $u_0$ (resp. of $\overline{u}$) on $p$ (resp. on $[t^n, t^{n+1}] \times a$). One discretizes the hyperbolic equation with an Euler scheme explicit in time and an upstream finite volumes scheme with respect to the flow, then using (8) the discretized equation can be written :

$$\begin{aligned} m(p)\, \frac{u_p^{n+1} - u_p^n}{k} &- \sum_{\substack{q\in N(p)\\ P_p<P_q}} \frac{P_q - P_p}{d_{pq}}\, m(\sigma_{pq}) \Big(f(u_q^n) - f(u_p^n)\Big) \\ &- \sum_{a\in\mathcal{A}_{ext}(p)} g_a^+ \Big(f(\overline{u}_a^n) - f(u_p^n)\Big) = 0 \quad \forall\ (p,n) \in \mathcal{T}\times\mathbb{N} \end{aligned} \tag{10}$$

## 3. Convergence for the elliptic equation

One shows in [Vi] the following result, which gives existence and "uniqueness" of $P_{\mathcal{T}}$ :

**Proposition 1** *One assumes (6). Let $\mathcal{T}$ be a mesh of $\Omega$ satisfying the properties 1, 2 and 3 of the section 2, then, there exist $P_\mathcal{T}$ solutions of (8), and all these solutions only differ from a constant.*

One proves an error estimate on a discrete $H^1$ norm in order $h$, and so the convergence of the approximate solution toward the exact solution of the elliptic equation :

**Theorem 1** *One assumes (6). Let $\mathcal{T}$ be a mesh of $\Omega$ satisfying the properties 1, 2, 3 and 4 of the section 2. One denotes by $P(\cdot)$ the exact solution of the problem (1), (3), such that $\sum_{p\in\mathcal{T}} m(p)\,P(x_p) = 0$. Let $(P_p)_{p\in\mathcal{T}}$ satisfying (8) and $\sum_{p\in\mathcal{T}} m(p)\,P_p = 0$. One defines the error on the cell $p$ by $e_p = P_p - P(x_p)$ for all $p \in \mathcal{T}$. Then there exist $C_1$ and $C_2$, depending only on $\alpha$, $\beta$, $\Omega$, $g$ and on the second order derivatives of $P$, such that :*

$$\left(\sum_{p\in\mathcal{T}}\sum_{q\in N(p)} \frac{(e_q - e_p)^2}{d_{pq}}\, m(\sigma_{pq})\right)^{1/2} \leq C_1\, h, \qquad \left(\sum_{p\in\mathcal{T}} m(p)\,|e_p|^2\right)^{1/2} \leq C_2\, h$$

**Proof :**

First one proves the consistency of the fluxes, i.e. that there exists $C$ which does not depend on the mesh such that for all $p$ in $\mathcal{T}$ and all $q$ in $N(p)$ :

$$\left|\frac{1}{m(\sigma_{pq})}\int_{\sigma_{pq}} \nabla P(\tau).n_p(\tau)\,d\tau - \frac{P(x_q) - P(x_p)}{d_{pq}}\right| \leq C\,h$$

Then using the scheme (10), the elliptic equation (1) integrated over $p \in \mathcal{T}$ and the consistency of the fluxes one proves the first result, i.e. the error estimate in $H^1_0$ norm.

To prove the last result (the error estimate in $L^2$ norm), one establishes a discrete Poincaré inequality :

**Lemma 1** *Let $\Omega$ be a convex set of $\mathbb{R}^2$, and $\mathcal{T}$ be a mesh of $\Omega$ satisfying the properties 1, 2, 3 and 4 of the section 2. Let $u$ be a function constant on each cell of $\mathcal{T}$ : $u(x) = u_p$ if $x \in p$ $(p \in \mathcal{T})$, one denotes by $\hat{u}_\Omega$ the mean value of $u$ on $\Omega$. Then there exists $C$, depending only on $\Omega$, such that :*

$$\sum_{p\in\mathcal{T}} m(p)\,|u_p - \hat{u}_\Omega|^2 \leq C \sum_{p\in\mathcal{T}}\sum_{q\in N(p)} \frac{(u_p - u_q)^2}{d_{pq}}\, m(\sigma_{pq})$$

## 4. Convergence and error estimate for the hyperbolic equation

In order to show the convergence of $u_{\mathcal{T},k}$ toward the entropy solution of (2), (4), (5), we pass to the limit in the discretized equations. The approximate solution converges, in a non linear weak star sense, to an entropy process solution (or a measure valued solution see [Di88]) of the hyperbolic equation. We prove that this solution is in fact the entropy solution. This step is developed by R. Eymard, T. Gallouët and R. Herbin in [EGH95] for a hyperbolic equation defined on $\mathbb{R}^N$ ($N \geq 1$). It can be generalized to the problem considered here. We have the following result :

