A BOLTZMANN MODEL FOR TRAPPED PARTICLES IN A SURFACE POTENTIAL [*]

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Abstract. In this article, we propose a model describing the transport of trapped particles in a surface potential. The potential confines particles close to the surface increasing the number of surface collisions. First, we consider the case of non charged particles. From a kinetic description, we rigorously derive a two dimensional Boltzmann equation. In the case of charged particles we introduce the coupling with the Poisson equation. We perform a formal asymptotic analysis which leads to a two dimensional Boltzmann equation coupled with a three dimensional Poisson equation. We illustrate the charged particle model with some numerical simulations of a gas discharge on a satellite solar array. We use a Particle In Cell (P.I.C.) scheme that is a particle discretization for the Boltzmann equation and a Fourier approximation for the Poisson equation.

Key words. Boltzmann equation, Poisson equation, Surface collisions, Asymptotic model, Simulation of gas discharge, P.I.C. scheme, Particle method.

AMS subject classifications. 82C70, 82C40, 82C21, 82B40, 82B21, 35J05, 76M28.

1. Introduction. In this article, we are interested in particles subject to a given external potential in a half space. We propose a mathematical model to describe their transport and we perform numerical simulations. The starting point of our model is a kinetic model constituted of the Vlasov equation set in the half space. It is completed by some boundary conditions for the description of the surface collisions.

In order to reduce the cost of the numerical simulations, it is classical to derive some asymptotic models with a smaller number of variables than the kinetic description. It is possible when the physical conditions allow to do it, for example when particles are subject to many collisions. The resulting model depends on the considered physical process (see [5] and the references given there).

Here, we want to establish an asymptotic model when the applied potential confines the particles close to the surface increasing the number of collisions with it. Furthermore, we assume that the dominant surface collisions process is specular; the other collisions are supposed to be only a perturbation. Although it is not true in practical situations we assume that the error is of the same order as the error we make with the asymptotic limit.

This work is a continuation of [15] in which the considered problem is the same as here but the dominant surface collisions are supposed diffusive. In [15], the resulting model is a diffusive model in two space dimensions; its variables are the time, the position on the surface and the total energy of the particles. These kinds of models are called in the literature Fokker-Planck models or "SHE model" (for Spherical Harmonics Expansion) because of its earlier derivation in [26]. The diffusive limits have been extensively studied, see for example [3], [6], [4] or [21] in different contexts: radiative transfer, semi-conductors, neutron transport.

Here, the resulting model is a Boltzmann model in two space dimensions; its variables are the time, the position on the surface, the parallel velocity to the surface and the total energy in the transverse direction. Furthermore the collision operator

of this equation describes the non specular surface collisions. This is an intermediate model between the kinetic description and the SHE model.

We begin with the rigorous derivation of the asymptotic model in the case of a purely specular surface collision process. Then we consider the case of charged particles. A coupling with the Poisson equation is introduced and the problem becomes non linear. We derive the asymptotic model considering a general surface collision process with a dominant specular collisions process. A similar work with a rigorous derivation has been done for quantum transport and in the case of the whole space in [7]. Here, the results obtained for the coupled model with the Poisson equation are formal. We defer the rigorous proof of the limit to a future work [27].

In order to illustrate the charged particle model, we perform some numerical simulations of gas discharge on a satellite solar array. High voltage solar arrays are subject to strong interactions with the plasma environment. According to the literature, see [11], [12], the scenario can be decomposed essentially into two phases. First a large density plasma is created by a primary discharge. In some cases this plasma triggers an electric arc called the secondary arc. Here we are concerned with the modeling of the primary discharge. We propose a simplified model for the electron motion during the primary discharge, based on the previously derived asymptotic model. According to M. Cho and D.E. Hasting [12], the primary discharge is due to a combination of three phenomena: an enhanced field emission at a particular point called the triple point where metallic parts, dielectrics and vacuum meet, electron secondary emission on the lateral side of the dielectric and an electron stimulated desorption of the neutral gas adsorbed by the surface. So we complete our model by some boundary conditions which describe the enhanced field emission at the triple point. Furthermore we propose a mathematical modeling for surface collisions describing the electron secondary emission process. We discretize this model using a P.I.C. (for Particle In Cell) scheme (see [13], [16], [23]), based on a particle discretization of the Boltzmann equation and a Fourier approximation of the the Poisson equation. The simulation of electron secondary emission operator involves a branching process (where numerical particles multiply) followed by a collapsing process. We conclude with the presentation of the numerical results.

2. The kinetic model. We are interested in the kinetic modeling of particles trapped in a surface potential and subject to specular collisions with this surface, see Fig. 2.1. We consider the domain $\Omega = \mathbb{R}^2 \times (-\infty, 0)$ and we suppose that a solid wall is located on the boundary of Ω: $\partial\Omega = \mathbb{R}^2 \times \{0\}$. We denote by $x = (\underline{x}, z) \in \mathbb{R}^2 \times \mathbb{R}^-$, $v = (\underline{v}, v_z) \in \mathbb{R}^2 \times \mathbb{R}$ the position and velocity vectors. Due to their geometric relation with the surface, $\underline{x}$ and $\underline{v}$ are called the parallel components of the position and velocity vectors and z and v_z are called the transverse components of these vectors.

Let f be the particle distribution function. It satisfies the Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + \frac{E}{m} \cdot \nabla_v f = 0, \tag{2.1}$$

for all $(x, v, t) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^+$, where m is the particle mass and where E is the force field. In Section 3, non charged particles are considered. In this case, E stands for a potential force given by $E = -\nabla_x \phi$ where ϕ is a given potential. In Section 4, the case of charged particles is discussed. Then, E is an electrostatic force given by $E = -q\nabla_x(\phi + \phi_s)$ where q is the charge of the particles, ϕ is the external applied electric potential and ϕ_s is the self-consistent potential given by (4.1).
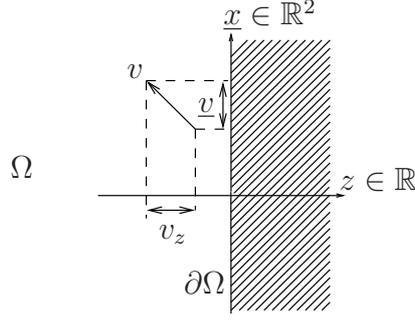
FIG. 2.1. *The domain is defined by* $\Omega = \mathbb{R}^2 \times \mathbb{R}^-$. *The position,* $x \in \Omega$, *and the velocity,* $v \in \mathbb{R}^3$, *are split in* $(\underline{x}, z) \in \mathbb{R}^2 \times \mathbb{R}^-$ *and* $(\underline{v}, v_z) \in \mathbb{R}^2 \times \mathbb{R}$. *Due to their geometric relation with the surface* $\partial\Omega = \mathbb{R}^2 \times \{0\}$, $\underline{x}$ *and* $\underline{v}$ *are called parallel components of the position and the velocity and* z *and* v_z *are called transverse components of the same vectors.*

We complete the problem with some initial and boundary conditions. We consider that, in $\Omega \times \mathbb{R}^3$

$$f(t = 0) = f_0, \tag{2.2}$$

where f_0 is a L^∞ given function with a compact support in $\bar{\Omega} \times \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^- \times \mathbb{R}^3$. In order to discuss the boundary conditions, we introduce the outgoing and incoming traces of f on the surface $\partial\Omega$, respectively denoted by $\gamma^-(f)(\underline{x}, v, t)$ (or $\gamma^-(f)(v_z)$) for $v_z < 0$ and by $\gamma^+(f)(\underline{x}, v, t)$ (or $\gamma^+(f)(v_z)$) for $v_z > 0$, for all $(\underline{x}, v, t) \in \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^+$. Then, we consider the boundary condition given by

$$\gamma^-(f)(v_z) = \beta \, \mathcal{S}\gamma^+(f)(v_z) + (1 - \beta) \, \mathcal{K}\gamma^+(f)(v_z), \tag{2.3}$$

for all $v_z < 0$ and where β is the probability for an incoming particle on the surface $\partial\Omega$ to be re-emitted specularly. The operator $\mathcal{S}$ is the specular reflection operator and $\mathcal{K}$ is a general collision operator on $\partial\Omega$. For all $v_z < 0$, we write

$$\mathcal{S}\gamma^+(f)(v_z) = \gamma^+(f)(-v_z), \tag{2.4}$$

and

$$\mathcal{K}\gamma^+(f)(v_z) = \int_{\{v' = (\underline{v}', v_z') \in \mathbb{R}^2 \times \mathbb{R}^+\}} K(v', v) \, \gamma^+(f)(v_z') \, v_z' \, dv', \tag{2.5}$$

where $K(v', v) \, |v_z| \, dv$ is the number of re-emitted particles in $[v, v+dv]$ for an incoming particle on the surface $\partial\Omega$, with a velocity given by v'.

The aim of this article is to obtain an asymptotic model in a situation where the potential ϕ confines the particles close to the surface. We denote by $L_\parallel$ and $L_\perp$ the characteristic lengths respectively in the parallel and transverse directions. We assume $\alpha = L_\perp / L_\parallel$ is a small dimensionless parameter. We denote by $E_c \geq 0$, $v_c = \sqrt{L_\parallel E_c / m}$, $t_c = L_\parallel / v_c$ and f_c the characteristic force field magnitude, velocity, time and distribution function. And, we introduce a re-scaling for the problem (2.1)-(2.3) denoted by $\hat{x} = (\hat{\underline{x}}, \hat{z})$, $\hat{v}$ and $\hat{t}$ being defined by

$$\underline{x} = L_\parallel \, \hat{\underline{x}}, \qquad z = L_\perp \hat{z} = \alpha \, L_\parallel \, \hat{z}, \qquad v = v_c \, \hat{v} \qquad \text{and} \qquad t = t_c \, \hat{t}. \tag{2.6}$$

We suppose that there exists $\hat{\phi}$ and $\hat{f}_0$, independent of α and such that $\phi(x,t) = L_\parallel E_c \, \hat{\phi}(\hat{x}, \hat{t})$, for $(x,t) \in \Omega \times \mathbb{R}^+$ and $f_0(x,v) = f_c \, \hat{f}_0(\hat{x}, \hat{v})$, for $(x,v) \in \Omega \times \mathbb{R}^3$. Finally we set $f(x,v,t) = f_c \, f^\alpha(\hat{x}, \hat{v}, \hat{t})$ for $(x,v,t) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^+$. Inserting this new scale in (2.1) and omitting the "hats", we obtain:

$$\partial_t f^\alpha + \underline{v} \cdot \nabla_{\underline{x}} f^\alpha - \nabla_{\underline{x}} \phi \cdot \nabla_{\underline{v}} f^\alpha + \frac{1}{\alpha} \left(v_z \, \partial_z f^\alpha - \partial_z \phi \, \partial_{v_z} f^\alpha \right) = 0,$$

in $\Omega \times \mathbb{R}^3 \times \mathbb{R}^+$.

We denote by ψ the transverse component of the potential ϕ and by ϕ_0 its parallel component, that is the functions such that for all $(\underline{x}, z, t) \in \bar{\Omega} \times \mathbb{R}^+$

$$\phi_0(\underline{x}, t) = \phi(\underline{x}, 0, t) \quad \text{and} \quad \psi(\underline{x}, z, t) = \phi(\underline{x}, z, t) - \phi_0(\underline{x}, t). \tag{2.7}$$

In order to model the case of an attractive potential for the particles, we make the following hypotheses:

ASSUMPTION 1. *For all $(\underline{x}, t) \in \mathbb{R}^2 \times \mathbb{R}^+$, the function $z \in \mathbb{R}^- \mapsto \psi(\underline{x}, z, t) \in \mathbb{R}$ is decreasing, continuous and satisfies $\lim_{z \to -\infty} \psi(\underline{x}, z, t) = +\infty$.*

Then remarking that $\partial_z \phi = \partial_z \psi$, we obtain the re-scaled Vlasov equation.

$$\partial_t f^\alpha + \underline{v} \cdot \nabla_{\underline{x}} f^\alpha - \nabla_{\underline{x}} (\phi_0 + \psi) \cdot \nabla_{\underline{v}} f^\alpha + \frac{1}{\alpha} \left(v_z \, \partial_z f^\alpha - \partial_z \psi \, \partial_{v_z} f^\alpha \right) = 0, \tag{2.8}$$

in $\Omega \times \mathbb{R}^3 \times \mathbb{R}^+$, where ϕ_0 and ψ are the parallel and transverse components of the given potential ϕ defined by (2.7).
The initial condition is given by

$$f^\alpha(t = 0) = f_0, \tag{2.9}$$

in $\Omega \times \mathbb{R}^3$. Finally, the boundary condition, on the surface $\partial \Omega$, is:

$$\gamma^-(f^\alpha)(v_z) = \beta \, \mathcal{S} \gamma^+(f^\alpha)(v_z) + (1 - \beta) \, \mathcal{K} \gamma^+(f^\alpha)(v_z) \quad \text{for all } v_z < 0, \tag{2.10}$$

where $\mathcal{S}$ and $\mathcal{K}$ are given by (2.4) and (2.5).

3. A rigorous asymptotic limit for specular collisions. In this section, we are interested in the rigorous limit $\alpha \to 0$ of system (2.8)-(2.10) when the collisions on the surface $\partial \Omega$ are only specular reflections. Then, in all this section, we will consider $\beta = 1$ in (2.10) and the boundary condition on $\partial \Omega$ for Eq. (2.8) is rewritten

$$\gamma^-(f^\alpha)(v_z) = \gamma^+(f^\alpha)(-v_z) \quad \text{for all } v_z < 0. \tag{3.1}$$

For all $\alpha > 0$, we have the following result (see for instance [2], [9, 10], [20] or [24])

THEOREM 3.1. *We suppose $f_0 \in L^\infty(\Omega \times \mathbb{R}^3)$ with a compact support in $\bar{\Omega} \times \mathbb{R}^3$ and we assume $\phi \in C^2(\bar{\Omega})$. Let ϕ_0 and ψ be the parallel and transverse components of the potential ϕ defined by (2.7). Let $\alpha > 0$, then there exists a unique $f^\alpha \in L^\infty(\Omega \times \mathbb{R}^3 \times [0, T]) \cap C([0, T]; L^1(\Omega \times \mathbb{R}^3))$ for all $T > 0$, the weak solution to (2.8), (2.7), (2.9) and (3.1), i.e. satisfying:*

$$\int_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^+} f^\alpha(x, v, t) \left(\partial_t + \underline{v} \cdot \nabla_{\underline{x}} - \nabla_{\underline{x}} (\phi_0(\underline{x}, t) + \psi(x, t)) \cdot \nabla_{\underline{v}} \right) \varphi(x, v, t) \, dx \, dv \, dt$$

$$+\frac{1}{\alpha}\int_{\Omega\times\mathbb{R}^3\times\mathbb{R}^+} f^\alpha(x,v,t)\left(v_z\,\partial_z - \partial_z\psi(x,t)\,\partial_{v_z}\right)\varphi(x,v,t)\,dx\,dv\,dt \tag{3.2}$$

$$+\int_{\Omega\times\mathbb{R}^3} f_0(x,v)\,\varphi(x,v,0)\,dx\,dv = 0$$

for all $\varphi \in C_c^1(\overline\Omega \times \mathbb{R}^3 \times \mathbb{R}^+)$ and such that $\varphi(\underline{x},0,\underline{v},v_z,t) = \varphi(\underline{x},0,\underline{v},-v_z,t)$ for all $v_z < 0$.

