PLASMA EXPANSION IN VACUUM: MODELING THE BREAKDOWN OF QUASINEUTRALITY *

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Abstract. We consider a system consisting of a vacuum gap delimited by two electrodes. A quasineutral plasma (ions and electrons) is injected from the cathode and expands. Electrons are emitted from the plasma-vacuum interface to the anode forming a beam. In this paper, from a two-fluid isentropic system coupled with the Poisson equation, we perform a formal asymptotic analysis. This leads to a quasineutral model for the plasma region and a Child-Langmuir model for the beam region. The main point of the analysis is the connection between these two models. This is done by studying a transmission layer problem. Finally, we numerically show the accuracy of the asymptotic model comparing it to the original two-fluid one.

Key words. Expanding plasma, Euler-Poisson model, quasineutral limit, Child-Langmuir law, travelling-wave analysis, Bohm sheath criterion

AMS subject classifications. 82D10, 76W05, 76X05, 76N10, 76N20, 76L05

1. Introduction.

In this paper, we are interested in the modeling of a quasineutral plasma expansion in the vacuum gap separating two electrodes. A high density quasineutral plasma, constituted of ions and electrons, is emitted from the cathode. This plasma undergoes a thermal expansion and simultaneously, electrons are accelerated from the plasma-vacuum interface towards the anode, forming an electron beam in the vacuum. The gap between the plasma-vacuum interface and the anode is diminishing with time, thereby increasing the emitted electron current in the beam.

By a formal asymptotic analysis of the two-fluid Euler-Poisson model for the electrons and the ions, we propose an asymptotic model which consists of a quasineutral one-fluid model for the plasma region and a Child-Langmuir type model for the beam. The main difficulty is to connect these two models through the moving plasma-vacuum interface. For this purpose, we introduce a transmission problem which views the plasma-vacuum interface at the microscopic level as a travelling-wave solution of the two-fluid Euler-Poisson model. From the transmission problem, we deduce connection relations between the plasma and the beam. We validate these relations by numerical comparisons between our model and the original two-fluid Euler model.

We study this phenomenon in relation with two physical applications. The first one concerns the development of plasma diodes (see e.g. [37]). In this case, the plasma is used to increase the beam current as compared with conventional plane parallel diodes which are limited by the Child-Langmuir law [10], [17]. The second one concerns the appearance of electrical discharges on satellite solar cells. In this case, the model can be used to describe the discharge to arc transition phenomenon (see [4], [19] for more detail).

The starting point of the modeling is a one-dimensional isentropic Euler system for each species (ions and electrons) coupled with the Poisson equation. In the plasma region, electrical effects occur on very short space scales, of the order of the Debye length. Hence, in numerical simulations, the space discretization of the Euler-Poisson

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model must resolve the Debye length, otherwise numerical instabilities develop. In our case, due to the large density in the plasma region, the Debye length is very small, which makes numerical simulations of the two-fluid model virtually impossible in practical cases. Therefore, the two-fluid Euler-Poisson model cannot be used for practical purposes.

For this reason, starting from the two-fluid system and using a convenient scaling of the equations, we derive a quasineutral model describing the plasma as a single fluid. This derivation is based on a formal asymptotic analysis. The noticeable point of our approach is that the plasma current is not supposed equal to zero, contrary to standard approaches. This is necessary to account for electron emission at the plasma-vacuum interface and the formation of the beam.

In the beam region, the quasineutrality hypothesis is obviously inadequate, since only one species (the electrons) is present in this region. Additionally, we expect that the self-consistent electric field plays an important role. Therefore, to analyze the beam region, we return to the Euler-Poisson problem and introduce different scaling assumptions. This new scaling allows to derive a Child-Langmuir type model. This model is a stationary pressureless Euler-Poisson model, and has analytic solutions.

The device is then divided in two zones which need to be connected through adequate transmission conditions. This is delicate because different scaling assumptions are used in the two zones, and none of the two models is appropriate to describe the region close to the interface. In addition, numerical simulations of the two-fluid Euler-Poisson model show that, contrary to a classical fluid expansion in vacuum, the plasma region is bordered by a shock. This is due to the counter-pressure exerted by the acceleration of the electrons onto the plasma. This counter pressure must be properly evaluated in order to correctly describe the interface motion. To this aim, we introduce and analyze the transition region as a traveling-wave solution of the two-fluid Euler-Poisson model, which connects on the plasma side to the quasineutral model and on the beam side, to the Child-Langmuir model. With the help of this transition model, we are able to propose convenient transmission conditions between the plasma and beam model, which make the overall system (at least formally) well-posed. One of these conditions relates the plasma current to the electron beam current. The other condition is nothing but Bohm's sheath criterion [3], [30] when the electron beam is viewed as a sheath. This condition tells us that the electrons must enter the beam region with supersonic speed.

This work is a continuation of earlier work [13], [15]. In these works, the same methodology was used but the plasma quasineutral model was current-free and the counter-pressure exerted by the electrons onto the plasma was concentrated at the interface. In the present paper (see also a summary in [14]), the major difference is that the quasineutral model for the plasma is a current-carrying one and that the connection relations between the plasma and the beam region are supported by the analysis of the transmission problem. The numerical simulations show an excellent agreement with the original two-fluid Euler-Poisson model, which the earlier models did not achieve (see section 4.2). There is still a slight discrepancy between the two models near the cathode. This difference certainly results from the presence of a boundary layer at this point. The boundary layer analysis which would allow to clarify this point is beyond the scope of the present work. It is important to note that this boundary layer does not damage the quality of the results in the rest of the domain. Indeed, the asymptotic and two-fluid models give similar results in the core of the plasma region as well as in the beam region.

The mathematical theory of the Euler-Poisson system has been investigated in [29], [5] and more recently, in [20]. Quasineutral limits have been investigated in [8] and [35] for the Euler equations and in [25], [28], [22] for the drift-diffusion model. One of the first references on the Child-Langmuir law is [26]. Its mathematical aspects have been studied in series of paper (see e.g. [17], [18], [1], [11], [12] and the review [10]). The travelling-wave transition problem bears similarities with the shock profile problem studied in [6] and [7]. The physical problem studied here has strong analogies with the ion sheath problem, which have received much interest [21], [30], [31], [32], [33], [34], [36], [2], [24] and references therein. The question of boundary conditions at sheath edges has been numerically investigated in [27].

The paper is organized as follows. In Section 2, we introduce the two-fluid Euler-Poisson model and the scaling. In Section 3, we present the asymptotic model, leaving the proof of technical points to section 5. In Section 4, we numerically validate the asymptotic model comparing it to the original two-fluid one.