**Theorem 2** *One assumes (6). Let $\mathcal{T}$ be a mesh of $\Omega$ satisfying the properties* 1, 2, 3 *and* 4 *of the section 2 and $k$ be a time step satisfying the stability condition (9). Let $u_{\mathcal{T},k}$ be the solution of the discretized problem (10).*

*Then there exists $u \in L^\infty(\Omega \times \mathbb{R}^+)$ such that $u_{\mathcal{T},k}$ goes to $u$, when $h$ goes to* 0*, in $L^p_{loc}(\Omega \times \mathbb{R}^+)$ for all $p < +\infty$, and $u$ is the entropy solution of the hyperbolic equation, i.e. $u$ satisfies (7).*

**Proof :**

First we prove, using the numerical scheme an $L^\infty$ estimate on the approximate solution :

$$\|u_{\mathcal{T},k}\|_{L^\infty(\Omega\times\mathbb{R}_+)} \leq \max\left(\|u_0\|_{L^\infty(\Omega)}, \|\overline{u}\|_{L^\infty(\Gamma^+\times\mathbb{R}_+)}\right) \tag{11}$$

One denotes by $\mathcal{M}(\overline{\Omega})$ (respectively $\mathcal{M}(\overline{\Omega}\times\mathbb{R}_+)$ and $\mathcal{M}(\overline{\Gamma}^+\times\mathbb{R}_+)$) the set of the positive measures on $\Omega$ (respectively on $\overline{\Omega}\times\mathbb{R}_+$ and $\overline{\Gamma}^+\times\mathbb{R}_+$), i.e. the set of linear positive and continuous forms on $C_c(\overline{\Omega})$ (respectively on $C_c(\overline{\Omega}\times\mathbb{R}_+)$ and $C_c(\overline{\Gamma}^+\times\mathbb{R}_+)$).

Then one establishes a continuous entropy estimates for the approximate solution proving the following lemma :

**Lemma 2** *One assumes (6). Let $\mathcal{T}$ be a mesh of $\Omega$ satisfying the properties* 1, 2, 3 *and* 4 *of the section 2 and $k$ be a time step satisfying the stability condition (9). Let $u_{\mathcal{T},k}$ be the solution of the discretized problem (10). Then there exist $\mu_{\mathcal{T},k} \in \mathcal{M}(\overline{\Omega}\times\mathbb{R}_+)$, $\mu_{\mathcal{T}} \in \mathcal{M}(\overline{\Omega})$ and $\overline{\mu}_{\mathcal{T},k} \in \mathcal{M}(\overline{\Gamma}^+\times\mathbb{R}_+)$ such that :*

$$\iint_{\Omega\times\mathbb{R}_+} |u_{\mathcal{T},k}(x,t)-\kappa|\,\varphi_t(x,t)\,dx\,dt + \int_\Omega |u_0(x)-\kappa|\,\varphi(x,0)\,dx$$

$$-\iint_{\Omega\times\mathbb{R}_+}\Big(f(u_{\mathcal{T},k}(x,t)\top\kappa)-f(u_{\mathcal{T},k}(x,t)\perp\kappa)\Big)\,\nabla P(x).\nabla\varphi(x,t)\,dx\,dt$$

$$+\iint_{\Gamma^+\times\mathbb{R}_+}\Big(f(\overline{u}(\tau,t)\top\kappa)-f(\overline{u}(\tau,t)\perp\kappa)\Big)\,g(\tau)\,\varphi(\tau,t)\,d\tau\,dt$$

$$\geq -\iint_{\Omega\times\mathbb{R}_+}\Bigg(|\varphi_t(x,t)|+|\nabla\varphi(x,t)|\Bigg)\,d\mu_{\mathcal{T},k}(x,t)$$

$$-\int_\Omega \varphi(x,0)\,d\mu_{\mathcal{T}}(x) - \iint_{\Gamma^+\times\mathbb{R}_+}\varphi(\tau,t)\,d\overline{\mu}_{\mathcal{T},k}(\tau,t) \qquad (12)$$

*for all $\kappa\in\mathbb{R}$ and all $\varphi\in C_c^\infty(\overline{\Omega}\times\mathbb{R}_+,\mathbb{R}_+)$.*