Furthermore, for all $T > 0$, there exists $C_T > 0$, depending only on T and f_0, such that for all $\alpha > 0$

$$\|f^\alpha\|_{L^\infty(\Omega\times\mathbb{R}^3\times[0,T])} \le C_T. \tag{3.3}$$

The aim of this section is the proof of the following result

THEOREM 3.2. . Let $\alpha > 0$, $f_0 \in L^\infty(\Omega \times \mathbb{R}^3)$ with a compact support in $\bar\Omega \times \mathbb{R}^3$ and $\phi \in C^2(\bar\Omega)$. We denote by ϕ_0 and ψ the parallel and transverse components of ϕ defined by (2.7). We suppose that ψ satisfies Assumption (1). Let f^α be the weak solution to (2.8), (2.7), (2.9) and (3.1), i.e. satisfying (3.2). Then there exists $f \in L^\infty(\Omega \times \mathbb{R}^3 \times [0,T])$ for all $T > 0$ such that

$$\lim_{\alpha\to 0} f^\alpha = f \quad \text{in } L^\infty(\Omega \times \mathbb{R}^3 \times [0,T]) \text{ for the weak-} \star \text{ topology} \tag{3.4}$$

and all $T > 0$.

Furthermore there exists $F \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times [0,T])$ for all $T > 0$ such that

$$f(\underline{x},z,\underline{v},v_z,t) = F(\underline{x},\underline{v},\varepsilon_z,t) \tag{3.5}$$

for almost all $(\underline{x},z,\underline{v},v_z,t) \in \mathbb{R}^2 \times \mathbb{R}^- \times \mathbb{R}^3 \times \mathbb{R}^+$, where

$$\varepsilon_z = |v_z|^2/2 + \psi(\underline{x},z,t). \tag{3.6}$$

The function F satisfies:

$$\left[\partial_t + \underline{v}\cdot\nabla_{\underline{x}} - \left(\nabla_{\underline{x}}\phi_0 + <\nabla_{\underline{x}}\psi>\right)\cdot\nabla_{\underline{v}}\right](N_z\,F)$$
$$+\partial_{\varepsilon_z}\left[\left(<\partial_t\psi> + \underline{v}\cdot<\nabla_{\underline{x}}\psi>\right)(N_z\,F)\right] = 0 \tag{3.7}$$

in $\mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}^2 \times (0,+\infty) \times (0,+\infty))$, where for all function g from $\mathbb{R}^-$ into $\mathbb{R}^n$ $(n \ge 1)$, $<g>$ is the mean value of g in the transverse direction and is given by

$$<g> = \frac{2}{N_z(\underline{x},\varepsilon_z,t)}\int_{\mathcal{Z}(\underline{x},\varepsilon_z,t)}^0 \frac{g(z)}{v_z(\underline{x},z,\varepsilon_z,t)}\,dz, \tag{3.8}$$

where

$$v_z(\underline{x},z,\varepsilon_z,t) = \sqrt{2(\varepsilon_z - \psi(\underline{x},z,t))}, \tag{3.9}$$

and $\varepsilon_z \mapsto \mathcal{Z}(\underline{x},\varepsilon_z,t)$ is the inverse function of the one to one function $z \mapsto \psi(\underline{x},z,t)$ from $\mathbb{R}^-$ onto $\mathbb{R}^+$, then

$$z = \mathcal{Z}(\underline{x},\varepsilon_z,t), \quad \text{for all } \varepsilon_z \in \mathbb{R}^+ \quad \Leftrightarrow \quad \varepsilon_z = \psi(\underline{x},z,t), \quad \text{for all } z \in \mathbb{R}^-. \tag{3.10}$$

The density of states, N_z, is the bounce time between two successive collisions of the particle with the surface and is given by

$$N_z(\underline{x}, \varepsilon_z, t) = 2 \int_{\mathcal{Z}(\underline{x}, \varepsilon_z, t)}^{0} \frac{1}{v_z(\underline{x}, z, \varepsilon_z, t)} \, dz. \tag{3.11}$$

Furthermore

$$F(\underline{x}, \underline{v}, \varepsilon_z, 0) = F_0(\underline{x}, \underline{v}, \varepsilon_z), \tag{3.12}$$

for almost all $(\underline{x}, \underline{v}, \varepsilon_z) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+$ where F_0 is the initial condition of the asymptotic model, given by

$$F_0(\underline{x}, \underline{v}, \varepsilon_z) = \frac{1}{N_z(\underline{x}, \varepsilon_z, 0)} \int_{Z(\underline{x}, \varepsilon_z, 0)}^{0} \frac{f_0(x, \underline{v}, v_z(\underline{x}, z, \varepsilon_z, 0)) + f_0(x, \underline{v}, -v_z(\underline{x}, z, \varepsilon_z, 0))}{v_z(\underline{x}, z, \varepsilon_z, 0)} \, dz, \tag{3.13}$$

where v_z is given by (3.9).

REMARK 1. *First, remark that $\varepsilon_z(v_z, z) = |v_z|^2/2 + \psi(z)$ is the transverse total energy. Let us introduce the changes of variables $(z, v_z) \in \mathbb{R}^- \times \mathbb{R}^+ \mapsto (z, \varepsilon_z)$ and $(z, v_z) \in \mathbb{R}^- \times \mathbb{R}^- \mapsto (z, \varepsilon_z)$. We have $dv_z \, dz = d\varepsilon_z \, dz / \sqrt{2(\varepsilon_z - \psi(z))}$. Let ε_0 be a given positive real number. We have*

$$\int_{\mathbb{R} \times \mathbb{R}^-} \delta(\varepsilon_z - \varepsilon_0) \, dz \, dv_z = 2 \int_{\mathbb{R}^-} \int_{\psi(z)}^{+\infty} \delta(\varepsilon_z - \varepsilon_0) \frac{d\varepsilon_z}{\sqrt{2(\varepsilon_z - \psi(z))}} \, dz.$$

Thanks to (2.7) and Assumption 1, for all $\underline{x} \in \mathbb{R}^2$ and $t \geq 0$, the function $z \mapsto \psi(\underline{x}, z, t)$ is one to one from $\mathbb{R}^-$ onto $\mathbb{R}^+$. Denoting by $\varepsilon_z \mapsto \mathcal{Z}(\underline{x}, \varepsilon_z, t)$ its inverse function and using Fubini's Theorem, we obtain

$$\int_{\mathbb{R} \times \mathbb{R}^-} \delta(\varepsilon_z - \varepsilon_0) \, dz \, dv_z = \int_{\mathbb{R}^+} \delta(\varepsilon_z - \varepsilon_0) \int_{\mathcal{Z}(\varepsilon_z)}^{0} \frac{2}{\sqrt{2(\varepsilon_z - \psi(z))}} \, dz \, d\varepsilon_z = N_z(\underline{x}, \varepsilon_0, t).$$

Thus $N_z(\varepsilon_0) = \int_{\mathbb{R} \times \mathbb{R}^-} \delta(\varepsilon_z - \varepsilon_0) \, dz \, dv_z$ and $N_z(\underline{x}, \varepsilon_z, t) F(\underline{x}, \underline{v}, \varepsilon_z, t) \, d\underline{x} \, d\underline{v} \, d\varepsilon_z$ is the number of particles with a parallel position in $[\underline{x}, \underline{x}+d\underline{x}]$, a parallel velocity in $[\underline{v}, \underline{v}+d\underline{v}]$, a total transverse energy in $[\varepsilon_z, \varepsilon_z + d\varepsilon_z]$ and a transverse position in $[\mathcal{Z}(\underline{x}, \varepsilon_z, t), 0]$.

REMARK 2. *Several extensions of this work can be considered*
1. *First, we could keep the general boundary condition (2.10). In this case it is necessary to introduce the trace of the distribution function on the surface (see [2], [9, 10], [20] or [24]) and the proof becomes more technical.*
2. *The next generalizations are related to Assumption 1. First, it is possible to relax the decay hypothesis on the transverse potential. Moreover, we can consider a transverse potential with a finite limit l for $z \to -\infty$. In this case the asymptotic model holds only for the total transverse energies ε_z in $[0, l]$.*
3. *There is no more difficulty if volume collisions are considered, for example collisions between the considered particles and neutral particles supposed present in the domain. In this case the right hand side of the rescaled Vlasov equation (2.8) is no more 0 but $Q(f^\alpha)$ where Q is a collision operator given by*

$$Q(f^\alpha)(v) = \int_{\mathbb{R}^3} [s(v', v) \, f^\alpha(v') - s(v, v') \, f^\alpha(v)] \, dv',$$

for all $v \in \mathbb{R}^3$ and where $s(v', v) \, dv$ is the probability per unit time for a given particle with velocity v' to be scattered into the volume dv around v by the collision.

4. *Finally, the rigorous proof when the potential is no more given but is calculated by the Poisson equation is a work in progress, see [27]. In this case the Vlasov equation becomes non linear since the potential depends on the particle distribution function. In section 4, the formal limit is performed and in Remark 3 we point out important difficulties which remain to be treated for the rigorous proof.*

Proof of Theorem 3.2:

Let $\alpha > 0$ and f^α be the weak solution to (2.8), (2.9) and (3.1), i.e. satisfying (3.2). Thanks to (3.3), there exists $f \in L^\infty(\Omega \times \mathbb{R}^3 \times [0, T])$ for all $T > 0$ such that f^α tends to f in $L^\infty(\Omega \times \mathbb{R}^3 \times [0, T])$ for the weak-$\star$ topology for all $T > 0$.

We begin this paragraph with the proof of (3.5). First, we use a change of variables in order to express the velocity in term of the total transverse energy ε_z. In this new set of coordinates, it is proved that the distribution function does not depend on the transverse position variable z. Furthermore, still in this coordinate system, the distribution function is even. Then, in the original coordinates the distribution function depends on the transverse position and velocity variables, z and v_z, only through the total transverse energy ε_z.

Finally taking into account these informations in the weak formulation (3.2), we prove (3.7).

Proof of (3.5):

Multiplying (3.2) by α and letting α go to zero, we obtain

$$\int_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^+} f \, (v_z \, \partial_z \varphi - \partial_z \psi \, \partial_{v_z} \varphi) \, dx \, dv \, dt = 0,$$

for all $\varphi \in C_c^1(\overline{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^+)$ and such that $\varphi(\underline{x}, 0, \underline{v}, v_z, t) = \varphi(\underline{x}, 0, \underline{v}, -v_z, t)$ for all $v_z < 0$. Choosing $\varphi(\underline{x}, z, \underline{v}, v_z, t) = \chi(\underline{x}, \underline{v}, t) \, \tilde{\varphi}(z, v_z)$ where $\chi \in C_c^1(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+)$ and $\tilde{\varphi} \in C_c^1((-\infty, 0] \times \mathbb{R})$ such that $\tilde{\varphi}(0, v_z) = \tilde{\varphi}(0, -v_z)$ for all $v_z < 0$, we obtain

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{0} f(\underline{x}, z, \underline{v}, v_z, t) \, (v_z \, \partial_z \tilde{\varphi}(z, v_z) - \partial_z \psi(x, t) \, \partial_{v_z} \tilde{\varphi}(z, v_z)) \, dz \, dv_z = 0 \quad (3.14)$$

for almost all $(\underline{x}, \underline{v}, t) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+$.

Let $\underline{x} \in \mathbb{R}^2$, $t \geq 0$ and $z \leq 0$ be fixed, we consider the changes of variables

$$
\begin{cases}
\mathbb{R}^- & \to & (-\infty, -\sqrt{2\,\psi(\underline{x}, z, t)}] \\
v_z & \mapsto & u_z^-(z, v_z) = -\sqrt{|v_z|^2 + 2\,\psi(\underline{x}, z, t)},
\end{cases}
$$
$$
\begin{cases}
\mathbb{R}^+ & \to & [\sqrt{2\,\psi(\underline{x}, z, t)}, +\infty), \\
v_z & \mapsto & u_z^+(z, v_z) = \sqrt{|v_z|^2 + 2\,\psi(\underline{x}, z, t)}.
\end{cases}
\qquad (3.15)
$$

First, let us remark that u_z^+ and u_z^- are the velocities associated to the total transverse energy $|v_z|^2/2 + \psi(\underline{x}, z, t)$. Then, these change of variables allow to take into account effect of the confining potential on the particle velocity. Note that when the kinetic energy is vanishing, that is when $v_z = 0$, then due to the non zero potential energy, we have, for all $z < 0$ $u_z^-(z, 0) = -u_z^+(z, 0) \neq 0$. Then in the transformation $(z, v_z) \mapsto (z, u_z^\pm)$ the line $v_z = 0$ has two different images (see Figure 3.1).
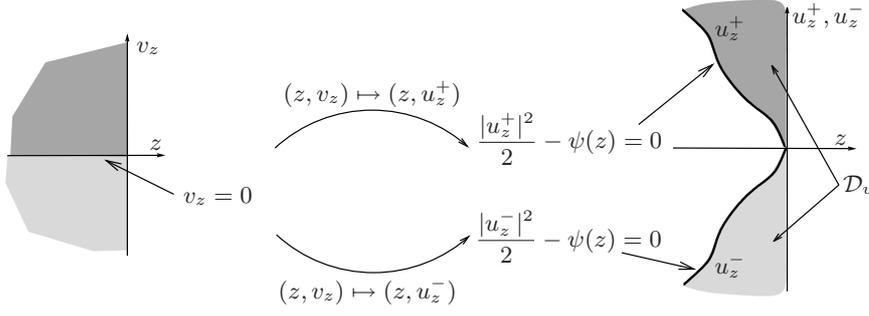
FIG. 3.1. *Changes of variables defined by* $(z, v_z) \mapsto (z, u_z^\pm(z, v_z))$. *The functions* $u_z^\pm$ *are defined by* $sign(v_z) = sign(u_z^\pm)$ *and* $|u_z^\pm|^2/2 = |v_z|^2/2 + \psi(\underline{x}, z, t)$.