2. The two-fluid Euler-Poisson model. We consider two electrodes separated by a vacuum gap of size L, with the cathode located at $x = 0$ and the anode at $x = L$. A potential difference ϕ_L is applied between these two electrodes. A quasineutral plasma constituted of one single charged species of ions and of electrons is injected at the cathode.

The electrons of mass m_e, of charge $q_e = -q$ are described by their density n_e and their velocity u_e; the ions of mass m_i, of charge $q_i = +q$ by n_i and u_i. Assuming the adiabaticity for each species, the pressure laws are given by $p_\alpha(n_\alpha) = c_\alpha(n_\alpha)^\gamma$, $\alpha = e, i$, where $\gamma > 1$ is the ratio of specific heats and c_i, c_e are given constants.

We consider the one-dimensional isentropic Euler system. Thus, n_e, n_i, u_e and u_i satisfy, for all $t > 0$, $x \in [0, L]$ and $\alpha = i, e$:

$$(n_\alpha)_t + (n_\alpha u_\alpha)_x = 0, \tag{2.1}$$

$$m_\alpha((n_\alpha u_\alpha)_t + (n_\alpha u_\alpha^2)_x) + (p_\alpha)_x = q_\alpha n_\alpha E\,. \tag{2.2}$$

The electric field $E = -\phi_x$ is given by the Poisson equation

$$-\varepsilon_0 \phi_{xx} = q(n_i - n_e)\,. \tag{2.3}$$

where ε_0 is the vacuum permittivity.

At the beginning of the process, there is no plasma in the gap. Therefore, the initial conditions are given by $n_i|_{t=0} = n_e|_{t=0} = 0$ in $[0, L]$, and $u_i|_{t=0}$, $u_e|_{t=0}$ are undefined. We assume that a quasineutral plasma is emitted from the cathode with the same velocities for the two species. So, the boundary conditions for (2.1), (2.2) are

$$n_i|_{x=0} = n_e|_{x=0} = n_0\,, \quad u_i|_{x=0} = u_e|_{x=0} = u_0\,, \tag{2.4}$$

on $\mathbb{R}^+$, where n_0 is the density of the injected plasma and u_0 its velocity. For a hyperbolic system, the number of boundary conditions which can be prescribed depends on the number of incoming characteristics. So, (2.4) should be understood as follows: suppose we introduce a discretization of (2.1), (2.2) by means of a Godunov scheme. Then, (2.4) will be used to compute the flux of the Riemann problem across the domain boundary.

Setting the origin of potential at the cathode, the boundary conditions for the Poisson equation are given by

$$\phi|_{x=0} = 0\,, \quad \phi|_{x=L} = \phi_L > 0\,, \tag{2.5}$$

where ϕ_L is the cathode potential.

We now introduce a scaling of system (2.1)-(2.3). Let us choose the size of the device L as characteristic length, the density n_0 and the velocity u_0 of the emitted plasma at the cathode as characteristic density and velocity respectively, the time $\tau = L/u_0$, the emitted ion pressure $p_0 = n_0 m_i u_0^2$ and the anode potential ϕ_L as characteristic time, pressure and potential respectively. We introduce three dimensionless parameters:

$$\varepsilon = \frac{m_e}{m_i}, \quad \eta = \frac{m_i u_0^2}{q\phi_L}, \quad \lambda = \frac{\varepsilon_0 \phi_L}{q n_0 L^2} = \frac{q\,\phi_L}{q^2/(\varepsilon_0\,(n_0\,L^2)^{-1})}\,, \tag{2.6}$$

respectively measuring the electron to ion mass ratio, the ratio of the thermal energy of the plasma to the applied potential energy and the ratio of the applied potential energy to the typical self-consistent potential energy. The scaled variables are defined by $\bar{x}$, $\bar{t}$, $\bar{\phi}$, $\bar{n}_\alpha$, $\bar{u}_\alpha$ and $\bar{p}_\alpha$ where $x = L\bar{x}$, $\phi = \phi_L\bar{\phi}$, $n_\alpha = n_0\bar{n}_\alpha$, $u_\alpha = u_0\bar{u}_\alpha$ and $p_\alpha = p_0\bar{p}_\alpha$. Then, omitting the bars, we get the scaled version of the two-fluid Euler-Poisson system posed on $[0,1]$:

$$(n_i)_t + (n_i u_i)_x = 0\,, \tag{2.7}$$

$$(n_i u_i)_t + \left(n_i u_i^2 + p_i(n_i)\right)_x = -\eta^{-1} n_i \phi_x\,, \tag{2.8}$$

$$(n_e)_t + (n_e u_e)_x = 0\,, \tag{2.9}$$

$$\varepsilon\left((n_e u_e)_t + (n_e u_e^2)_x\right) + (p_e(n_e))_x = \eta^{-1} n_e \phi_x\,, \tag{2.10}$$

$$-\lambda\phi_{xx} = n_i - n_e\,, \tag{2.11}$$

with the following dimensionless boundary conditions:

$$n_i|_{x=0} = n_e|_{x=0} = 1\,, \quad u_i|_{x=0} = u_e|_{x=0} = 1\,, \quad \phi|_{x=0} = 0\,, \quad \phi|_{x=1} = 1\,. \tag{2.12}$$

We are interested in the situation where λ is of order 1 and η very small. These values agree with those observed in some high current diodes (see [37]). In practice, ε is small, but cannot be neglected. Indeed, although the inertia of the electrons is small, they undergo large accelerations. Therefore, ε will be considered as an $O(1)$ quantity.

The numerical resolution of the two-fluid model (2.7)-(2.11) presents a very restrictive constraint related to the coupling with the Poisson equation. Indeed, the dimensionless Debye length $\bar{\lambda}_D = (\varepsilon_0 m_i u_0^2/(q^2 n_0 L^2))^{1/2} = (\eta\lambda)^{1/2}$ is the scale length of electrostatic interactions in the plasma. It is a well established fact in the physics literature that the space discretization Δx must satisfies $\Delta x \leq \bar{\lambda}_D$ for a numerical scheme to be stable. Therefore, for small η, a very fine mesh is needed to solve the problem, which is a very severe constraint for practical cases of interest. In order to bypass this numerical restriction, we investigate the limit $\eta \to 0$ of this model. This program is performed in the next section.

3. The asymptotic model. We now present the asymptotic model obtained when $\eta \to 0$ in (2.7)-(2.11). In this section, we concentrate on the main steps of this derivation and defer technical proofs to section 5. The asymptotic model is constituted of a quasineutral model for the plasma region and a Child-Langmuir law for the beam one.