*Moreover for all $T\in\,]0,+\infty[$, there exist $C$, $D$ and $E$, depending only on $\alpha$, $\beta$, $\eta$, $\Omega$, $T$, $P$ and on the data of the problem, such that :*

$$\mu_{\mathcal{T},k}\Big(\overline{\Omega}\times[0,T]\Big)\leq C\,\Big(\sqrt{h}+\sqrt{k}\Big) \text{ and } \overline{\mu}_{\mathcal{T},k}\Big(\overline{\Gamma}^+\times[0,T]\Big)\leq D\,h \qquad (13)$$

$$\begin{cases}\mu_{\mathcal{T}}(\overline{\Omega})\longrightarrow 0 \text{ when } h \text{ goes to } 0\\ \text{furthermore if } u_0\in L^\infty\cap BV(\Omega),\text{ then } \mu_{\mathcal{T}}(\overline{\Omega})\leq E\,h\end{cases} \qquad (14)$$

**Proof :**

To prove this result, one first establishes, using the non decreasing property of $f$ and the stability condition (9), the following discrete entropy inequality :

$$m(p)\,\frac{|u_p^{n+1}-\kappa|-|u_p^n-\kappa|}{k} - \sum_{\substack{q\in N(p)\\ P_p<P_q}}\frac{P_q-P_p}{d_{pq}}\,m(\sigma_{pq})\times$$
$$\times\Bigg(f(u_q^n\top\kappa)-f(u_q^n\perp\kappa)-\Big(f(u_p^n\top\kappa)-f(u_p^n\perp\kappa)\Big)\Bigg) \qquad (15)$$
$$-\sum_{a\in\mathcal{A}_{ext}(p)} g_a^+\Bigg(f(\overline{u}_a^n\top\kappa)-f(\overline{u}_a^n\perp\kappa)-\Big(f(u_p^n\top\kappa)-f(u_p^n\perp\kappa)\Big)\Bigg)\leq 0$$

for all $p\in\mathcal{T}$, all $n\in\mathbb{N}$ and all $\kappa\in\mathbb{R}$.

Multiplying (12) by $\displaystyle\frac{1}{k}\int_{t^n}^{t^{n+1}}\int_p \varphi(x,t)\,dx\,dt$, (with $\varphi\in C_c^\infty(\overline{\Omega}\times\mathbb{R}_+,\mathbb{R}_+)$), one proves (12). One establishes (13) and (14) proving the following result which gives weak estimates on the variation of $u_{\mathcal{T},k}$.

There exists $C_w$, depending on $\eta$, $\alpha$, $\beta$, $\Omega$, $P$, $u_0$, $\overline{u}$, $f$ and $g$ such that :

$$\sum_{n=0}^{N}\sum_{p\in\mathcal{T}} k \left[ \sum_{\substack{q\in N(p)\\ P_q>P_p}} \frac{P_q-P_p}{d_{pq}}\, m(\sigma_{pq}) \left|f(u_q^n)-f(u_p^n)\right| \right. \\ \left. + \sum_{a\in\mathcal{A}_{ext}(p)} g_a^+ \left|f(\overline{u}_a^n)-f(u_p^n)\right| \right] \leq \frac{C_w}{\sqrt{h}} \tag{16}$$

and

$$\sum_{n=0}^{N}\sum_{p\in\mathcal{T}} m(p)\, \left|u_p^{n+1}-u_p^n\right| \leq \frac{C_w}{\sqrt{k}} \tag{17}$$

Then using the following lemma of non linear weak $\star$ compactness (see [EGH95]) :

**Lemma 3** *(Eymard, Gallouët, Herbin)*
*Let $E\subset\mathbb{R}^N$ $(N\geq 1)$. Let $(u_n)_{n\in\mathbb{N}}$ in $L^\infty(E)$, $\|u_n\|_\infty\leq U$ for all $n\in\mathbb{N}$. Then there exist $\mu\in L^\infty(E\times]0,1[)$ and a subsequence still denoted by $(u_n)_{n\in\mathbb{N}}$ such that :*

$$\lim_{n\to+\infty} u_n = \mu \text{ non linear weak } \star$$

*i.e. one has :*

$$\lim_{n\to+\infty}\int_E g(u_n(x))\,\varphi(x)\,dx = \int_E\int_0^1 g(\mu(x,\alpha))\,\varphi(x)\,d\alpha\,dx$$