Using these changes of variables in (3.14), we obtain

$$\int_{-\infty}^{0} \int_{-\infty}^{-\sqrt{2\,\psi(z)}} f(z, v_z(z, u_z^-))\, \left(u_z^- \, \partial_z \tilde\varphi(z, v_z(z, u_z^-)) \right.$$

$$\left. -\partial_z \psi \, \partial_{v_z} \tilde\varphi(z, v_z(z, u_z^-)) \frac{u_z^-}{v_z(z, u_z^-)} \right)\, du_z^- \, dz$$

$$+ \int_{-\infty}^{0} \int_{\sqrt{2\,\psi(z)}}^{+\infty} f(z, v_z(z, u_z^+))\, \left(u_z^+ \, \partial_z \tilde\varphi(z, v_z(z, u_z^+)) \right.$$

$$\left. -\partial_z \psi \, \partial_{v_z} \tilde\varphi(z, v_z(z, u_z^+)) \frac{u_z^+}{v_z(z, u_z^+)} \right)\, du_z^+ \, dz = 0,$$

where $v_z(z, u_z^\pm) = sign(u_z^\pm) \sqrt{|u_z^\pm|^2 - 2\,\psi(z)}$.

Now, we express the test function as well as the distribution function in terms of the new variables. We define $\mathcal{D}_u = \{(z, u_z) \in \mathbb{R}^- \times \mathbb{R} \; ; \; |u_z|^2/2 - \psi(\underline{x}, z, t) \geq 0\}$, then let $\overline\varphi \in C_c^1(\mathcal{D}_u)$ such that

$$\begin{cases} \overline\varphi(0, u_z) = \overline\varphi(0, -u_z), \; \forall \, u_z \in \mathbb{R}, \\ \overline\varphi(z, -\sqrt{2\,\psi(z)}) = \overline\varphi(z, \sqrt{2\,\psi(z)}), \; \forall \, (\underline{x}, z, t). \end{cases} \tag{3.16}$$

Since $u_z^\pm(0, v_z) = v_z$ and $u_z^+(z, 0) = \sqrt{2\,\psi(z)}$ and $u_z^-(z, 0) = -\sqrt{2\,\psi(z)}$, we can choose $\tilde\varphi(z, v_z) = \overline\varphi(z, u_z(z, v_z))$ where $u_z(z, v_z) = u^+(z, v_z)$ if $v_z \geq 0$ and $u^-(z, v_z)$ otherwise. Remarking that $\partial_z \tilde\varphi = \partial_z \overline\varphi + (\partial_z \psi / u_z)\, \partial_{u_z} \overline\varphi$ and $\partial_{v_z} \tilde\varphi = (v_z / u_z)\, \partial_{u_z} \overline\varphi$, we obtain:

$$\int_{\mathcal{D}_u} \overline{f}(\underline{x}, \underline{v}, z, u_z, t)\, u_z \, \partial_z \overline\varphi(z, u_z)\, du_z \, dz = 0, \tag{3.17}$$

for all $\overline\varphi \in C_c^\infty(\mathcal{D}_u)$ such that $\overline\varphi(0, u_z) = \overline\varphi(0, -u_z)$ for all u_z and $\overline\varphi(z, -\sqrt{2\,\psi(z)}) = \overline\varphi(z, \sqrt{2\,\psi(z)})$ for all z and where $\overline{f}$ is given by $\overline{f}(\underline{x}, \underline{v}, z, u_z, t) = f(\underline{x}, z, \underline{v}, v_z, t)$. This gives

$$\partial_z \overline{f} = 0,$$

for almost all $(z, u_z) \in \mathcal{D}_u$ and almost all $(\underline{x}, \underline{v}, t) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+$ since in $\mathcal{D}_u$, $u_z \neq 0$ almost everywhere. So $\overline{f}$ does not depend on z almost everywhere in $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{D}_u \times \mathbb{R}^+$.

Now let us prove that $\overline{f}$ is even with respect to the variable u_z. We recall that thanks to Assumption 1, for all $\underline{x} \in \mathbb{R}^2$ and $t \geq 0$, the function $z \mapsto \psi(\underline{x}, z, t)$ is one to one from $\mathbb{R}^-$ onto $\mathbb{R}^+$. Then, using Fubini's Theorem, Eq. (3.17) can be rewritten

$$\int_{\mathbb{R}} \overline{f}(\underline{x}, \underline{v}, u_z, t)\, u_z \int_{\mathcal{Z}(\underline{x}, |u_z|^2/2, t)}^{0} \partial_z \overline{\varphi}(z, u_z)\, dz\, du_z$$
$$= \int_{\mathbb{R}} \overline{f}(\underline{x}, \underline{v}, u_z, t)\, u_z \left(\overline{\varphi}(0, u_z) - \overline{\varphi}(\mathcal{Z}(\underline{x}, |u_z|^2/2, t), u_z) \right) du_z = 0,$$

where $\mathcal{Z}$ is defined by (3.10).

Now, we choose $\overline{\varphi}(z, u_z) = z\, \varphi(u_z)$ with $\varphi \in C^1_c(\mathbb{R})$ and φ even. We note that this function is admissible since it satisfies (3.16). Then we have

$$\int_{\mathbb{R}} \overline{f}(\underline{x}, \underline{v}, u_z, t)\, u_z\, \mathcal{Z}(\underline{x}, |u_z|^2/2, t)\, \varphi(u_z)\, du_z = 0.$$

Using the change of variables $u_z \mapsto -u_z$ from $\mathbb{R}^+$ onto $\mathbb{R}^-$, we obtain

$$\int_{-\infty}^{0} \mathcal{Z}(\underline{x}, |u_z|^2/2, t)\, u_z \left(\overline{f}(\underline{x}, \underline{v}, u_z, t) - \overline{f}(\underline{x}, \underline{v}, -u_z, t) \right) \varphi(u_z)\, du_z = 0$$

for all $\varphi \in C^1_c(\mathbb{R}^-)$. This means that $u_z \mapsto \overline{f}(\underline{x}, \underline{v}, u_z, t)$ is almost everywhere even. So, there exists $F \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times [0, T])$ for all $T > 0$ such that

$$f(\underline{x}, z, \underline{v}, v_z, t) = \overline{f}(\underline{x}, \underline{v}, u_z, t) = F(\underline{x}, \underline{v}, |u_z|^2/2, t)$$

for almost all $(x, v, t) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^+$. This concludes the proof of (3.5).

Proof of (3.7):
In (3.2) we choose an even test function with respect to the variable v_z and we use the change of variables $v_z \mapsto -v_z$ from $\mathbb{R}^-$ onto $\mathbb{R}^+$, this gives

$$\int_{\Omega \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+} \left(f^\alpha(v_z) + f^\alpha(-v_z) \right) \left(\partial_t + \underline{v} \cdot \nabla_{\underline{x}} - \nabla_{\underline{x}}(\phi_0 + \psi) \cdot \nabla_{\underline{v}} \right) \varphi\, dx\, d\underline{v}\, dv_z\, dt \quad (3.18)$$
$$+ \frac{1}{\alpha} \int_{\Omega \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+} \left(f^\alpha(v_z) - f^\alpha(-v_z) \right) \left(v_z\, \partial_z - \partial_z \psi\, \partial_{v_z} \right) \varphi\, dx\, d\underline{v}\, dv_z\, dt$$
$$+ \int_{\Omega \times \mathbb{R}^2 \times \mathbb{R}^+} \left(f_0(v_z) + f_0(-v_z) \right) \varphi(x, \underline{v}, v_z, 0)\, dx\, d\underline{v}\, dv_z = 0$$

for all $\varphi \in C^1_c(\bar{\Omega} \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+)$,

We introduce the total transverse energy ε_z defined by (3.6). It defines a change of variables:

$$\begin{array}{rcl} \mathbb{R}^+ & \to & [\psi(\underline{x}, z, t), +\infty) \\ v_z & \mapsto & \varepsilon_z = |v_z|^2/2 + \psi(\underline{x}, z, t). \end{array} \quad (3.19)$$

In (3.18) we choose test functions depending on v_z only through the transverse total energy, that is $\varphi(\underline{x}, z, \underline{v}, v_z, t) = \tilde{\varphi}(\underline{x}, \underline{v}, z, \varepsilon_z, t)$ for all $(\underline{x}, z, \underline{v}, v_z, t) \in \bar{\Omega} \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+$. Note that $\varphi \in C^1_c(\Omega \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+)$ is equivalent to $\tilde{\varphi} \in C^1_c(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{D}_\varepsilon \times \mathbb{R}^+)$ where $\mathcal{D}_\varepsilon = \{(z, \varepsilon_z) \in \mathbb{R}^- \times \mathbb{R}^+ \; ; \; \varepsilon_z - \psi(\underline{x}, z, t) \geq 0\}$.

Using the change of variables (3.19), Eq. (3.18) becomes

$$
\int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{D}_\varepsilon \times \mathbb{R}^+} \Big(f^\alpha(v_z) + f^\alpha(-v_z) \Big) \times
$$

$$
\times \Big(\partial_t + \underline{v} \cdot \nabla_{\underline{x}} - \nabla_{\underline{x}}(\phi_0 + \psi) \cdot \nabla_{\underline{v}} + \big(\partial_t \psi + \underline{v} \cdot \nabla_{\underline{x}} \psi \big) \partial_{\varepsilon_z} \Big) \, \tilde{\varphi} \, d\underline{x} \, d\underline{v} \, dz \, \frac{d\varepsilon_z}{v_z} \, dt
$$

$$
+ \frac{1}{\alpha} \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{D}_\varepsilon \times \mathbb{R}^+} \Big(f^\alpha(v_z) + f^\alpha(-v_z) \Big) \partial_z \tilde{\varphi} \, d\underline{x} \, d\underline{v} \, dz \, d\varepsilon_z \, dt
$$

$$
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{D}_\varepsilon} \Big(f_0(v_z(\underline{x}, z, \varepsilon_z, 0)) + f_0(-v_z(\underline{x}, z, \varepsilon_z, 0)) \Big) \, \tilde{\varphi} \, d\underline{x} \, d\underline{v} \, dz \, \frac{d\varepsilon_z}{v_z(\underline{x}, z, \varepsilon_z, 0)} \, dt = 0,
$$

with $v_z = v_z(\underline{x}, z, \varepsilon_z, t) = \sqrt{2\left(\varepsilon_z - \psi(\underline{x}, z, t)\right)}$.

We recall that $\mathcal{Z}(\underline{x}, \varepsilon_z, t)$ is defined by (3.10). Then using Fubini's Theorem, choosing as a test function $\tilde{\varphi}(\underline{x}, z, \underline{v}, \varepsilon_z, t) = \bar{\varphi}(\underline{x}, \underline{v}, \varepsilon_z, t)$ where $\bar{\varphi} \in C_c^1(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+)$, we obtain

$$
\int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+} \left(\int_{\mathcal{Z}(\underline{x}, \varepsilon_z, t)}^0 \frac{f^\alpha(v_z) + f^\alpha(-v_z)}{\sqrt{2(\varepsilon_z - \psi(\underline{x}, z, t))}} \, dz \right) \Big(\partial_t + \underline{v} \cdot \nabla_{\underline{x}} - \nabla_{\underline{x}} \phi_0 \cdot \nabla_{\underline{v}} \Big) \bar{\varphi} \, d\underline{x} \, d\underline{v} \, d\varepsilon_z \, dt
$$

$$
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+} \left[- \left(\int_{\mathcal{Z}(\underline{x}, \varepsilon_z, t)}^0 \frac{f^\alpha(v_z) + f^\alpha(-v_z)}{\sqrt{2(\varepsilon_z - \psi(\underline{x}, z, t))}} \, \nabla_{\underline{x}} \psi(\underline{x}, z, t) \, dz \right) \cdot \nabla_{\underline{v}} \bar{\varphi} \right.
$$

$$
\left. + \left(\int_{\mathcal{Z}(\underline{x}, \varepsilon_z, t)}^0 \frac{f^\alpha(v_z) + f^\alpha(-v_z)}{\sqrt{2(\varepsilon_z - \psi(\underline{x}, z, t))}} \big(\partial_t \psi + \underline{v} \cdot \nabla_{\underline{x}} \psi \big) \, dz \right) \partial_{\varepsilon_z} \bar{\varphi} \right] d\underline{x} \, d\underline{v} \, d\varepsilon_z \, dt
$$

$$
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+} \left(\int_{\mathcal{Z}(\underline{x}, \varepsilon_z, 0)}^0 \frac{f_0(v_z(\underline{x}, z, \varepsilon_z, 0)) + f_0(-v_z(\underline{x}, z, \varepsilon_z, 0))}{\sqrt{2(\varepsilon_z - \psi(\underline{x}, z, 0))}} \, dz \right) \bar{\varphi} \, d\underline{x} \, d\underline{v} \, d\varepsilon_z = 0.
$$

We pass to the limit α tends to zero in the previous equality, using (3.4) and (3.5). This gives, for all $\bar{\varphi} \in C_c^1(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+)$:

$$
\int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+} F \, N_z \left(\partial_t + \underline{v} \cdot \nabla_{\underline{x}} - \nabla_{\underline{x}} \phi_0 \cdot \nabla_{\underline{v}} \right) \bar{\varphi} \, d\underline{x} \, d\underline{v} \, d\varepsilon_z \, dt +
$$

$$
\int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+} F \, N_z \left(- <\nabla_{\underline{x}} \psi> \cdot \nabla_{\underline{v}} \varphi + \Big(<\partial_t \psi> + \underline{v} \cdot <\nabla_{\underline{x}} \psi> \Big) \partial_{\varepsilon_z} \bar{\varphi} \right) dt \, d\varepsilon_z \, d\underline{v} \, d\underline{x}
$$

$$
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+} N_z(\underline{x}, \varepsilon_z, 0) \, F_0(\underline{x}, \underline{v}, \varepsilon_z) \, \bar{\varphi}(\underline{x}, \underline{v}, \varepsilon_z, 0) \, d\underline{x} \, d\underline{v} \, d\varepsilon_z = 0,
$$

where N_z is the density of states and is given by (3.11), and where $<\nabla_{\underline{x}} \psi>$ and $<\partial_t \psi>$ are defined by (3.8), finally F_0 is the function given by (3.13).

This is the weak form of (3.7) and (3.12) and concludes the proof of Theorem 3.2.

4. The formal asymptotic limit for charged particles. In this section, we are interested in the formal limit of problem (2.8), (2.7), (2.9) and (2.10) for charged particles. In this case the potential is no longer given but has to be calculated using the Poisson equation. The problem becomes non linear with respect to the distribution function, since the potential depends linearly on the particle distribution function. This section is organized as follows. In Section 4.1, we present the Vlasov-Poisson model in dimensionless variables. In Section 4.2 we formally pass to the limit in the re-scaled problem and then we derive the asymptotic model in the non linear case.