At time $t = 0$, both the electron and ion fluids are going to penetrate into the diode. Because of the adiabatic law with $\gamma > 1$, the interface of each fluid with the vacuum is moving with finite speed. Let us denote by $X_e(t)$ and $X_i(t)$ the positions

of the electron-vacuum and ion-vacuum interfaces. Since the electrons are accelerated towards the anode by an electric field of order $O(1/\eta)$, $\eta \ll 1$, the electron-vacuum interface $X_e(t)$ is going to reach the anode $x = 1$ after a very short transient. We shall neglect this transient and suppose that $X_e(t) = 1$ as soon as $t > 0$. On the other hand, $X_i(t)$ stays inside the gap $(0, 1)$. As $\eta \to 0$, we assume that $X_i(t) \to X(t)$. The quasineutral model corresponds to the limit $\eta \to 0$ in the interval $[0, X(t)]$, where both fluids are present, while the Child-Langmuir model describes the electron fluid in the beam $[X(t), 1]$.

3.1. The plasma region. The quasineutral plasma model is obtained as the formal limit $\eta \to 0$ in (2.7)-(2.11) in the interval $[0, X(t)]$.

PROPOSITION 3.1. *We denote by n_e^η, n_i^η, u_e^η, u_i^η, ϕ^η the solutions of the two-fluid Euler-Poisson model (2.7)-(2.11). In $[0, X(t)]$, the formal limit $\eta \to 0$ gives:*

$$n_e^\eta, n_i^\eta \to n, \quad u_i^\eta \to u, \quad u_e^\eta \to u - j/n, \quad \phi^\eta \to 0$$

where n, u, j and ϕ are solutions to the following quasineutral Euler model with non-zero current:

$$n_t + (nu)_x = 0\,, \quad j_x = 0\,, \quad \phi = 0\,, \tag{3.1}$$

$$(1+\varepsilon)\left((nu)_t + (nu^2)_x\right) + \left(p_i(n) + p_e(n)\right)_x + \varepsilon\left(-2uj + \frac{j^2}{n}\right)_x = \varepsilon j_t\,, \tag{3.2}$$

The (formal) proof of proposition 3.1 is deferred to section 5. The quantity j is the plasma current and is the limit of $n_i^\eta u_i^\eta - n_e^\eta u_e^\eta$. Its value is unknown at this stage and will be specified further. The formal proof does not ensure that the boundary conditions (2.12) are kept. We shall assume that this holds true for n and u:

$$n|_{x=0} = 1\,, \quad u|_{x=0} = 1\,. \tag{3.3}$$

Note that, if $j \neq 0$, the boundary conditions for u_e^η is lost in the limit, since $u_e|_{x=0} = 1 - j \neq 1$. This is due to the appearance of a boundary layer at the cathode (see Figure 4.8). The study of this boundary layer is deferred to future work.

In (3.2), the additional flux term $\varepsilon(-2uj + j^2/n)_x$ is a reaction term due to the electron acceleration. If $j \neq 0$, the system (3.1)-(3.2) is strictly hyperbolic if and only if the plasma density n satisfies $n > n_H(j)$ with

$$n_H(j) = \left(\frac{\varepsilon j^2}{(c_i + c_e)\gamma(1+\varepsilon)}\right)^{1/(\gamma+1)}. \tag{3.4}$$

In the domain of hyperbolicity $n \in [n_H(j), \infty)$, the characteristic velocities are given by

$$l_\pm = u - \frac{\varepsilon\, j}{(1+\varepsilon)\, n} \pm \left(\frac{\gamma\,(c_i + c_e)\, n^{\gamma-1}}{1+\varepsilon} - \frac{\varepsilon\, j^2}{(1+\varepsilon)^2\, n^2}\right)^{1/2}.$$

Since, there is no ion mass flux through the interface $X(t)$, the plasma interface must move with velocity u:

$$\frac{dX}{dt} = u(X(t), t)\,. \tag{3.5}$$

Now, we look at how many boundary conditions we must impose to the quasineutral model (3.1)-(3.2) at $x = X(t)$. In a frame moving with the plasma-vacuum interface, the boundary becomes fixed and the hyperbolic problem has eigenvalues $l_- - u$ and $l_+ - u$. We shall be considering cases where $j < 0$. Indeed, electron emission at the plasma-vacuum interface requires that electrons are accelerated to velocities larger than the ion ones, thereby leading to a negative current. Then, obviously, $l_+ + u > 0$ and the corresponding characteristic field is outgoing relative to the domain $[0, X(t)]$. Now, an easy computation shows that $l_- - u > 0$ if and only if $n < n_P(j)$ where

$$n_P(j) = \left(\frac{\varepsilon j^2}{(c_i + c_e)\gamma}\right)^{1/(\gamma+1)} > n_H(j)\,. \tag{3.6}$$

Therefore, if $n > n_P(j)$, we must impose one additional boundary condition at the interface, while if $n_H(j) < n < n_P(j)$, no additional boundary condition is necessary. The determination of the additional condition when $n > n_P(j)$ will require the analysis of the transition problem (see below).

3.2. The beam region. In the beam region $[X(t), 1]$, there are no ions and the electron fluid satisfies a one-fluid Euler-Poisson system, which is deduced from the two-fluid one by letting $n_i = 0$:

$$(n_e)_t + (n_e u_e)_x = 0\,, \quad -\lambda\phi_{xx} = -n_e\,, \tag{3.7}$$

$$\varepsilon\left((n_e u_e)_t + (n_e u_e^2)_x\right) + (p_e(n_e))_x = \eta^{-1} n_e \phi_x\,, \tag{3.8}$$

Since the anode electric field is of order $O(1/\eta)$, electrons are accelerated to large velocities. To estimate this velocity, we let the electron kinetic energy εu_e^2 be of the order of the applied potential energy $(1/\eta)\phi|_{x=1} = 1/\eta$. Then, in the beam, u_e reaches values of the order of $1/\sqrt{\varepsilon\eta} \gg 1$. Therefore, the appropriate scaling to describe the beam is not the same as for the plasma and we need to rescale the electron velocity. We set $u_e = (\varepsilon\eta)^{-\frac{1}{2}}\bar{u}_e$ in (3.7)-(3.8). We obtain the following system on $[X(t), 1]$:

$$\sqrt{\varepsilon\eta}\,(n_e)_t + (n_e\bar{u}_e)_x = 0\,, \quad -\lambda\phi_{xx} = -n_e\,, \tag{3.9}$$

$$\varepsilon\left(\sqrt{\varepsilon\eta}\,(n_e\bar{u}_e)_t + (n_e\bar{u}_e^2)_x\right) + \varepsilon\eta\,(p_e(n_e))_x = \varepsilon\, n_e\phi_x\,, \tag{3.10}$$

where the boundary conditions at $x = X(t)$ are such that n_e, $\bar{u}_e$, ϕ match the values of the corresponding quantities for the 2-fluid Euler-Poisson model in the plasma region. Note that since $u_e = O(1)$ in the plasma region, we have $\bar{u}_e|_{x=X(t)} = (\varepsilon\eta)^{1/2}u_e|_{x=X(t)} = O((\varepsilon\eta)^{1/2}) \to 0$ as $\eta \to 0$. Similarly, since $\phi \to 0$ in the plasma region, we have $\phi|_{x=X(t)} \to 0$ as $\eta \to 0$. Finally, we recall that $\phi|_{x=1} = 1$.