*for all $\varphi\in L^1(E)$ and all $g\in C([-U,U],\mathbb{R})$.*

one passes to the limit in (12) and one proves that there exists $\mu\in L^\infty(\Omega\times\mathbb{R}_+\times]0,1[)$ and a subsequence such that $u_{\mathcal{T},k}$ goes to $\mu$ in the non linear weak $\star$ sense and $\mu$ is an entropy process solution of (2), (4), (5), i.e. $\mu$ satisfies the following inequality :

$$\iint_{\Omega\times\mathbb{R}_+}\int_0^1 |\mu(x,t,\alpha)-\kappa|\,\varphi_t(x,t)\,d\alpha\,dx\,dt + \int_\Omega |u_0(x)-\kappa|\,\varphi(x,0)\,dx \\ -\iint_{\Omega\times\mathbb{R}_+}\int_0^1\Big(f\big(\mu(x,t,\alpha)\top\kappa\big)-f\big(\mu(x,t,\alpha)\bot\kappa\big)\Big)\nabla P(x).\nabla\varphi(x,t)\,d\alpha\,dx\,dt \\ +\iint_{\Gamma^+\times\mathbb{R}_+}\Big(f\big(\overline{u}(\tau,t)\top\kappa\big)-f\big(\overline{u}(\tau,t)\bot\kappa\big)\Big)\,g(\tau)\,\varphi(\tau,t)\,d\tau\,dt\geq 0 \tag{18}$$

for all $\kappa\in\mathbb{R}$ and all $\varphi\in C_c^\infty(\overline{\Omega}\times\mathbb{R}_+,\mathbb{R}_+)$. Then using the Kruzkov technique one proves that if $\mu$ and $\nu$ are two entropy process solutions of (2), (4), (5) then :

$$\mu(x,t,\alpha)=\nu(x,t,\beta) \qquad a.e.\ (x,t,\alpha,\beta)\in\Omega\times\mathbb{R}_+\times]0,1[\times]0,1[$$

So $\mu$ does not depend on its third argument and it is the entropy solution solution of (2), (4), (5).

Moreover one can prove an error estimate, as R. Eymard, T. Gallouët, M. Ghilani and R. Herbin do it in [EGGH] for $\Omega = \mathbb{R}^N$ $(N \geq 1)$. The essential difference between [EGGH] and this work comes from the control of boundary terms. One obtains the following error estimate :

**Theorem 3** *One assumes (6) and one supposes $u_0$ in $BV(\Omega)$. Let $\mathcal{T}$ be a mesh of $\Omega$ satisfying the properties* 1, 2, 3 *and* 4 *of the section 2 and $k$ be a time step satisfying the stability condition (9). Let $u_{\mathcal{T},k}$ be the solution of the discretized problem (10) and $u \in L^\infty(\Omega \times \mathbb{R}_+) \cap BV(\Omega \times [0,T])$ (for all $T \in ]0,+\infty[$) the entropy solution of the hyperbolic equation (i.e. satisfying (7)). Then for all $T \in ]0,+\infty[$, there exists $C$, independent of the mesh and the time step, such that :*

$$\iint_{\Omega\times[0,T]} |u_{\mathcal{T},k}(x,t) - u(x,t)| \, dx \, dt \leq C \, h^{1/4} \tag{19}$$

To prove this theorem one use the Kruzkov's technique with (7) and (12). One obtains :

$$\iint_{\Omega\times[0,T]} |u_{\mathcal{T},k}(x,t) - u(x,t)| \, dx \, dt \leq$$
$$C\left(\mu_0(\overline{\Omega}) + \overline{\mu}(\overline{\Gamma}^+ \times [0,T]) + \mu(\overline{\Omega} \times [0,T]) + \left(\mu(\overline{\Omega} \times [0,T])\right)^{1/2}\right)$$

One concludes using (13) and (14).

## 5. Bibliography


[Di88] DiPerna R.,1985, ¡¡Measure-Valued Solutions to Conservation Laws¿¿, Arch. Rat. Mech. Anal., 88, 223–270.

[EGH95] Eymard R., Gallouët T., Herbin R., 1995, ¡¡Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation¿¿, Chin. Ann. of Math., Vol 16 B : 1,p. 1–14.

[EGGH] Eymard R. and Gallouët T. and Ghilani M. and Herbin R., ¡¡Error estimates for the approximate solutions of a nonlinear hyperbolic equation given by some finite volume schemes¿¿, submitted.

[He93] Herbin R., 1993, ¡¡An error estimate for a finite volume scheme for a diffusion convection problem on a triangular mesh¿¿, Num. Meth. in P.D.E.

[Sz89] Szepessy A., 1989, ¡¡Measure valued solution of scalar conservation laws with boundary conditions¿¿, Arch. Rat. Mech. Anal. Vol. 107, No. 2, p.182-193.

[Vi] VIGNAL M.-H., ¡¡Convergence of a Finite Volume Scheme for an elliptic-hyperbolic system¿¿, to appear in M$^2$AN.