4.1. The Vlasov-Poisson model in physical and re-scaled variables. We are interested in the motion of trapped charged particles in a surface potential and subject to collisions with this surface. We are interested in the asymptotic model obtained when the specular collisions are the dominant collision process on the surface and when the applied potential confines the particles close to the surface.

Then we consider the Vlasov equation for the particle distribution function in physical variables

$$\partial_t f + v \cdot \nabla_x f - \frac{q}{m} \nabla_x (\phi + \phi_s) \cdot \nabla_v f = 0,$$

for all $x = (\underline{x}, z) \in \Omega = \mathbb{R}^2 \times (-\infty, 0)$, all $v = (\underline{v}, v_z) \in \mathbb{R}^3$, all $t \in \mathbb{R}^+$ and where q and m are respectively the charge and the mass of the particles and ϕ is a given external potential. It is coupled to the Poisson equation for the self-consistent potential

$$-\varepsilon_0 \Delta_x \phi_s = q \int_{\mathbb{R}^3} f(v)\, dv, \tag{4.1}$$

for $(x, t) \in \Omega \times \mathbb{R}^+$ and where ε_0 is the vacuum permittivity.

We add to this system some boundary and initial conditions for the Vlasov equation

$$f(t = 0) = f_0, \text{ in } \Omega \times \mathbb{R}^3$$
$$\gamma^-(f)(v_z) = \beta\, \mathcal{S}\gamma^+(f)(v_z) + (1 - \beta)\, \mathcal{K}\gamma^+(f)(v_z), \quad \text{for } v_z < 0,$$

where f_0 is a given function and where we recall that $\gamma^+(f)$ and $\gamma^-(f)$ are the incoming and outgoing traces of f on $\partial\Omega = \mathbb{R}^2 \times \{0\}$. Furthermore $\mathcal{S}$ and $\mathcal{K}$ are the surface collisions operators given by (2.4) and (2.5) and β is the probability for an incoming particle on the surface $\partial\Omega$ to be re-emitted specularly.

Finally, we must specify the boundary conditions for the Poisson equation. Several choices are possible, they depend on the considered physical problem. Here we choose an homogeneous Neumann boundary condition on the surface $\partial\Omega$ for which we have an analytic solution, then we set:

$$\begin{cases} \partial_z \phi_s(\underline{x}, 0, t) = 0 \text{ for } (\underline{x}, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\ \lim_{|x| \to +\infty} \phi_s(x, t) = 0 \text{ for } t \in \mathbb{R}^+. \end{cases}$$

We recall that in the regime we are interested in, the external potential ϕ confines the particles close to the surface $\partial\Omega$. Then as in the linear case (see Section 3) we denote by $L_\parallel$ and $L_\perp$ the characteristic lengths respectively in the parallel and transverse directions and we assume $\alpha = L_\perp/L_\parallel$ is a small dimensionless parameter. We assume the specular collisions dominant on the surface. Then we set $\beta = 1 - \alpha$. We denote by ϕ_c, $v_c = \sqrt{|q\,\phi_c|/m}$, $t_c = L_\parallel/v_c$ and f_c the characteristic potential, velocity, time and distribution function. We introduce a re-scaling for the Vlasov-Poisson system denoted by $\hat{x} = (\hat{\underline{x}}, \hat{z})$, $\hat{v}$ and $\hat{t}$ being defined by (2.6). We suppose that there exist $\hat{f}_0$ and $\hat{\phi}$, independent of α such that:

$$\phi(x, t) = \phi_c\, \hat{\phi}(\hat{x}, \hat{t}), \quad \text{for } (x, t) \in \Omega \times \mathbb{R}^+,$$
$$f_0(x, v) = f_c\, \hat{f}_0(\hat{x}, \hat{v}), \quad \text{for } (x, v) \in \Omega \times \mathbb{R}^3.$$

Finally following [7], we remark that the Poisson equation is a non local operator. Thus the support of the self-consistent potential is not necessarily of the same order

as the support of the distribution function. Then we define f^α and ϕ_s^α by

$$f(x, v, t) = f_c \, f^\alpha(\hat{x}, \hat{v}, \hat{t}) \quad \text{for } (x, v, t) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^+,$$

$$\phi_s(x, t) = \phi_c \, \phi_s^\alpha(\underline{\hat{x}}, \alpha \hat{z}, \hat{t}) \quad \text{for } (x, t) \in \Omega \times \mathbb{R}^+.$$

This new scale leads to a dimensionless parameter λ. It is the ratio of two densities and is defined by

$$\lambda = f_c \, v_c^3 \, \frac{q \, |L_\parallel|^2}{\varepsilon_0 \, \phi_c}$$

Since, the particles are confined close to the surface, all the mass is localized in a small volume near $\partial\Omega$. Then, the number of particles is very large in the domain and we assume that $\alpha \, f_c \, v_c^3 = \frac{\varepsilon_0 \, \phi_c}{q \, |L_\parallel|^2}$, that is $\lambda = 1/\alpha$. Inserting this new scale in the Vlasov-Poisson system and omitting the "hats", we obtain the re-scaled Vlasov equation for the particle distribution function

$$\partial_t f^\alpha(z) + \underline{v} \cdot \nabla_{\underline{x}} f^\alpha(z) - \left(\nabla_{\underline{x}}\phi(z) + \nabla_{\underline{x}}\phi_s^\alpha(\alpha \, z)\right) \cdot \nabla_{\underline{v}} f^\alpha(z) - \partial_z \phi_s^\alpha(\alpha \, z) \, \partial_{v_z} f^\alpha(z) \quad (4.2)$$

$$+ \frac{1}{\alpha}\left(v_z \, \partial_z f^\alpha(z) - \partial_z \psi(z) \, \partial_{v_z} f^\alpha(z)\right) = 0,$$

where ϕ is a given external potential and ψ is its transverse component defined by (2.7). It is coupled to the re-scaled Poisson equation

$$-\Delta_{\underline{x}}\phi_s^\alpha(\underline{x}, \alpha \, z, t) - \partial_{zz}^2 \phi_s^\alpha(\underline{x}, \alpha \, z, t) = \frac{1}{\alpha}\int_{\mathbb{R}^3} f^\alpha(\underline{x}, z, v, t) \, dv, \qquad (4.3)$$

for $(\underline{x}, z) \in \Omega$ and all $t \geq 0$.

The boundary and initial conditions are given by

$$\gamma^-(f^\alpha)(v_z) = (1 - \alpha) \, \mathcal{S}\gamma^+(f^\alpha)(v_z) + \alpha \, \mathcal{K}\gamma^+(f^\alpha)(v_z), \quad \text{for } v_z < 0, \qquad (4.4)$$

$$\partial_z \phi_s^\alpha(z = 0) = 0 \text{ on } \partial\Omega \times \mathbb{R}^+, \qquad (4.5)$$

$$\lim_{|x|\to+\infty} \phi_s^\alpha(x, \cdot) = 0 \text{ in } \mathbb{R}^+, \qquad (4.6)$$

$$f^\alpha(t = 0) = f_0, \text{ in } \Omega \times \mathbb{R}^3, \qquad (4.7)$$

where $\mathcal{S}$ is given by (2.4) and $\mathcal{K}$ by (2.5).

4.2. The formal non linear asymptotic limit. In this section we present the asymptotic model obtained by formally passing to the limit $\alpha \to 0$ in (4.2)-(4.7).

THEOREM 4.1 (Formal). *Let $\alpha > 0$, f_0 and ϕ be given functions, we denote by ϕ_0 and ψ the parallel and transverse components of ϕ defined by (2.7). We suppose that ψ satisfies Assumption 1. Let f^α, ϕ_s^α be the solutions to the re-scaled Vlasov-Poisson model (4.2)-(4.7). Then, the formal limit $\alpha \to 0$ gives $f^\alpha \to f$ and $\phi_s^\alpha \to \phi_s$ where*

$$f(x, v, t) = F(\underline{x}, \underline{v}, \varepsilon_z, t), \qquad (4.8)$$

with ε_z defined by (3.6) and where F satisfies, for $(\underline{x}, \underline{v}, \varepsilon_z, t) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+$,

$$\partial_t(N_z F) + \underline{v} \cdot \nabla_{\underline{x}}(N_z F) - \left(\nabla_{\underline{x}}\tilde{\phi}_s + \nabla_{\underline{x}}\phi_0 + < \nabla_{\underline{x}}\psi >\right) \cdot \nabla_{\underline{v}}(N_z F) \qquad (4.9)$$

$$+ \partial_{\varepsilon_z}\left(\left(< \partial_t \psi > + \underline{v} \cdot < \nabla_{\underline{x}}\psi >\right) N_z F\right) = \mathcal{K}(F) - F,$$

where N_z is the density of states given by (3.11), the function $\tilde{\phi}_s$ is the trace of the self consistent potential ϕ_s on the surface $\partial\Omega$, that is for all $t \geq 0$ we have

$$\begin{cases} \tilde{\phi}_s(\underline{x}, t) = \phi_s(\underline{x}, 0, t), & \text{for all} \quad \underline{x} \in \mathbb{R}^2, \\ -\Delta_{\underline{x}, z}\phi_s = 0, & \text{in} \quad \mathbb{R}^2 \times (-\infty, 0), \\ \partial_z \phi_s(\underline{x}, 0, t) = \rho(\underline{x}, t), & \text{for all} \quad \underline{x} \in \mathbb{R}^2, \\ \lim_{|(\underline{x}, z)| \to +\infty} \phi_s(\underline{x}, z, t) = 0, \end{cases} \tag{4.10}$$

where the density of particles is given by

$$\rho(\underline{x}, t) = \int_{\mathbb{R}^2 \times \mathbb{R}^+} N_z(\underline{x}, \varepsilon_z, t) \, F(\underline{x}, \underline{v}, \varepsilon_z, t) \, d\varepsilon_z \, d\underline{v}.$$

For all functions g from $\mathbb{R}^-$ into $\mathbb{R}^n$ $(n \geq 1)$, $< g >$ is the mean value of g in the transverse direction and is given by (3.8).

Finally the operator $\mathcal{K}$ is a collision operator which takes into account the non specular part of surface collisions. It is such that:

$$\mathcal{K}(F)(\underline{x}, \underline{v}, \varepsilon_z, t) = \int_{\mathbb{R}^2} \int_0^{+\infty} K\left((\underline{v}', \sqrt{2\,\varepsilon_z'}), (\underline{v}, -\sqrt{2\,\varepsilon_z}) \right) F(\underline{x}, \underline{v}', \varepsilon_z', t) \, d\varepsilon_z' \, d\underline{v}', \tag{4.11}$$

for all $(\underline{x}, \underline{v}, \varepsilon_z, t) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+$.

REMARK 3. *Let us assume that* $\underline{x} \mapsto \rho(\underline{x}, t) = \int N_z \, F \, d\varepsilon_z \, d\underline{v}$ *is in* $L^1(\mathbb{R}^2)$ *for all* $t \geq 0$. *We introduce, for all* $t \geq 0$ *the function* $V \in W_{loc}^{1,1}(\mathbb{R}^2 \times \mathbb{R})$ *defined by*

$$V(\underline{x}, z) = \frac{2}{4\,\pi} \int_{\mathbb{R}^3} \frac{\delta(z')\,\rho(\underline{x}', t)}{\sqrt{|\underline{x} - \underline{x}'|^2 + |z - z'|^2}} \, dz' \, d\underline{x}' = \frac{1}{2\,\pi} \int_{\mathbb{R}^2} \frac{\rho(\underline{x}', t)}{\sqrt{|\underline{x} - \underline{x}'|^2 + |z|^2}} \, d\underline{x}', \tag{4.12}$$

for all $(\underline{x}, z) \in \mathbb{R}^3$ *and where* δ *is the Dirac delta distribution. This function satisfies the following elliptic problem:*

$$\begin{cases} -\Delta_{\underline{x}, z}V = 2\,\delta(z)\,\rho(\underline{x}, t), & \text{for } (\underline{x}, z) \in \mathbb{R}^2 \times \mathbb{R}, \\ \lim_{|(\underline{x}, z)| \to +\infty} V = 0. \end{cases}$$

Now, remark that V *is symmetric that is* $V(\underline{x}, z) = V(\underline{x}, -z)$ *for all* $(\underline{x}, z) \in \mathbb{R}^3$. *Then, choosing a symmetric test function in the weak formulation of the previous problem, we obtain*

$$\int_{\mathbb{R}^2 \times \mathbb{R}^-} \nabla_{\underline{x}, z}V(\underline{x}, z) \cdot \nabla_{\underline{x}, z}\varphi(\underline{x}, z) \, d\underline{x}, dz - \int_{\mathbb{R}^2} \rho(\underline{x}, t)\, \varphi(\underline{x}, 0) \, d\underline{x} = 0,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^-)$. *This is the weak formulation of (4.10) and then*

$$\phi_s = V_{|\mathbb{R}^2 \times \mathbb{R}^-}. \tag{4.13}$$

Remark that $\phi_s \in W_{loc}^{1,1}(\mathbb{R}^2 \times \mathbb{R}^-)$ *for all* $t \geq 0$ *and then the trace of* ϕ_s *exists in* $L^1(\partial\Omega)$ *that is* $\phi_s(\underline{x}, 0, t) = \tilde{\phi}_s(\underline{x}, t)$ *exists for almost all* $(\underline{x}, t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

It is important to note that this L^1 regularity is not sufficient to ensure the existence of a weak solution to the system (4.9), (4.10). This existence result is the first

step of the rigorous proof. Then, like in [14] for the case of the whole space, some additional estimates are needed.

Proof of Theorem 4.1: (formal)

Let $\alpha > 0$ and f^α, ϕ_s^α be the solutions to (4.2)-(4.7). We assume that ϕ_s^α tends to ϕ_s and f^α tends to f when α tends to 0.