Let us denote by n_e^η, $\bar{u}_e^\eta$, ϕ^η a solution of system (3.9)-(3.10) with the above specified boundary conditions. The limit $\eta \to 0$ is analyzed in the following:

PROPOSITION 3.2. . *As $\eta \to 0$, n_e^η, $\bar{u}_e^\eta$, ϕ^η converge to n_e, $\bar{u}_e$, ϕ, a solution of the Child-Langmuir problem on $[X(t), 1]$:*

$$(n_e\bar{u}_e)_x = 0\,, \quad (n_e\bar{u}_e^2)_x = n_e\phi_x\,, \quad -\lambda\phi_{xx} = -n_e\,, \tag{3.11}$$

with the following boundary conditions

$$\bar{u}_e|_{x=X(t)} = 0\,, \quad \phi|_{x=X(t)} = 0\,, \quad \phi|_{x=1} = 1\,. \tag{3.12}$$

Denote by $\bar{j}_e = n_e\bar{u}_e$. Then, $\bar{j}_e$ is independent of x on $[X(t), 1]$. We introduce the Child-Langmuir current:

$$\bar{j}_{CL}(t) = \frac{4\sqrt{2}\lambda}{9\,(1 - X(t))^2}\,. \tag{3.13}$$

Then, for any value $\bar{j}_e \in [0, \bar{j}_{CL}(t)]$, *there exists a unique solution* n_e, $\bar{u}_e$, ϕ, *with* ϕ *strictly positive on* $(X(t), 1]$, *given by* $n_e = \bar{j}_e/\sqrt{2\phi}$, $\bar{u}_e = \sqrt{2\phi}$. *The potential* ϕ *is implicitly determined by*

$$\int_0^{\phi(x)} \frac{d\psi}{\sqrt{\mu^2 + 2\sqrt{2}\,\lambda^{-1}\bar{j}_e\sqrt{\psi}}} = x - X(t)\,, \tag{3.14}$$

where μ *is determined such that* $\phi|_{x=1} = 1$ *or equivalently, such that (3.14) holds with* $x = 1$ *and* $\phi = 1$. *In particular, for* $\bar{j}_e = \bar{j}_{CL}(t)$, *we have* $\mu = 0$ *and* $\phi = ((x - X)/(1 - X))^{4/3}$. *There is no solution for* $\bar{j}_e \notin [0, \bar{j}_{CL}(t)]$.

The formal proof of the convergence of the solutions of system (3.9)-(3.10) towards those of system (3.11) is obvious. That the solutions of (3.11) are given by (3.14) is classical. We refer the reader for instance to [17].

We note that $\mu = (\phi_x)|_{x=X(t)}$. We have $\mu \in [0, 1/(1 - X)]$. Indeed, μ cannot become negative because ϕ has to stay positive. Similarly, if $\mu > 1/(1 - X)$, then the potential cannot remain a convex function and match the boundary condition $\phi = 1$ at $x = 1$. But, if ϕ becomes concave, this means that n_e is negative, which is forbidden. Relation (3.14) (for $x = 1$ and $\phi = 1$) defines a one-to-one, onto mapping $\mu \to \bar{j}_e$ which is a decreasing function from $[0, 1/(1 - X)]$ to $[0, \bar{j}_{CL}(t)]$. Going back to the scaling used for the quasineutral model, the electron flux j_e and the Child-Langmuir current j_{CL} are given by $j_e, j_{CL} = (\varepsilon\eta)^{-1/2}\bar{j}_e, \bar{j}_{CL}$.

The Child-Langmuir problem has been thoroughly analyzed in the literature (see e.g. [10], [17]).

3.3. The transition region. Now, the key point of the modeling consists in connecting the quasineutral model (3.1)-(3.2) with the Child-Langmuir one (3.11). So far, none of the two models is closed because certain data are still to be specified. These lacking relations are: the value of the current j and the additional boundary condition(s) at $x = X(t)$ for the quasineutral model (see the discussion at the end of section 3.1) and the value of the electron flux j_e for the Child-Langmuir model. We recall that the plasma boundary $X(t)$ moves with the velocity u according to (3.5). The values of j, j_e and the additional boundary condition at $x = X(t)$ is found through the analysis of a transmission problem and will connect the two models.

The transmission layer problem is deduced from the original two-fluid model by stretching the space variable about the plasma-vacuum interface. We change the position variable x to the stretched variable $\xi = (x - X(t))/\eta^{1/2}$ and we let $n_e(x, t) = \tilde{n}_e(\xi, t)$ and similarly for n_i, u_e and u_i. The potential is rescaled according to $\phi(x, t) = \eta\tilde{\phi}(\xi, t)$. Inserting this change of variables into (2.7)-(2.11) and denoting by $\sigma = dX/dt$ the interface velocity, we get (dropping the tildes for simplicity):

$$\eta^{1/2}(n_i)_t + (n_i(u_i - \sigma))_\xi = 0\,, \tag{3.15}$$

$$\eta^{1/2}(n_i u_i)_t + (n_i u_i(u_i - \sigma) + p_i(n_i))_\xi = -n_i\phi_\xi\,, \tag{3.16}$$

$$\eta^{1/2}(n_e)_t + (n_e(u_e - \sigma))_\xi = 0\,, \tag{3.17}$$

$$\varepsilon\eta^{1/2}(n_e u_e)_t + (\varepsilon n_e u_e(u_e - \sigma) + p_e(n_e))_\xi = n_e\phi_\xi\,, \tag{3.18}$$

$$-\lambda\phi_{xx} = n_i - n_e\,, \tag{3.19}$$

Neglecting terms of order $O(\eta^{1/2})$, we obtain the following travelling-wave problem

for the two-fluid Euler-Poisson problem:

$$(n_i(u_i - \sigma))' = 0\,, \tag{3.20}$$
$$(n_i u_i(u_i - \sigma) + p_i(n_i))' = -n_i\phi'\,, \tag{3.21}$$
$$(n_e(u_e - \sigma))' = 0\,, \tag{3.22}$$
$$(\varepsilon n_e u_e(u_e - \sigma) + p_e(n_e))' = n_e\phi'\,, \tag{3.23}$$
$$-\lambda\phi'' = n_i - n_e\,, \tag{3.24}$$

where primes denote ξ-derivatives for simplicity.