The proof of (4.8) is similar to that of (3.2), (3.5) in Theorem 3.2. Then we do not detail it. Let us prove (4.9). For all $\underline{x} \in \mathbb{R}^2$, $z \in \mathbb{R}^-$, $v_z \in \mathbb{R}$, and $t \geq 0$ we define ε_z by (3.6). Let $\varepsilon_z' \in [0, +\infty)$, we multiply (4.2) by $\delta(\varepsilon_z - \varepsilon_z')$ and we integrate the result on $\mathbb{R}^- \times \mathbb{R}$ with respect to z and v_z, we obtain

$$B_1^\alpha + B_2^\alpha = 0, \tag{4.14}$$

where

$$B_1^\alpha = \int_{\mathbb{R}^- \times \mathbb{R}} \left[\partial_t + \underline{v} \cdot \nabla_{\underline{x}} + \left(\nabla_{\underline{x}} \phi(z) + \nabla_{\underline{x}} \phi_s^\alpha(\alpha\, z) \right) \cdot \nabla_{\underline{v}} + \partial_z \phi_s^\alpha(\alpha\, z)\, \partial_{v_z} \right] \times$$
$$\times f^\alpha(z)\, \delta(\varepsilon_z - \varepsilon_z')\, dv_z\, dz,$$

and

$$B_2^\alpha = \frac{1}{\alpha} \int_{\mathbb{R}^- \times \mathbb{R}} \left(v_z\, \partial_z f^\alpha(z) + \partial_z \psi(z)\, \partial_{v_z} f^\alpha(z) \right) \delta(\varepsilon_z - \varepsilon_z')\, dv_z\, dz.$$

Now, using the same arguments as in Remark 3, we can prove that $\phi_s^\alpha(\underline{x}, z, t) = V^\alpha(\underline{x}, z, t)$ where for all $t \geq 0$, $V^\alpha(\cdot, \cdot, t)$ is the solution to

$$\begin{cases} -\Delta_{\underline{x}, z} V^\alpha(\underline{x}, z, t) = \dfrac{1}{\alpha}\, \overline{\rho}^\alpha(\underline{x}, z/\alpha, t), & \text{for } (\underline{x}, z) \in \mathbb{R}^3, \\ \lim\limits_{|(\underline{x}, z)| \to +\infty} V^\alpha = 0, \end{cases}$$

with $\overline{\rho}^\alpha(\underline{x}, z, t) = \int_{\mathbb{R}^3} f^\alpha(\underline{x}, z, v, t)\, dv$ if $z \leq 0$ and $\overline{\rho}^\alpha(\underline{x}, z, t) = \int_{\mathbb{R}^3} f^\alpha(\underline{x}, -z, v, t)\, dv$ otherwise.

An easy calculation gives

$$\partial_z \phi_s^\alpha(\underline{x}, \alpha\, z, t) = \frac{\alpha}{4\,\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^-} \left(\int_{\mathbb{R}^3} f^\alpha(\underline{x}, z, v, t)\, dv \right) \times$$
$$\left(\frac{z - z'}{\left(|\underline{x} - \underline{x}'|^2 + |\alpha z - \alpha z'|^2 \right)^{3/2}} + \frac{z + z'}{\left(|\underline{x} - \underline{x}'|^2 + |\alpha z + \alpha z'|^2 \right)^{3/2}} \right) d\underline{x}'\, dz',$$

and then, we remark that

$$\lim_{\alpha \to 0} \partial_z \phi_s^\alpha(\alpha\, z) = 0.$$

Passing to the limit $\alpha \to 0$ in B_1^α, using Remark 1 as well as definitions (3.8) and (3.11), we obtain (see the proof of (3.7) for more details)

$$\lim_{\alpha \to 0} B_1^\alpha = \left[\partial_t + \underline{v} \cdot \nabla_{\underline{x}} - \left(\nabla_{\underline{x}} \phi_s(\underline{x}, 0, t) + < \nabla_{\underline{x}} \phi > (\underline{x}, t) \right) \cdot \nabla_{\underline{v}} \right] (N_z\, F)(\underline{x}, \underline{v}, \varepsilon_z', t)$$
$$+ \partial_{\varepsilon_z} \left(\left(< \partial_t \psi > (\underline{x}, t) + \underline{v} \cdot < \nabla_{\underline{x}} \psi > (\underline{x}, t) \right) N_z\, F \right)(\underline{x}, \underline{v}, \varepsilon_z', t).$$

It remains to pass to the limit in B_2^α. We denote by $\bar{f}^\alpha$ the function defined by $f^\alpha(\underline{x}, z, \underline{v}, v_z, t) = \bar{f}^\alpha(\underline{x}, z, \underline{v}, u_z, t)$ for all $(\underline{x}, z) \in \Omega$, all $v \in \mathbb{R}^3$ and all $t \geq 0$. We begin by transforming B_2^α with the changes of variables (3.15) and we use Fubini's Theorem. We obtain

$$B_2^\alpha = \frac{1}{\alpha} \int_\mathbb{R} \left[\int_{\mathcal{Z}(\underline{x}, |u_z|^2/2, t)}^0 \partial_z \bar{f}^\alpha(z, u_z) \, dz \right] \delta(|u_z|^2/2 - \varepsilon_z') \, u_z \, du_z.$$

This can be rewritten, using the change of variables $u_z \mapsto -u_z$,

$$B_2^\alpha = \frac{1}{\alpha} \int_0^{+\infty} \left[\gamma^+(\bar{f}^\alpha)(u_z) - \bar{f}^\alpha(\mathcal{Z}(|u_z|^2/2), u_z) - \gamma^-(\bar{f}^\alpha)(-u_z) + \bar{f}^\alpha(\mathcal{Z}(|u_z|^2/2), -u_z) \right]$$
$$\delta(|u_z|^2/2 - \varepsilon_z') \, u_z \, du_z.$$

Let us remark that $z = \mathcal{Z}(|u_z|^2/2) \Leftrightarrow \psi(z) = |u_z|^2/2 \Leftrightarrow v_z = 0$. Then

$$\bar{f}^\alpha(\mathcal{Z}(|u_z|^2/2), u_z) = f^\alpha(\mathcal{Z}(|u_z|^2/2), 0) = \bar{f}^\alpha(\mathcal{Z}(|u_z|^2/2), -u_z).$$

Furthermore, if $z = 0$, we have $\psi(0) = 0$ and then $u_z = v_z$. So $\gamma^+(\bar{f}^\alpha)(u_z) = \gamma^+(f^\alpha)(u_z)$ and $\gamma^-(\bar{f}^\alpha)(-u_z) = \gamma^-(f^\alpha)(-u_z)$ for all $u_z > 0$. Using boundary condition (4.4) and definition (2.4), we deduce

$$B_2^\alpha = \int_0^{+\infty} \left(\mathcal{K}(\gamma^+(f^\alpha))(-u_z) - \gamma^+(f^\alpha)(u_z) \right) \delta(|u_z|^2/2 - \varepsilon_z') \, u_z \, du_z.$$

Letting α tend to zero, using (4.8) and the change of variables $u_z \mapsto \varepsilon_z = |u_z|^2/2$ from $\mathbb{R}^+$ to $\mathbb{R}^+$, we get

$$\lim_{\alpha \to 0} B_2^\alpha = \int_0^{+\infty} \left(\mathcal{K}(F)(-\sqrt{2\,\varepsilon_z}) - F(\varepsilon_z) \right) \delta(\varepsilon_z - \varepsilon_z') \, d\varepsilon_z = \mathcal{K}(F)(-\sqrt{2\,\varepsilon_z'}) - F(\varepsilon_z'),$$

where $\mathcal{K}(F)$ is given by (4.11). We conclude the proof of (4.9) by passing to the limit in (4.14).

Now, we turn to the proof of (4.10). Let ϕ_s^α be the solution to the re-scaled Poisson equation (4.3), (4.5), (4.6). It satisfies the weak formulation

$$\int_{\mathbb{R}^2 \times \mathbb{R}^-} \nabla_{\underline{x}} \phi_s^\alpha(\underline{x}, \alpha\,z, t) \cdot \nabla_{\underline{x}} \varphi(\underline{x}, z) \, d\underline{x} \, dz + \frac{1}{\alpha} \int_{\mathbb{R}^2 \times \mathbb{R}^-} \partial_z \phi_s^\alpha(\underline{x}, \alpha\,z, t) \, \partial_z \varphi(\underline{x}, z) \, d\underline{x} \, dz$$
$$= \frac{1}{\alpha} \int_{\mathbb{R}^2 \times \mathbb{R}^-} \rho^\alpha(\underline{x}, z, t) \, \varphi(\underline{x}, z) \, d\underline{x} \, dz,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^2 \times (-\infty, 0])$ and where $\rho^\alpha = \int_{\mathbb{R}^3} f^\alpha \, dv$.

Let $\bar{\varphi} \in C_c^\infty(\mathbb{R}^2 \times (-\infty, 0])$. In the weak formulation, we choose $\varphi(\underline{x}, z, t) = \bar{\varphi}(\underline{x}, \alpha\,z)$ for all $(\underline{x}, z) \in \mathbb{R}^2 \times \mathbb{R}^-$. In the left hand side term, we use the change of variables $z \mapsto \bar{z} = \alpha\,z$. Omitting the "overhead bars", we obtain

$$\int_{\mathbb{R}^2 \times \mathbb{R}^-} \nabla_{(\underline{x}, z)} \phi_s^\alpha(\underline{x}, z, t) \cdot \nabla_{(\underline{x}, z)} \varphi(\underline{x}, z) \, d\underline{x} \, dz = \int_{\mathbb{R}^2 \times \mathbb{R}^-} \rho^\alpha(\underline{x}, z, t) \, \varphi(\underline{x}, \alpha\,z) \, d\underline{x} \, dz.$$

Passing to the limit $\alpha \to 0$ yields

$$\int_{\mathbb{R}^2 \times \mathbb{R}^-} \nabla_{(\underline{x}, z)} \phi_s(\underline{x}, z, t) \cdot \nabla_{(\underline{x}, z)} \varphi(\underline{x}, z) \, d\underline{x} \, dz = \int_{\mathbb{R}^2} \int_{\mathbb{R}^- \times \mathbb{R}^3} f(\underline{x}, z, v, t) \, dv \, dz \, \varphi(\underline{x}, 0) \, d\underline{x}.$$

Now, using (4.8) and change of variables (3.19), we remark that

$$\int_{\mathbb{R}^-} \int_{\mathbb{R}^3} f(\underline{x}, z, \underline{v}, v_z, t) \, dv \, dz = \int_{\mathbb{R}^2} \int_{\mathbb{R}^- \times \mathbb{R}} F(\underline{x}, \underline{v}, |v_z|^2/2 + \psi(z), t) \, dv_z \, dz \, d\underline{v}$$

$$= \int_{\mathbb{R}^2 \times \mathbb{R}^+} N_z \, F(\underline{x}, \underline{v}, \varepsilon_z, t) \, d\varepsilon_z \, d\underline{v} = \rho(\underline{x}, t).$$

Then for all $\varphi \in C_c^\infty(\mathbb{R}^2 \times (-\infty, 0])$, ϕ_s satisfies

$$\int_{\mathbb{R}^2 \times \mathbb{R}^-} \nabla_{(\underline{x}, z)} \phi_s(\underline{x}, z, t) \cdot \nabla_{(\underline{x}, z)} \varphi(\underline{x}, z) \, d\underline{x} \, dz - \int_{\mathbb{R}^2} \rho(\underline{x}, t) \, \varphi(\underline{x}, 0) \, d\underline{x} = 0.$$

This means that ϕ_s is the weak solution to (4.10) and this concludes the proof of Proposition 4.1.

5. Numerical simulations of a surface discharge. In this section, we are interested in a physical application for which we can use the previous asymptotic model and perform numerical simulations. The physical application concerns surface discharges. These phenomena occur during the triggering of electrical arcs on high voltage satellite solar generators.

This section is organized as follows: we begin with the presentation of the physical scenario and of the model. Then we describe the numerical discretization. We finish with the presentation of the numerical results.

5.1. The physical scenario of the discharge. Satellite solar arrays are constituted of strings of photoelectric cells. Each string contains several tens of cells and deliver about 50 volts. In order to respond to high power demands, constructors increase the number of cells and so the voltage. When this voltage exceeds 50 volts some electrical arcs may occur and damage the satellite. According to the literature (see [11, 12]), there are two main steps in the physical process. First, a primary discharge occurs and creates a high density plasma in the gap separating two solar cells. When the potential difference between the two solar cells is sufficient and when the plasma has filled the gap, an electrical arc appears. Here, we want to describe the beginning of the process that is the primary discharge. We consider the model geometry, in two space dimensions, drawn in Figure 5.1. Following [11], we assume
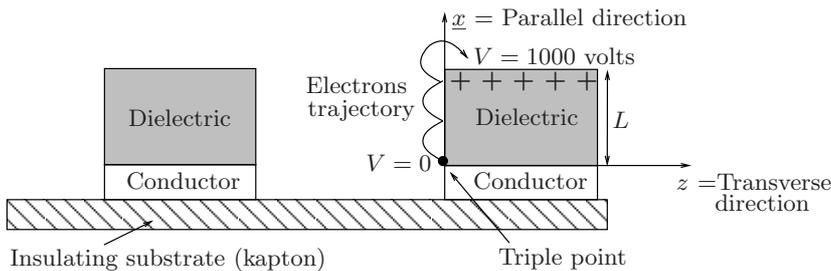
Fig. 5.1. *Side view of the three dimensional model geometry of the device*

that the primary discharge is due to a differential charging of the dielectric surface and of the conducting media (semiconductor or metallic interconnectors). Indeed the plasma environment of the satellite contributes to positively charging the dielectric surface compared with the conductor with potential difference ranging up to 1000

volts. This large electric field triggers an enhanced field electron emission at the triple point. This local increase of the electric field has two origins: the contact between metallic parts, dielectric and vacuum and the presence of micro-dielectric impurities at the triple point. Due to the electric field parallel to the dielectric boundary, the emitted electrons reach the top of the dielectric. On their way, they are attracted by the positive potential of the surface and collide with it. These collisions generate secondary electrons from the dielectric surface as well as desorbed neutral particles. Furthermore, during their way to the top of the dielectric, the electrons collide with the neutral particles and create some ions. Due to the increase of the electron density by secondary emission, an avalanche breakdown occurs and generates a high density plasma.

5.2. The mathematical model. The above described physical situation is an ideal example to apply our transport theory of trapped particles in a surface potential as presented in Section 4. However, additionally we must describe models for the enhanced field electron emission and the secondary emission process. This is a first step towards a description of the whole primary discharge. In Section 5.2.1, we scale Eqs (4.9), (4.10) back to physical dimensions . We make several assumptions in order to obtain a simplified one dimensional model (phenomenon independent of one parallel component, external transverse potential only depending on transverse components). We precise the non-specular surface collision operator in Section 5.2.2 by describing electron secondary emission at the surface. Finally, in Section 5.2.3 we complete the model by specifying the initial and boundary conditions.

5.2.1. The electron transport equation. We begin with the presentation of the asymptotic model in physical variables.

Let $L > 0$ be the height of the dielectric lateral surface (see Figure 5.1), in this section, we denote by $x = (\underline{x}, z) = ((\underline{x}_1, \underline{x}_2), z) \in [0, L] \times [0, L] \times \mathbb{R}^-$ the position and by $v = (\underline{v}, v_z) = ((\underline{v}_1, \underline{v}_2), v_z) \in \mathbb{R}^2 \times \mathbb{R}$ the velocity (see Figure 5.2). Let ϕ be the given
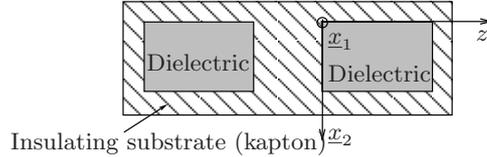
FIG. 5.2. *Above view of the three dimensional model geometry of the device*

external potential. We denote by ϕ_0 and ψ its parallel and transverse components defined by (2.7). On account of the negative electron charge, Assumption (1) becomes

ASSUMPTION 2. *For all $(\underline{x}, t) \in \mathbb{R}^2 \times \mathbb{R}^+$, the function $z \in \mathbb{R}^- \mapsto \psi(\underline{x}, z, t) \in \mathbb{R}^-$ is increasing, continuous and satisfies $\lim_{z \to -\infty} \psi(\underline{x}, z, t) = -\infty$.*

Note that thanks to (2.7), Assumption 2 implies that $\psi(z) \leq 0$ for all $z \leq 0$.

Let f be the distribution function of electrons , we set

$$N_z(\underline{x}, \varepsilon_z, t) \, F(\underline{x}, \underline{v}, \varepsilon_z, t) = \int_{\mathbb{R} \times \mathbb{R}^-} f(\underline{x}, z, \underline{v}, v_z, t) \, \delta(m \, |v_z|^2/2 - e\psi(\underline{x}, z, t) - \varepsilon_z) \, dv_z \, dz,$$

where $e > 0$ is the elementary charge, m is the electron mass and where the density

of states in physical variables N_z is given by (see Remark 1 for details)

$$
N_z(\varepsilon_z) = \int_{\mathbb{R}\times\mathbb{R}^-} \delta(m\,|v_z|^2/2 - e\psi(\underline{x},z,t) - \varepsilon_z)\,dv_z\,dz
$$
$$
= \frac{2}{m}\int_{\mathcal{Z}(\varepsilon_z)}^{0} \frac{1}{\sqrt{(2/m)(\varepsilon_z + e\,\psi(z))}}\,dz,
\tag{5.1}
$$

with $\mathcal{Z}$ defined by (3.10). Note that $N_z\,F\,d\underline{x}\,d\underline{v}\,d\varepsilon_z$ is the number of electrons with a parallel position in $[\underline{x},\underline{x}+d\underline{x}]$, a parallel velocity in $[\underline{v},\underline{v}+d\underline{v}]$ and total transverse energy in $[\varepsilon_z,\varepsilon_z+d\varepsilon_z]$.

Using (4.9) and re-scaling the problem in order to express it in physical variables, F satisfies:

$$
\partial_t(N_z F) + \underline{v}\cdot\nabla_{\underline{x}}(N_z F) + \frac{e}{m}\Big(\nabla_{\underline{x}}\tilde{\phi}_s + \nabla_{\underline{x}}\phi_0 + <\nabla_{\underline{x}}\psi>\Big)\cdot\nabla_{\underline{v}}(N_z F)
$$
$$
+\Big(<\partial_t\psi> + \underline{v}\cdot<\nabla_{\underline{x}}\psi>\Big)\partial_{\varepsilon_z}(N_z F) = \frac{1}{m}\left(\mathcal{K}(F) - F\right).
$$

Operator $\mathcal{K}$ models electron secondary emission and is given by (4.11) where K is precised in Section 5.2.2. The function $\tilde{\phi}_s$ is the trace of the self consistent potential on the dielectric surface and is given later.

In order to simplify the model, we make the following assumptions:

ASSUMPTION 3.
1. *The functions F, ϕ_0, ψ and ϕ_s are independent of the variable $\underline{x}_2$.*
2. *For an incident electron on the surface with the velocity $(\underline{v}', \sqrt{2\,\varepsilon_z/m})$, the number of secondary electrons re-emitted with a parallel velocity in $[\underline{v},\underline{v}+d\underline{v}]$ and a transverse energy in $[\varepsilon_z,\varepsilon_z+d\varepsilon_z]$ is given by $K((\underline{v}', \sqrt{2\,\varepsilon_z'}),(\underline{v}, -\sqrt{2\,\varepsilon_z}))$ $d\underline{v}\,d\varepsilon_z$. We assume this number is independent of $\underline{v}_2$.*
3. *The transverse component of the external potential ψ is independent of the variables $\underline{x}$ and t.*

Note that the third assumption implies that the density of states N_z is independent on $\underline{x}$ and t. Then, using Assumptions 3, setting $x = \underline{x}_1$, $v = \underline{v}_1$ and still denoting by F the moment $\int_{\mathbb{R}} F(\underline{v}_2)\,d\underline{v}_2$, we obtain

$$
\partial_t(N_z F) + v\,\partial_x(N_z F) + \frac{e}{m}\partial_x\Big(\phi_0 + \tilde{\phi}_s\Big)\partial_v(N_z F) = \frac{1}{m}\left(\mathcal{K}(F) - F\right),
\tag{5.2}
$$

for all $x \in [0,L]$, $v \in \mathbb{R}$, $\varepsilon_z \in \mathbb{R}^+$ and $t \geq 0$ and where the density of states N_z is given by (5.1).

Now we remark that the variable ε_z is only a parameter in Eq. (5.2). The particles have a constant transverse energy during their way to the top of the dielectric. Then the integral of F in the transverse direction: $G(x,v,t) = \int_{\mathbb{R}} N_z(\varepsilon_z)\,F(x,v,\varepsilon_z,t)\,d\varepsilon_z$ appears as a natural unknown. After integration with respect to ε_z of (5.2) the left hand side becomes an equation for G. In order to close the problem, we need to express the right hand side in terms of this new unknown. Then, we assume that, due to a large number of collisions, the transverse energy can be approximated by half the temperature of the parallel motion, that is:

ASSUMPTION 4. *We assume that F satisfies*

$$
F(x,v,\varepsilon_z,t) = \frac{\delta\left(\varepsilon_z - k_B T(x,t)/2\right)}{N_z(k_B T(x,t)/2)}\,G(x,v,t),
$$

where k_B is the Boltzmann constant and T is the temperature of the parallel motion given by

$$k_B T = \frac{1}{\rho} \int_{\mathbb{R}} m \, |v - u|^2 \, G \, dv, \qquad (5.3)$$

with the electron density, ρ, and mean velocity, u, given by

$$\rho = \int_{\mathbb{R}} G \, dv \qquad and \qquad u = \frac{1}{\rho} \int_{\mathbb{R}} v \, G \, dv. \qquad (5.4)$$

Under this assumption we obtain the following model for the moment G:

$$\left(\partial_t + v \partial_x + \frac{e}{m} \, \partial_x(\phi_0 + \tilde{\phi}_s) \, \partial_v \right) G = Q(x, v, t), \qquad (5.5)$$

for all $(x, v, t) \in [0, L] \times \mathbb{R} \times \mathbb{R}^+$ and where the collision operator Q is given by

$$Q(x, v, t) = \frac{1}{m} \int_{\mathbb{R}^+} \left(\mathcal{K}(F)(v, \varepsilon_z) - F(v, \varepsilon_z) \right) d\varepsilon_z. \qquad (5.6)$$

The function $\tilde{\phi}_s$ is the trace on the dielectric surface of the self-consistent potential ϕ_s. For all $t \geq 0$, we have:

$$
\begin{cases}
\tilde{\phi}_s(x, t) = \phi_s(x, 0, t), & \text{for all} \quad x \in [0, L], \\
-\Delta_{x,z} \phi_s = 0, & \text{in} \quad [0, L] \times (-\infty, 0), \\
\partial_z \phi_s(x, 0, t) = \frac{e}{\varepsilon_D} \rho_+(x, t) - \frac{e}{\varepsilon_0} \rho(x, t) & \text{for all} \quad x \in [0, L], \\
\lim_{|(x,z)| \to +\infty} \phi_s(x, z, t) = 0,
\end{cases}
\qquad (5.7)
$$

where ε_0 and ε_D are respectively the vacuum and the dielectric permittivities. The electron density ρ is given by (5.4). Finally ρ_+ is the density of the positive charges created on the surface when a secondary electron is emitted. It is given by

$$\frac{\partial \rho_+}{\partial t} = \int_{\mathbb{R}} Q(x, v, t) \, dv. \qquad (5.8)$$

For the closure of the model, we have to specify the collision operator which describes electron secondary emission and the initial and boundary conditions for the transport equation (5.2). We discuss these points in the following sections.

5.2.2. The electron secondary emission operator. In this section, we precise the collision operator Q given by (5.6) where $\mathcal{K}$ is given by (4.11). This operator models electron secondary emission on the surface $\partial \Omega = [0, L] \times \{0\}$. We have to express the number of re-emitted secondary electrons, K, in terms of the unknown G.

Let us first consider an incident electron arriving on the surface with the parallel velocity $(v, \underline{v}_2) \in \mathbb{R}^2$ and the transverse energy $\varepsilon_z \in \mathbb{R}^+$. In [11] the mean yield, $\mathcal{N}_m$, that is the mean number of re-emitted electrons after a collision with the surface for an incident electron, is fitted to the following formula

$$\mathcal{N}_m(v, \varepsilon_z) = \mathcal{N}_{max} \frac{\varepsilon_{cin}}{\varepsilon_{max}} \exp \left(2 - 2 \sqrt{\frac{\varepsilon_{cin}}{\varepsilon_{max}}} \right) \exp \left(2 \left(1 - \cos \theta \right) \right) \qquad (5.9)$$

where $\mathcal{N}_{max}$ and ε_{max} are given real number. $\mathcal{N}_{max}$ is the mean yield for an incoming electron with a normal trajectory ($\theta = 0$) and an energy ε_{max}, and $\cos\theta = \sqrt{2\,\varepsilon_z/m}/\sqrt{|v|^2 + 2\,\varepsilon_z/m}$ is the cosine of the angle between the normal vector of the surface and the electron velocity. Finally $\varepsilon_{cin} = m\,|v|^2/2 + m\,|\underline{v}_2|^2/2 + \varepsilon_z$ is the kinetic energy of the incident electron.

Since in our model, we have removed the $\underline{v}_2$ component of the velocity, we need to make a certain number of assumptions in order to construct a collision operator $\mathcal{K}$ from model (5.9). First, we approximate the velocity $\underline{v}_2$ by the thermal velocity:

$$m\,|\underline{v}_2|^2 = k_B T(x,t), \tag{5.10}$$

where $k_B T$ is given by (5.3).

Then, we make the following assumptions

ASSUMPTION 5.
  1. *A fraction $\chi \in\,]0,1[$ of the incident electron energy is transferred to the secondary emission.*
  2. *The number of re-emitted electrons per incident electron equals the mean yield $\mathcal{N}_m$.*
  3. *For an incoming electron of parallel velocity v and transverse energy ε_z, if $\mathcal{N}_m \geq \chi$ then we assume that $\chi/2$ electrons are re-emitted with the same parallel velocity v and the same transverse energy ε_z, $\chi/2$ electrons are re-emitted with parallel velocity $-v$ and transverse energy ε_z and $\mathcal{N}_m - \chi$ electrons are re-emitted with a parallel velocity $(0,0)$ and zero transverse energy. If $\mathcal{N}_m < \chi$ then we suppose that $\mathcal{N}_m/2$ electrons are re-emitted with the same parallel velocity v and the same transverse energy ε_z and $\mathcal{N}_m/2$ electrons are re-emitted with parallel velocity $-v$ and transverse energy ε_z.*

This yields to

$$K\left((v', \sqrt{2\,\varepsilon_z'}), (v, -\sqrt{2\,\varepsilon_z})\right) = \mathbf{1}_{(\mathcal{N}_m',+\infty)}(\chi)\frac{\mathcal{N}_m'}{2}\Big(\delta(v'-v) + \delta(v'+v)\Big)\otimes\delta(\varepsilon_z' - \varepsilon_z)$$
$$+\mathbf{1}_{[0,\mathcal{N}_m']}(\chi)\left[\frac{\chi}{2}\Big(\delta(v'-v) + \delta(v'+v)\Big)\otimes\delta(\varepsilon_z' - \varepsilon_z) + \Big(\mathcal{N}_m' - \chi\Big)\delta(v)\otimes\delta(\varepsilon_z)\right],$$

where $\mathcal{N}_m' = \mathcal{N}_m(v',\varepsilon_z')$ and where $\mathbf{1}_A$ is the indicator function of the set $A \subset \mathbb{R}$. Using (4.11), Assumption 4 and (5.6), we obtain the following collision operator:

$$Q(x,v,t) = \nu\left(\gamma(v,k_B\,T)\Big(G(v) + G(-v)\Big) - G(v) + \delta(v)\int_{\mathbb{R}}\gamma_0(v',k_B\,T)\,G(v')\,dv'\right), \tag{5.11}$$

where the collision frequency, ν, is defined by

$$\nu = \frac{1}{m\,N_z(k_B T/2)},$$

and

$$\gamma_0(v',k_B\,T) = \mathbf{1}_{[0,\mathcal{N}_m']}(\chi)\,(\mathcal{N}_m' - \chi), \qquad \gamma(v,k_B\,T) = \mathbf{1}_{(\mathcal{N}_m,+\infty)}(\chi)\,\frac{\mathcal{N}_m}{2} + \mathbf{1}_{[0,\mathcal{N}_m]}(\chi)\,\frac{\chi}{2},$$

with the following definitions for the mean yields $\mathcal{N}_m = \mathcal{N}_m(v,k_B T)$ and $\mathcal{N}_m' = \mathcal{N}_m(v',k_B T)$:

$$\mathcal{N}_m(v,k_B T) = \mathcal{N}_{max}\,\frac{m\,v^2/2 + k_B T}{\varepsilon_{max}}\exp\left(2 - 2\sqrt{\frac{m\,v^2/2 + k_B T}{\varepsilon_{max}}}\right)\exp\left(2\,(1 - \cos\theta)\right),$$

changing v into v' for the definition of $\mathcal{N}'_m$, and where $\cos\theta = \sqrt{k_B T}/\sqrt{m|v|^2 + k_B T}$ is the cosine of the angle between the normal vector to the surface and the electron velocity.

5.2.3. The initial and boundary conditions for the transport equation. At the beginning of the process there is no electron in the gap and no positive charge created on the surface, so the initial conditions are given by

$$G(t = 0) = 0 \quad \text{and} \quad \rho_+(t = 0) = 0. \tag{5.12}$$

At the point $x = L$ we assume that no particle enters the domain. Then for all $t \geq 0$

$$G(L, v, t) = 0, \quad \text{for all } v < 0. \tag{5.13}$$

Now, we want to precise the incoming electron flux in the domain at the triple point that is the electron distribution function at point $x = 0$ for $v > 0$. Following [11], we suppose that there is an enhanced field electron emission at the triple point $x = 0$ due to the presence of dielectric impurities or due to the microscopic structure of the conductor surface. Then the electron current intensity follows a Fowler-Nordheim law (see [11]) given by

$$J(t) = A \left(\beta \max(-E(0,t), 0) \right)^2 \exp\left(\frac{B}{\beta E(0,t)} \right) (A/m^2), \tag{5.14}$$

with $E(0,t) = -\partial_x(\phi_0 + \tilde{\phi}_s)(0,t)$, and where A and B are given constants. Finally β is the field enhancement factor. It is fixed empirically and varies between 500 and 1000.