We look for solutions of this travelling-wave problem which reconnect to the solutions of the quasineutral model on the left-hand side, i.e. for $\xi \to -\infty$ and to the solutions of the Child-Langmuir problem on the right-hand side, i.e. for $\xi \to \infty$. Additionally, since the interface is located at $\xi = 0$, we let $n_i = 0$ for $\xi > 0$. Furthermore, we demand that n_i be continuous at $\xi = 0$. Indeed, the ion fluid in the two-fluids Euler-Poisson model is an ordinary fluid. For an ordinary fluid, we know that no shock can border the vacuum [38], otherwise the Rankine-Hugoniot condition for the momentum equation cannot be satisfied across the interface. Therefore, we must disregard solutions of the transition problem for which n_i exhibits a discontinuity at the interface $\xi = 0$.

The boundary conditions are, for $\xi \to -\infty$:

$$n_i\,,\, n_e \to n_-\,, \quad u_i \to u_-\,, \quad u_e \to u_- - j/n_-\,, \quad \phi\,,\, \phi_\xi \to 0\,, \tag{3.25}$$

where we denote by $n_- = n|_{X(t)-0}$, $u_- = u|_{X(t)-0}$ the limit values taken by the solution of the quasineutral model at the interface, and j is the plasma current of the quasineutral model.

For $\xi \to \infty$, u_e must reconnect to the Child-Langmuir velocity, which is large of the order $O(\eta^{-1/2})$, because of the change of scale in the beam region (see discussion at the beginning of section 3.2). Therefore, for $\xi \to \infty$, $u_e \sim \eta^{-1/2}$ while σ is of order $O(1)$. We deduce that $n_e(u_e - \sigma) \sim n_e u_e$ as $\xi \to \infty$ and the reconnection to the Child-Langmuir region implies therefore that

$$n_e(u_e - \sigma) \to j_e\,, \qquad \text{as } \xi \to \infty\,, \tag{3.26}$$

where j_e is the electron flux of the Child-Langmuir problem. Finally, we shall be looking for a solution such that n_e is bounded as $\xi \to \infty$, otherwise the travelling-wave solution cannot be used to reconnect the two models.

Now, for the existence of solutions of travelling-wave solutions, we have the following

PROPOSITION 3.3. *Let σ, $n_- > 0$, u_-, j and j_e by given. Let us suppose additionally that $\gamma \geq 2$. Then, there exists a (smooth or unsmooth) solution to the travelling-wave problem (3.20)-(3.24), satisfying (3.25) for $\xi \to -\infty$, (3.26) for $\xi \to +\infty$, and such that n_e is bounded, $n_i = 0$ for $\xi \geq 0$, and n_i is continuous across $\xi = 0$, if and only if the following relations hold:*

$$j = -j_e\,, \quad u_- = \sigma\,, \quad n_- \in [n_H(j), n_P(j)]\,, \tag{3.27}$$

where $n_H(j)$ and $n_P(j)$ are given by (3.4) and (3.6).

This proposition is proved in section 5. The condition $\gamma \geq 2$ is a technical one. It can probably be removed at the expense of more analytical work.

Eqs. (3.27) provide some of the closure relations which connect the plasma and the beam models, but not all of them. That u_- must be equal to σ is obvious. It is just a rephrasing of (3.5) and states that there is no ion flux through the interface. The two other relations of (3.27) are more informative. The first one says that the plasma current of the quasineutral model must be equal to the electron flux in the beam region (up to a sign, due to the definitions of these quantities). However, it does not provide the exact value of this current. From the Child-Langmuir problem (see proposition 3.2), we know that $0 \leq j_e \leq j_{CL}$.

The last relation (3.27) provides an inequality constraint for the boundary condition at $x = X(t)$ of the quasineutral model. In view of the discussion at the end of section 3.1, if $n_- \leq n_P(j)$, no additional condition at $X(t)$ is necessary since then, the fluid is supersonic. Therefore, condition (3.27) fully determines the boundary conditions for the quasineutral model at the interface. This condition is nothing but the Bohm sheath criterion when the beam region is viewed as a sheath (see e.g. [3], [30] and the bibliography on plasma sheaths given in the introduction).

In order to close the model, we now have to assign a value to $j_e = -j \in [0, j_{CL}]$, since this value cannot be found from the formal asymptotic analysis. Based on our numerical experiments, we shall assume the following:

$$j_e = -j = j_{CL} = (\varepsilon\eta)^{-1/2} \frac{4\sqrt{2}\lambda}{9\,(1 - X(t))^2}\,. \tag{3.28}$$

3.4. The asymptotic model: summary. We now summarize our asymptotic model for the plasma expansion problem. It consists of:
(i) The quasineutral model (3.1)-(3.2) on $[0, X(t)]$ with boundary conditions (3.3) at $x = 0$;
(ii) The Child-Langmuir model (3.11) on $[X(t), 1]$ with boundary conditions (3.12);
(iii) Eq. (3.5) for the motion of $X(t)$;
(iv) Eqs. (3.28) which specifies the value of the currents in the quasineutral and Child-Langmuir models and the third condition (3.27) which constrains the limit value of n at the interface $X(t)$ for the quasineutral model.

4. Numerical simulations .

4.1. Numerical method. In this section, we present the numerical methods. We shall concentrate on the resolution of the asymptotic model (section 3.4). A few remarks about the two-fluid Euler-Poisson model will be made at the end of this section.

First, we write the quasineutral model (3.1), (3.2) in conservative form:

$$W_t + F(W)_x = S \tag{4.1}$$

where $W = (n, nu)^T$, $F(W) = (nu, nu^2 + (1+\varepsilon)^{-1}((p_i + p_e)(n) + \varepsilon(-2uj + j^2/n)))^T$ and $S = (0, (1+\varepsilon)^{-1}\varepsilon j_t)^T$.