In order to close the model, we assume a Maxwellian distribution, then we set for all $v > 0$:

$$F(0, v, \varepsilon_z, t) = \sqrt{\frac{2\,m}{\pi\,k_B\,T_{inj}}}\, \rho_{inj}(t) \exp\left(-\frac{mv^2/2 + \varepsilon_z}{k_B\,T_{inj}} \right).$$

Using the relation $G = \int_{\mathbb{R}^+} N_z(\varepsilon_z)\,F(\varepsilon_z)\,d\varepsilon_z$, this gives for all $v > 0$, and all $t \geq 0$,

$$G(0, v, t) = G_{lim}(v, t) = \sqrt{\frac{2\,m}{\pi\,k_B\,T_{inj}}}\, \rho_{inj}(t) \exp\left(-\frac{m\,v^2/2}{k_B\,T_{inj}} \right), \tag{5.15}$$

where k_B is the Boltzmann constant, T_{inj} is the injection temperature. We suppose a cold injection that is the injection temperature is chosen such that $T_{inj} = 1$ eV. Finally ρ_{inj} is the injected density given by the formula

$$\int_{\mathbb{R}^+} v\,G_{lim}(v, t)\,dv = \frac{J(t)}{e}. \tag{5.16}$$

5.3. The numerical method. In this section, we discretize the model given in Section 5.2 using a Particle In Cell method (P.I.C.) (see [8], [13], [16], [19], [23]). That is, we use a deterministic particle discretization for the Boltzmann equation (5.5) and the Poisson equation (5.7) is solved using a Fourier series approximation. The coupling of these discretizations is done through a fixed grid in order to introduce some smoothing and decrease numerical oscillations.

5.3.1. Principle of the scheme. Let $0 < t_1 < \cdots < t^n < t^{n+1}, \cdots$ a sequence of positive real points. For all $n \geq 0$, we denote by Δt^n the difference between t^{n+1} and t^n and we set $t^{n+1/2} = t^n + \Delta t^n/2$. A particle method (see [13], [16], [19]) for (5.5) consists in approximating the distribution function G by a linear combination of Dirac delta distributions centered on the numerical particles. During each time step Δt^n, we have a finite given number I^n of numerical particles. They move in the phase space along the characteristic curves of Eq. (5.5). We set

$$G(x,v,t) \approx G_{app}(x,v,t) = \sum_{n \geq 0} \sum_{i=1}^{I^n} w_i \, G_i^n \, \mathbf{1}_{[t^n,t^{n+1})}(t) \, \delta(x - x_i^n) \otimes \delta(v - v_i^{n+1/2}),$$

where for all $n \geq 0$ and all $i \in \{1, \cdots, I^n\}$, w_i is the control volume, in the phase space (x,v), associated with the ith particle. The quantities x_i^n, $v_i^{n+1/2}$ and G_i^n are the approximate position, velocity and weight of ith particle on the interval $[t^n, t^{n+1})$. The calculation of G_n^i will be presented in Section 5.3.3. In our model, see Section 5.2, the initial condition equals zero. Then all electrons come either from injection at the triple point $x = 0$ or from collisions with the surface, through the electron secondary emission process. For the ith particle, let us denote by $t_i^s \in (t^n)_{n \geq 0}$ the time of entry in the domain $[0, L]$ if the particle comes from injection and by $t_i^s \in (t^n)_{n \geq 0}$ its creation time if it results from secondary emission. For both situations, we denote by x_i^s, v_i^s and G_i^s the position, velocity and weight of this particle at the time t_i^s. They will be precised in Sections 5.3.2 and 5.3.3. We use a leapfrog scheme (see [19]) then we have

$$x_i^{n+1} = x_i^n + \Delta t^n \, v_i^{n+1/2}, \qquad v_i^{n+1/2} = v_i^{n-1/2} - \frac{\Delta t^n + \Delta t^{n-1}}{2} \, \frac{e}{m} \, E_i^n, \quad (5.17)$$

for all $n \geq 0$ such that $t^n \geq t_i^s$ and all $i \in \{1, \cdots, I^n\}$ and where $v_i^{s-1/2} = v_i^s$ and $\Delta t^{s-1} = 0$. We denote by E_i^n the approximate electric field ($E = -\partial_x(\phi_0 + \tilde{\phi}_s)$) at time t^n and at position x_i^n. We recall that ϕ_0 is the parallel component of the given external potential defined by (2.7) and $\tilde{\phi}_s$ is given by (5.7).

We introduce a fixed grid in order to present the calculation of E_i^n. We consider a uniform fixed grid of size Δx on the spatial domain $[0, L]$ with K cells $M_k = [X_{k-1/2}, X_{k+1/2}) = [(k-1)\Delta x, k\Delta x)$ and we denote by $X_k = (k - 1/2)\Delta x$ the center of M_k for $k = 1, \cdots, K$. We use a C.I.C. scheme (Cloud in Cell) and a Fourier approximation to solve (5.7) (see [8], [16], [19]). Thus we set for all $n \geq 0$ and all $k = 1, \cdots, K$

$$\rho_k^n = \sum_{i=1}^{I^n} \frac{w_i \, G_i^n}{\Delta x} \, W\left(\frac{x_i^n - X_k}{\Delta x}\right),$$

$$E_k^n = -\partial_x \phi_0(X_k, t^n) - \sum_{p=0}^N e \left(\frac{\rho_{+,p}^n}{\varepsilon_D} - \frac{\rho_p^n}{\varepsilon_0}\right) \cos\left(\frac{p \pi X_k}{L}\right),$$

where $W(x) = 1 + x$ if $-1 \leq x \leq 0$, $W(x) = 1 - x$ if $0 \leq x \leq 1$ and $W(x) = 0$ otherwise. $N \in \mathbb{N}^\star$ is given and

$$\rho_p^n = \sum_{k=1}^K \frac{2\rho_k^n}{p\,\pi} \Big[\cos(p\pi X_{k-1/2}/L) - \cos(p\pi X_{k+1/2}/L)\Big].$$

The same formula holds for the positive density $\rho_{+,p}^n$ with $\rho_{+,k}^n$ given by (5.19). We interpolate the electric field on each particle position, by setting

$$E_i^n = \sum_{k=1}^K E_k^n \, W\left(\frac{x_i^n - X_k}{\Delta x}\right).$$

It remains to discretize the boundary condition and the collision operator on the surface. We do it in the next sections.

5.3.2. The boundary condition. For the discretization of the boundary condition (5.15) at the point $x = 0$, we inject N_{inj} particles at each time $t^{n+1/2}$. Then we set

$$G_{lim}(v,t) \approx G_{lim,app}(v,t) = \sum_{n\geq 0} \sum_{i=1}^{N_{inj}} \bar{w}_i^n \, G_{lim,i}^{n+1/2} \, \delta(t - t^{n+1/2}) \otimes \delta(v - \bar{v}_i),$$

where $\bar{w}_i^n$ is the control volume of the ith particle at time $t^{n+1/2}$ in the space (v,t), $G_{lim,i}^{n+1/2}$ is an approximation of $G_{lim}(\bar{v}_i, t^{n+1/2})$ and $\bar{v}_i$ is the velocity of the ith injected particle.

We choose uniformly distributed velocities with respect to the normalized Maxwellian distribution. That is, we set

$$\bar{v}_i = g^{-1}\left(\frac{i - 1/2}{N_{inj}}\right), \quad \text{for all } i = 1, \cdots, N_{inj},$$

with g^{-1} the inverse function of the one to one function g from $\mathbb{R}^+$ onto $[0, 1[$ defined by $g(v) = \sqrt{\frac{2m}{\pi k_B T_{inj}}} \int_0^v \exp(-m\,w^2/(2\,k_B\,T_{inj}))\,dw$ for all $v \in \mathbb{R}^+$. Similarly, we define $\bar{v}_{i+1/2} = 0$ if $i = 0$, $\bar{v}_{i+1/2} = g^{-1}(i/N_{inj})$ if $1 \leq i \leq N_{inj} - 1$ and $\bar{v}_{i+1/2} = +\infty$ if $i = N_{inj}$. Note that $\int_{v_{i-1/2}}^{v_{i+1/2}} g'(v)\,dv = 1/N_{inj}$ and $v_{i-1/2} \leq \bar{v}_i \leq v_{i+1/2}$.

Then, discretizing (5.15), we have

$$\int_{t^n}^{t^{n+1}} \int_{v_{i-1/2}}^{v_{i+1/2}} G_{lim,app}(v,t)\,dt\,dv = \bar{w}_i^n \, G_{lim,i}^{n+1/2} \approx$$

$$\int_{t^n}^{t^{n+1}} \rho_{inj}(t^{n+1/2})\,dt \int_{v_{i-1/2}}^{v_{i+1/2}} g'(v)\,dv = \frac{\Delta t^n \, \rho_{inj}(t^{n+1/2})}{N_{inj}}.$$

Combining this relation with a similar discretization of (5.16) yields to

$$\bar{w}_i^n \, G_{lim,i}^{n+1/2} = \frac{\Delta t^n \, J^n}{e \sum_{j=1}^{N_{inj}} v_j^{n+1/2}},$$

where the approximate current flux, J^n, at the triple point $x = 0$ is given by discretizing (5.14), that is changing $E(0,t)$ in $E_{1/2}^n$.

We assume that the injected particles are not subject to collisions with the surface between the time $t^{n+1/2}$ and t^{n+1}. Then using the leapfrog scheme, we obtain the position and the velocity of the ith particle at time t^{n+1}: $v_i^{n+1/2} = \bar{v}_i$, $x_i^{n+1} = \Delta t^n \, v_i^{n+1/2}/2$. Finally the product of its control volume and its weight at time t^{n+1} is given by $w_i G_i^{n+1} = v_i^{n+1/2} \, \bar{w}_i^n \, G_{lim,i}^{n+1/2}$.

5.3.3. Surface collisions and positive charges. The surface collisions are described by the electron secondary emission operator given by (5.11), then the particle weights satisfy

$$\frac{dG_i(t)}{dt} = \begin{cases} \nu\,(\gamma_i - 1)\,G_i(t) + \nu\,\gamma_i\,G_{i_{sym}}(t) & \text{if } v_i(t) \neq 0, \\ \nu\,(\gamma_i - 1)\,G_i(t) + \nu\,\gamma_i\,G_{i_{sym}}(t) + \nu \sum_{j \in I_i(t)} \gamma_{0,j}\,G_j(t) & \text{if } v_i(t) = 0, \end{cases}$$
(5.18)

where $G_{i_{sym}}$ is the weight of the symmetric particle that is the particle such that $v_{i_{sym}} = -v_i(t)$ and where $I_i(t)$ is the set of particles with position $x_i(t)$ at time t. Furthermore $\gamma_i = \gamma(v_i(t), k_B\,T_i(t))$ where $T_i(t)$ is the temperature at time t and at position $x_i(t)$. Finally $\gamma_{0,j} = \gamma_0(v_j(t), k_B\,T_i(t))$.

Let us remark that usually, in numerical simulations, the particle i_{sym} does not exist in $\{1, \cdots, I^n\}$ ($n \geq 0$ fixed). The same remark holds for the particles of $I_i(t)$. Then, we proceed as follows:

For all $n \geq 0$ and all particles $i \in \{1, \cdots, I^n\}$, we create at time t^n: one particle i_{sym} with position, velocity, control volume and weight given by $x_{i_{sym}}^n = x_i^n$, $v_{i_{sym}}^{n-1/2} = -v_i^{n-1/2}$, $w_{i_{sym}} = w_i$ and $G_{i_{sym}}^n = 0$ at time t^n and one particle $i_0 \in I_i(t^n)$ with position, velocity, control volume and weight given by $x_{i_0}^n = x_i^n$, $v_{i_0}^{n-1/2} = 0$, $w_{i_0} = w_i$ and $G_{i_0}^n = 0$. These particles, i, i_{sym} and i_0, move in the phase space during Δt^n, then their positions at time t^{n+1} and their velocities at time $t^{n+1/2}$ are given using the leapfrog scheme (5.17). Using (5.18) and the property $w_i = w_{i_{sym}} = w_{i_0}$, their weight will be given by

$$\begin{cases} w_i\,G_i^{n+1} = w_i\,G_i^n + \Delta t^n\,\nu(T_i^n)\,(\gamma_i^n - 1)\,w_i\,G_i^n, \\ w_i\,G_{i_{sym}}^{n+1} = \Delta t^n\,\nu(T_i^n)\,\gamma_i^n\,w_i\,G_i^n, \\ w_i\,G_{i_0}^{n+1} = \Delta t^n\,\nu(T_i^n)\,\gamma_{0,i}^n\,w_i\,G_i^n, \end{cases}$$

where T_i^n is defined by

$$T_i^n = \sum_{k=1}^{K} T_k^n\,W\left(\frac{X_k - x_i^n}{\Delta x}\right),$$

where the temperature and the mean velocity on the cell k are defined by

$$\begin{pmatrix} u_k^n \\ T_k^n \end{pmatrix} = \frac{1}{\rho_k^n} \sum_{i=1}^{I^n} \begin{pmatrix} v_i^{n+1/2} \\ |v_i^{n+1/2} - u_k^n|^2 \end{pmatrix} \frac{w_i\,G_i^n}{\Delta x}\,W\left(\frac{X_k - x_i^n}{\Delta x}\right).$$

Furthermore if $\mathcal{N}_m(v_i^{n+1/2}, T_i^n) > \chi$, $\gamma_i^n = \chi/2$ and $\gamma_{0,i}^n = \mathcal{N}_m(v_i^{n+1/2}, T_i^n) - \chi$, otherwise $\gamma_i^n = \mathcal{N}_m(v_i^{n+1/2}, T_i^n)/2$ and $\gamma_{0,i}^n = 0$.