We define a uniform grid of size Δx on the spatial domain $[0, 1]$ with K cells $M_k =]x_{k-1/2}, x_{k+1/2}[$, $k = \{1...K\}$. Let Δt be the time step, we set $t^n = n\Delta t$ for all $n \in \mathbb{N}$. Let us assume that a piecewise constant approximation of the system (4.1) is given. Thus, we denote by W_k^n the approximation of W on $M_k \times [t^n, t^{n+1}[$ and we use a finite volume method [23], [38]. We assume that at time t^n, the interface position X^n is in M_{k_0+1}. We use a Godunov type solver subject to a CFL stability condition. Thus, on the cell M_{k_0+1}, Δt should be of the order of $(X^n - x_{k_0-1/2})$ which can be

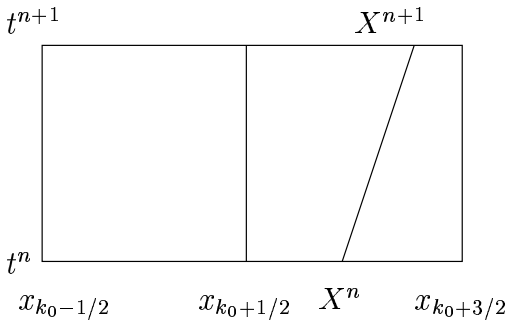

FIG. 4.1. *Last cell of the mesh:* $M_{\tilde{k}_0}$

very small. For this reason, we rather define $M_{\tilde{k}_0}$, the last cell filled by the plasma, as the union of M_{k_0} and M_{k_0+1} (see figure 4.1). Moreover, we assume that $W^n_{\tilde{k}_0}$ is the approximation of W on $[x_{k_0-1/2}, X^n]$ where we recall that $x_{k_0-1/2}$ is the interface between the cells M_{k_0-1} and M_{k_0}.

Now, let us describe the discretization of the system (4.1). The first step consists in defining the interface motion. From (3.5), the interface position at time t^{n+1} is approximated by $X^{n+1} = X^n + \Delta t\, \bar{u}^n$, where the interface velocity $\bar{u}^n$ is computed by solving a Riemann problem, i.e. by solving (4.1) with a piecewise constant initial data $W^n_L = W^n_{\tilde{k}_0}$ if $x < 0$, and $W^n_R = (0,0)^T$ if $x > 0$. Therefore, we have to specify how we solve this Riemann problem, and in particular, how we enforce the third condition (3.27).

Suppose first that $n^n_L \leq n^n_P := n_P(j^n)$, where n_P is given by (3.6). Then the solution of the Riemann problem is just the constant state W^n_L separated from the vacuum W^n_R by a shock moving with speed u^n_L. If $n^n_L > n^n_P$, the solution of the Riemann problem involves a rarefaction wave associated with the first characteristic field connecting n^n_L to n^n_P. The state associated with n^n_P in this rarefaction wave has velocity denoted by u^n_V. Then, the rarefaction wave is directly followed by a shock of velocity u^n_V which relates the state (n^n_P, u^n_V) to the vacuum W^n_R.

Next, we make a few remarks about the finite volume method to solve (4.1). We have used, either the HLLE scheme [38] or the polynomial upwind scheme [16]. This is a standard point, except for the treatment of the last cell. For that purpose, we use an integration of the system (4.1) on the trapeze T defined by the points $(x_{k_0-1/2}, t^n), (X^n, t^n), (X^{n+1}, t^{n+1}), (x_{k_0-1/2}, t^{n+1})$. The flux along $[(X^n, t^n), (X^{n+1}, t^{n+1})]$ is computed from the exact solution of the Riemann problem as developed above.

The last step of the numerical method concerns the treatment of the source term S in (4.1). Since the component of S on the mass conservation equation is 0, we can update the plasma density first. We also compute the interface position at X^{n+1} at time t^{n+1} because the knowledge of $\bar{u}^n$ only requires the knowledge of the solution at time t^n. From X^{n+1}, using (3.13), we compute $j^{n+1} := j(t^{n+1})$. Then we can update the plasma velocity with a semi-implicit treatment of the source term. It turns out that such a semi-implicit treatment greatly improves the stability of the scheme.

The numerical simulation of the two-fluid Euler-Poisson model is straightforward. We use the same treatment of the interface, just replacing the Riemann solver by an exact solver for the Euler equation when one of the states is the vacuum [38]. The treatment of the source term in the momentum equation (which now depends on

the electric field through the resolution of the Poisson equation) is done in a similar implicit way: again, we exploit the fact that there is no source term in the mass conservation equations.

4.2. Numerical results. In the first simulation corresponding to Figures 4.2, 4.3, 4.4, 4.5 and 4.6, we compare the asymptotic model with the original two-fluid one. For the problems we have in mind, namely the plasma diode problem, the physical values of the parameters should be $\gamma = 2$, $\lambda \sim 10^{-3}$, $\varepsilon \sim 5.10^{-4}$ and $\eta \sim 10^{-6}$. In this simulation, we rather use $\varepsilon = 5.10^{-1}$, $\eta = 10^{-4}$ essentially because of the numerical problems occurring during the simulation of the two-fluid model.

Figures 4.2 and 4.3 show the density and velocity for the ions and the electrons respectively in the plasma region, i.e. between the cathode and the interface $X(t)$. The present asymptotic model leads to the right interface velocity and the right density and velocity profiles.

Figure 4.4 displays the electron density and velocity in the beam region. The singularity of the density profile close to the interface as well as the electron acceleration from the interface to the anode is well approximated by the Child-Langmuir law. On Figure 4.5 we compare the electrostatic potential given by the Poisson equation and the one obtained by the Child-Langmuir law. We can see again that the beam part is well modeled. Finally, on Figure 4.6 we compare the current computed from the two-fluid model with the Child-Langmuir current computed with the asymptotic model. The left figure displays the current as a function of x for three sampled values of time. The right picture shows the anode current as a function of time. We can see that the Child-Langmuir current gives a very good approximation of the two-fluid current. The two-fluid model exhibits a current peak close to the interface.The existence of this peak can be explained thanks to the solution of the transmission problem. Indeed, the solution of the transmission problem satisfies (see section 5.2) $n_e(u_e - \sigma) = j_e$, $n_i(u_i - \sigma) = 0$, where j_e and σ are constant with rapport to x. Thus $j = n_i u_i - n_e u_e = -j_e + (n_i - n_e)\sigma$. Therefore, j is, up to constant, proportional to $n_i - n_e$. An analysis of the monotony of $n_i - n_e$ as a function of x easily shows that it has the same behaviour as that exhibited by the curves shown on Figure 4.6 (left). The details are left to the reader.

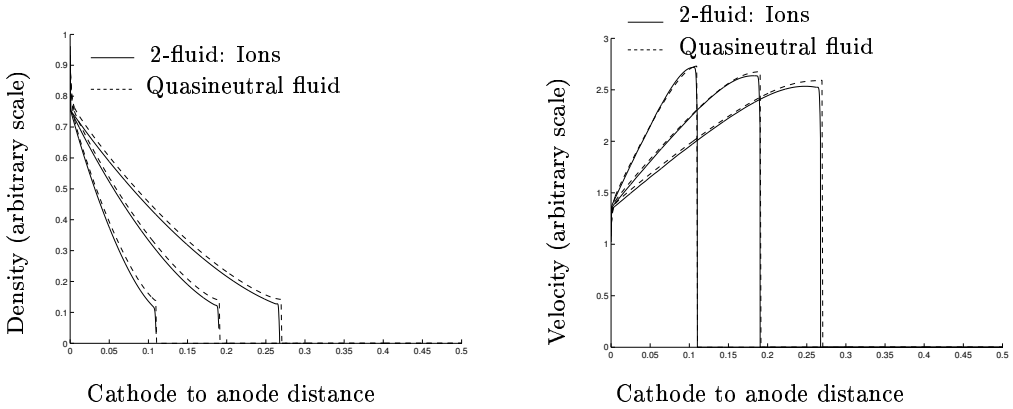

FIG. 4.2. *Densities and velocities of the ion fluid (2-fluid Euler-Poisson model) compared with the asymptotic model. The values are observed between the cathode and the interface $x = X(t)$ at times $t = 0.04\tau, t = 0.07\tau, t = 0.1\tau$, τ being the time scale (see section 2).*

In the second simulation corresponding to Figure 4.7, the physical parameters are

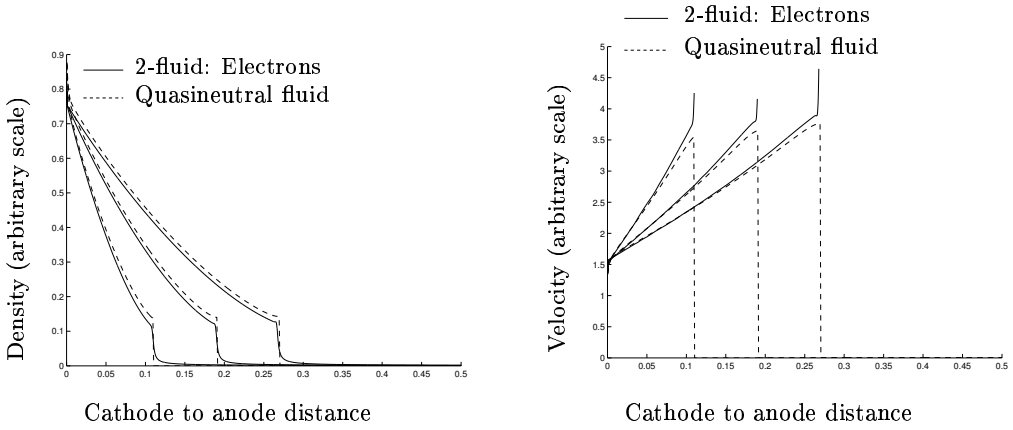

FIG. 4.3. *Densities and velocities of the electron fluid (2-fluid Euler-Poisson model) compared with the asymptotic model. The values are observed between the cathode and the interface $x = X(t)$ at times $t = 0.04\tau, t = 0.07\tau, t = 0.1\tau$.*

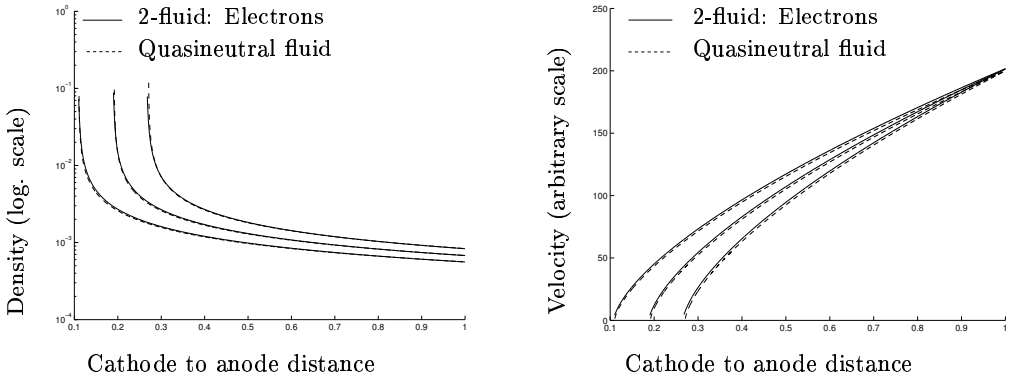

FIG. 4.4. *Densities and velocities of the electron fluid (2-fluid Euler-Poisson model) compared with the Child-Langmuir model: values observed between the interface $x = X(t)$ and the anode $x = 1$ at times $t = 0.04\tau, t = 0.07\tau, t = 0.1\tau$.*

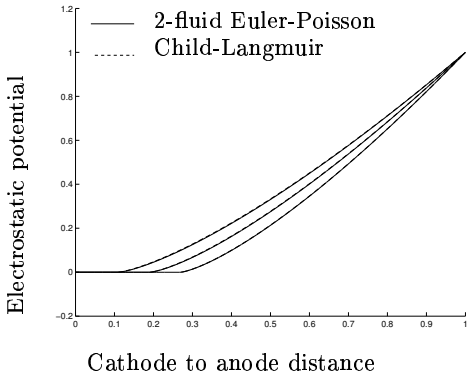

FIG. 4.5. *Electrostatic potential computed from the Poisson equation of the two-fluid model and from the asymptotic model at times $t = 0.04\tau, t = 0.07\tau, t = 0.1\tau$.*

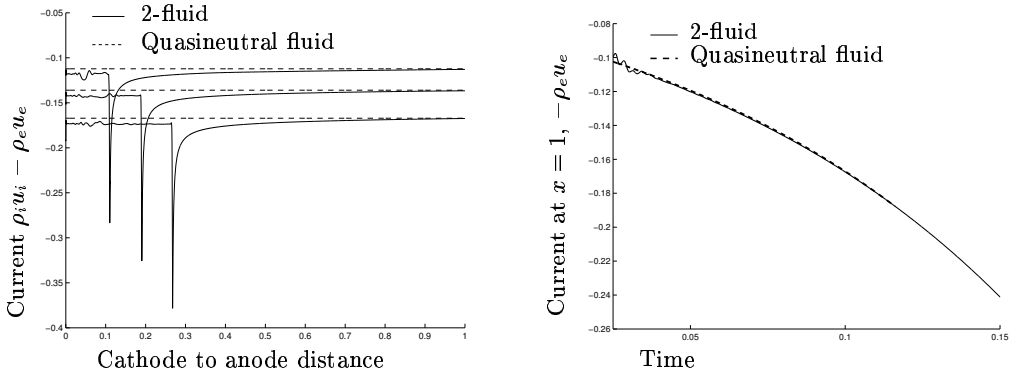

FIG. 4.6. *Left picture: current as a function of distance at time $t = 0.04\tau$, $t = 0.07\tau$ and $t = 0.1\tau$. Solid line: current j of the two-fluid model; dashed line: Child-Langmuir current j_{CL} of the asymptotic model. Right picture: anode current $j(1)$ as a function of time. Solid line: two fluid model; dashed line: quasineutral model.*

still $\gamma = 2$, $\lambda = 10^{-3}$ and $\eta = 10^{-4}$ but we reverse the role of the electrons and the ions. Rather than reversing the anode potential, we change the mass ratio to $\varepsilon = 2$, which is the reciprocal of the previous value of ε. On Figure 4.7, we can observe that the behaviour is about the same as in the previous simulation, which shows that the model is quite insensitive to the value of ε.