REMARK 4.
1. *It is clear that $w_i\,G_{i_{sym}}^{n+1} \geq 0$ and $w_i\,G_{i_0}^{n+1} \geq 0$ but it is possible that $w_i\,G_i^{n+1} < 0$. In this case, we assume that the incident electron has been attached on the surface and then the numerical particle i is eliminated.*
2. *This method generates too many particles, we solve this problem using a collapsing process at each time step. We describe it in section 5.3.4.*

Now, let us describe the discretization of the positive charges equation (5.8). Like for the electron density, at each time t^n, we approximate the density ρ_+ by a piece-wise constant function on the uniform fixed grid, with $\rho_k^0 = 0$ and

$$(\rho_+)_k^{n+1} = (\rho_+)_k^n + \sum_{i \in I^n} \frac{W\left(\frac{X_k - x_i^n}{\Delta x}\right)}{\Delta x} \Big[\mathbf{1}_{\mathbb{R}^+}(w_i\, G_i^{n+1}) \Delta t^n\, \nu(T_i^n) \big(2\,\gamma_i^n + \gamma_{0,i}^n - 1\big)\, w_i\, G_i^n$$
$$+ \mathbf{1}_{\mathbb{R}^-}(w_i\, G_i^{n+1}) \left(-w_i\, G_i^n + \Delta t^n\, \nu(T_i^n)\, \big(\gamma_i^n + \gamma_{0,i}^n\big)\, w_i\, G_i^n\right) \Big].$$
$$(5.19)$$

5.3.4. The collapsing process. The discretization of the secondary electron emission operator creates too many numerical particles. Then, in this section, following [1] we propose a collapsing process in order to decrease the number of numerical particles.

We define a dual mesh, that is for $k = 1, \cdots, K$, we set $D_1 = [0, X_1]$, $D_k = [X_{k-1}, X_k]$ if $2 \leq k \leq K - 1$ and $D_K = [X_K, L]$. We decide to apply the collapsing process on each dual cell D_k if the number of particles in D_k exceeds the given threshold value N_{tol}.

We decide to keep N_{min} numerical particles in each dual cell, where N_{min} is a fixed positive integer for all dual cells and all discrete time. In order to choose the remaining particles, first let us remark that the collision process on the surface is such that the secondary electrons are emitted with a very small weight compared with the incident electron weight (see Section 5.3.2). It is classical that particle methods are characterized by some noise in the numerical results when the variations of the particle weights are too important, see [8] or [19]. But we want a random process for the choice of the remaining particles so that we do not keep all the biggest particles. Since we want to minimize the variations of the weights we do not use a completely random process. We propose a random algorithm which prevents important differences between the weights.

Let $n \geq 0$ and $k \in \{1, \cdots, K\}$ be fixed, we denote by $i_1, i_2, \cdots, i_{N_k^n} \in \{1, \cdots, I^n\}$ the indices of the particles in the dual cell D_k at time t^n. We define the local mass, momentum and total energy in D_k at time t^n by:

$$m_k = \sum_{j=1}^{N_k^n} w_{i_j}\, G_{i_j}^n, \quad q_k^n = \sum_{j=1}^{N_k^n} v_{i_j}^{n+1/2}\, w_{i_j}\, G_{i_j}^n, \quad e_k^n = \sum_{j=1}^{N_k^n} \left|v_{i_j}^{n+1/2}\right|^2 w_{i_j}\, G_{i_j}^n.$$

We introduce for all $l \in \{1, \cdots, N_k^n\}$, $\beta_l = \sum_{j=1}^l \frac{w_{i_j}\, G_{i_j}^n}{m_k} \in]0, 1]$. Then we take a random number $R \in]0, 1[$ and we keep the first particle i_{j_0} such that $\beta_{i_{j_0}} \geq R$. Note that the probability to keep a particle i_j is given by $w_{i_j}\, G_{i_j}^n / m_k^n$ and so the bigger the particle, the larger the probability to keep it. We repeat this process N_{min} times.

Now, following [1] we calculate new positions, velocities and weights for the N_{min} remaining particles, preserving the mass, momentum and total energy in D_k and the masses in the cells M_k and M_{k+1} of the fixed grid.

Let us denote by $l_1, \cdots, l_{N_{min}} \in \{1, \cdots, I^n\}$ the indices of the particles kept in the dual cell D_k at time t^n. For $i \in \{l_1, \cdots, l_{N_{min}}\}$, we denote by $\tilde{x}_i^n$, $\tilde{v}_i^{n+1/2}$, $\tilde{w}_i$ and $\tilde{G}_i^n$ the new position, velocity, control volume and weight of the particle i.

Let $i \in \{l_1, \cdots, l_{N_{min}}\}$, first, we assign the lost mass uniformly to the remaining

particles

$$\tilde{w}_i \, \tilde{G}_i^n = w_i \, G_i^n + \frac{m_k^n - \sum_{j=1}^{N_{min}} w_{l_j} \, G_{l_j}^n}{N_{min}}.$$

We conserve the momentum q_k^n and the energy e_k^n using a translation and a homothety, we set

$$\tilde{v}_i^{n+1/2} = A \, v_i^{n+1/2} + B,$$

where A and B are calculated such that

$$\sum_{j=1}^{N_{min}} \tilde{v}_{l_j}^{n+1/2} \, \tilde{w}_{l_j} \, \tilde{G}_{l_j}^n = q_k^n \quad \text{and} \quad \sum_{j=1}^{N_{min}} \left| \tilde{v}_{l_j}^{n+1/2} \right|^2 \tilde{w}_{l_j} \, \tilde{G}_{l_j}^n = e_k^n.$$

Now, we determine the new positions $\tilde{x}_i^n$ for $i \in \{l_1, \cdots, l_{N_{min}}\}$ in order to conserve the mass contributions to the cells M_k and M_{k-1}. We denote by $\rho_{k-1,r}^n$ and by $\rho_{k,l}^n$ the mass contributions, of the particles initially in D_k, respectively to ρ_{k-1}^n (the mass in the cell M_{k-1}) and ρ_k^n (the mass in the cell M_k). Thanks to Section 5.3.1, we have

$$\rho_{k-1,r}^n = \sum_{j=1}^{N_k^n} \frac{w_{i_j} \, G_{i_j}^n}{\Delta x} \left(\frac{x_{i_j}^n - X_k}{\Delta x} \right) \quad \text{and} \quad \Delta x \, \rho_{k-1,r}^n + \Delta x \, \rho_{k,l}^n = m_k.$$

We want to choose the positions $(\tilde{x}_{l_1}^n, \cdots, \tilde{x}_{l_{N_{min}}}^n)$ of the N_{min} remaining particles, in the dual cell D_k and such that the mass contributions $\rho_{k-1,r}^n$ and $\rho_{k,l}^n$ do not change. Since m_k is already conserved, we just have to conserve the value of $\rho_{k-1,r}^n$ (or $\rho_{k,l}^n$). To do this, we first look at the old positions, then we introduce

$$\bar{\rho}_{k-1,r}^n = \sum_{j=1}^{N_{min}} \frac{\tilde{w}_{l_j} \, \tilde{G}_{l_j}^n}{\Delta x} \left(\frac{x_{i_j}^n - X_k}{\Delta x} \right).$$

If $\bar{\rho}_{k-1,r}^n > \rho_{k-1,r}^n$ this means that the particles are too far from X_k, then we choose for $\tilde{x}_i^n$ $(i = l_1, \cdots, l_{N_{min}})$ a linear interpolation between the old position x_i^n and X_k. We set $\tilde{x}_i^n = \theta \, x_i^n + (1 - \theta) \, X_k$ with $\theta \in]0, 1[$. Similarly, if $\bar{\rho}_{k-1,r}^n < \rho_{k-1,r}^n$ this means that the particles are too far from X_{k-1}, then we set $\tilde{x}_i^n = \theta \, x_i^n + (1 - \theta) \, X_{k-1}$. In both cases, θ is calculated such that

$$\sum_{j=1}^{N_{min}} \frac{\tilde{w}_{l_j} \, \tilde{G}_{l_j}^n}{\Delta x} \left(\frac{\tilde{x}_{l_j}^n - X_k}{\Delta x} \right) = \rho_{k-1,r}^n.$$

This concludes the Section 5.3.4 and the description of the numerical scheme.

5.4. The numerical results. For the numerical results, we use linear applied potentials in parallel and transverse directions. We set $\psi(z) = E_\perp z$ and $\phi_0(\underline{x}) = U \, (x - L)/L$, where the magnitude of the transverse field, $E_\perp = 5 \times 10^7$ V/m, the size of the dielectric surface $L = 10^{-4}$ m and the difference of potential in the parallel direction $U = 500$ V. This choice gives an explicit density of states:

$$N_z(\varepsilon_z) = \frac{2\sqrt{2 \, \varepsilon_z}}{e \, E_\perp \, \sqrt{m}}.$$

Following [11], we choose values for Silicon in the mean yield formula (5.9), that is $\mathcal{N}_{max} = 2.9$ and $\varepsilon_{max} = 420$ eV and we set $\chi = 100\%$.

In Eq. (5.7) the dielectric permittivity is given by $\varepsilon_D = 5,5\,\varepsilon_0$. Furthermore, for the boundary condition, we choose the field enhancement factor such that $\beta = 500$ in the electron current intensity (5.14). The real numbers A and B determined by the work function of the conductor surface (see [11]) are given by $B = 6,2\;10^{10}\,V/m$, $A = S_{fn}\,A'/S_{real}$ where $A' = 4,6\;10^{-5}\,A/V^2$ and S_{fn}/S_{real} is a ratio of emission surfaces which expresses the diffusion of electrons when they cross the dielectric impurity from the conductor to the vacuum. They are chosen such that $S_{fn} = 10^{-15}\,m^2$ and $S_{real} = 10^{11}\,m^2$.

All the results are given at three different times $t = 5 \times 10^{-11}$s , $t = 15 \times 10^{-11}$s and $t = 20 \times 10^{-11}$s . Figure 5.3 on the left shows the surface electron density along the dielectric in logarithm scale and on the right their temperature measured in eV. We clearly see the avalanche phenomenon and the creation of a very dense electronic cloud. On Figure 5.4 we can see the density of positive charges created on the surface by the secondary electron process (left figure) and the electric potential in the domain (right figure). We remark that many positive charges are created on the surface and so many secondary electrons in the domain. This confirms the important role of this collision process in the creation of the high density plasma. Figure 5.5 on the left shows the mean velocity and on the right the mean collision frequency. Finally Figure 5.6 shows the injected electron current intensity at $x = 0$. Note that at the end of the simulation, it becomes very small. Then no more (or few) electron are injected in the domain at the triple point. The increase of electrons in the domain results from secondary electron emission.
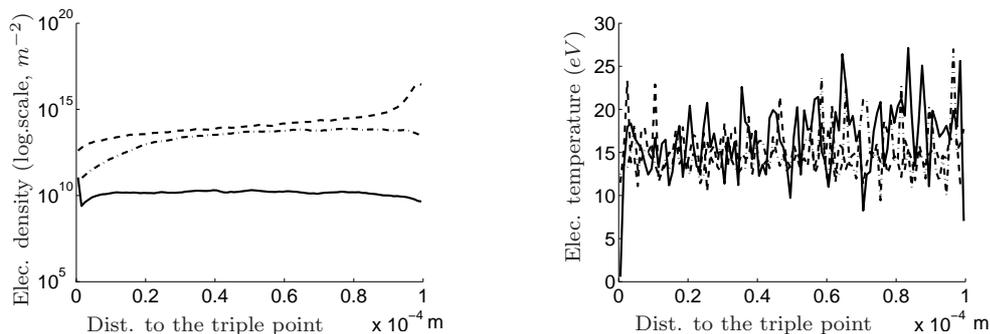
FIG. 5.3. *Surface electron density in log. scale (left fig.) and electron temperature measured in eV (right fig.) at times $t = 5 \times 10^{-11}s$ (solid line), $t = 15 \times 10^{-11}s$ (dashed-dotted line) and $t = 20 \times 10^{-11}s$ (dashed line).*

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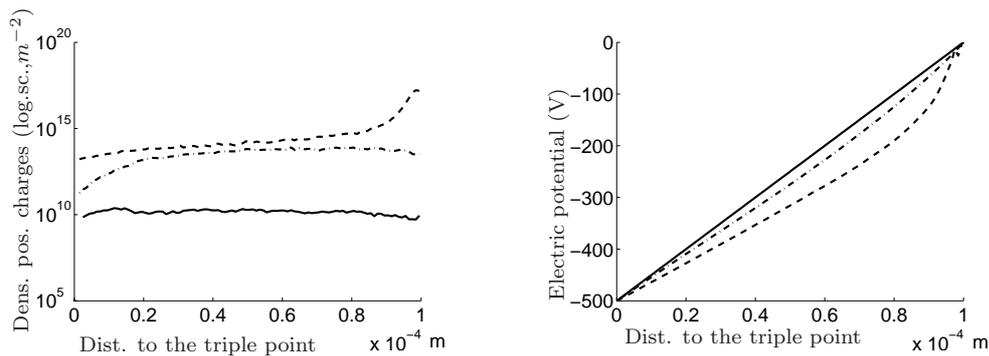
FIG. 5.4. *Left fig.: density (in log scale) of positive charges created on the surface by the secondary electron emission process. Right fig.: electric potential. The results are given at times $t = 5 \times 10^{-11}s$ (solid line), $t = 15 \times 10^{-11}s$ (dashed-dotted line) and $t = 20 \times 10^{-11}s$ (dashed line).*
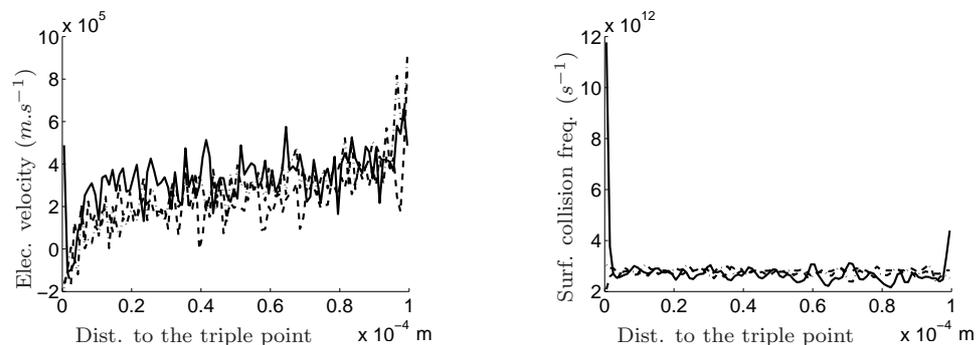
FIG. 5.5. *Electron mean velocity (left fig.) and electron collision frequency with the dielectric surface (right fig.) at times $t = 5 \times 10^{-11}s$ (solid line), $t = 15 \times 10^{-11}s$ (dashed-dotted line) and $t = 20 \times 10^{-11}s$ (dashed line).*

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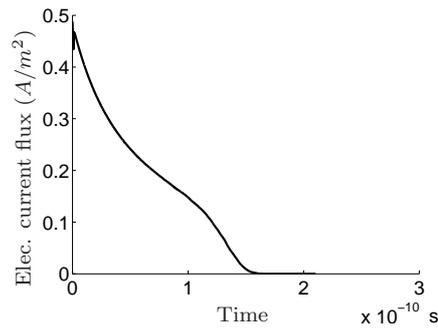
Fig. 5.6. *Electronic current injected at the triple point ($x = 0$) as a function of the time*

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