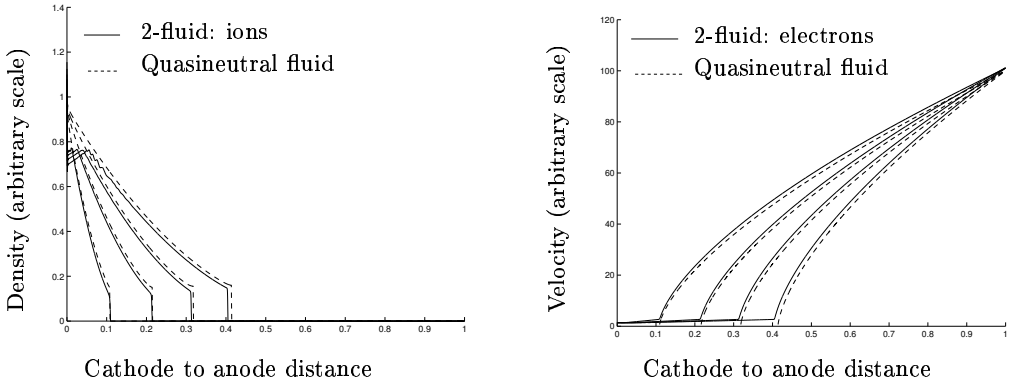

FIG. 4.7. *Behaviour of the 2-fluid Euler-Poisson model when $\varepsilon = 2$ in comparison with the asymptotic model at time $t = 0.05\tau$, $t = 0.1\tau$, $t = 0.15\tau$, $t = 0.2\tau$: ion and quasineutral densities (left picture); electron and Child-Langmuir velocities (right picture)*

In the third simulation (Figures 4.8 and 4.9), we study the behaviour of the solution of the two-fluid system when the dimensionless parameter η varies. We consider $\gamma = 2$, $\varepsilon = 5.10^{-1}$, $\lambda = 10^{-3}$. Figure 4.8 shows the plasma region on the left and the beam region on the right. We see the effect of the electron acceleration on the plasma motion. We notice that the electron acceleration to the anode is increasing as η decreases to 0 due to the factor $1/\eta$ in the source term of the momentum equation of the two-fluid system. Moreover, the faster electrons are in the beam, the slower the plasma motion is. That is why the plasma region cannot be well-described by a standard quasineutral limit. This effect is taken into account in the asymptotic model through the additional pressure term in (3.2). Figure 4.9 is a zoom-in near the

cathode of the left picture of Figure 4.8. We can remark the existence of a boundary layer at the cathode which results from the loss of a boundary condition in the limit $\eta \to 0$. Indeed, in the two-fluid model, the electron velocity equals 1 at the cathode when in the asymptotic model it equals $1 - j/n < 1$. Note that the width of this boundary layer is a decreasing function of η, as it should.

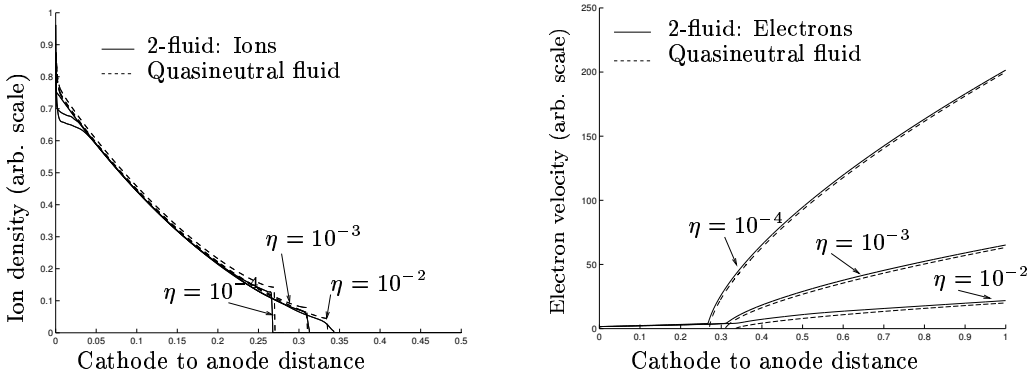

FIG. 4.8. *Ion density in the plasma region (left picture) and electron velocity (right picture): comparison between the Euler-Poisson and quasineutral models (right picture) for* $\eta = 10^{-2}$, $\eta = 10^{-3}$, $\eta = 10^{-4}$ *at time* $t = 0.1\tau$

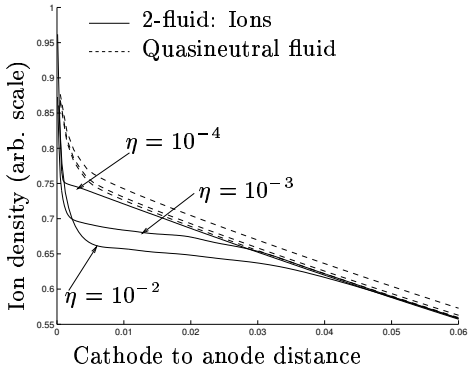

FIG. 4.9. *Zoom-in near the cathode of the ion density: comparison between the Euler-Poisson and quasineutral models for* $\eta = 10^{-2}$, $\eta = 10^{-3}$, $\eta = 10^{-4}$ *at time* $t = 0.1\tau$

In the fourth simulation (Figure 4.10), we look at the asymptotic model when η becomes smaller. Figure 4.10 shows the plasma region on the left picture and the beam part on the right picture. We note that contrary to the two-fluid model, the asymptotic model allows simulations of the plasma expansion even for physically realistic values of η (in our case, $\eta \sim 10^{-6}$).

Let us note that in the two-fluid model simulations, we use a space step $\Delta x \leq 2.10^{-4}$ with a CFL coefficient equal to 0.8. In those of the asymptotic model, $\Delta x = 10^{-3}$ is sufficient. Thus, the limit model gives a good approximation of the original two-fluid model with a lower numerical cost. However, our numerical method breaks down outside the domain of hyperbolicity of the system (3.1)-(3.2) i.e. when the density becomes smaller than n_H defined by (3.4). However, numerically, the solution can actually leave the domain of hyperbolicity. Indeed, on Figure 4.11, at time $t =$

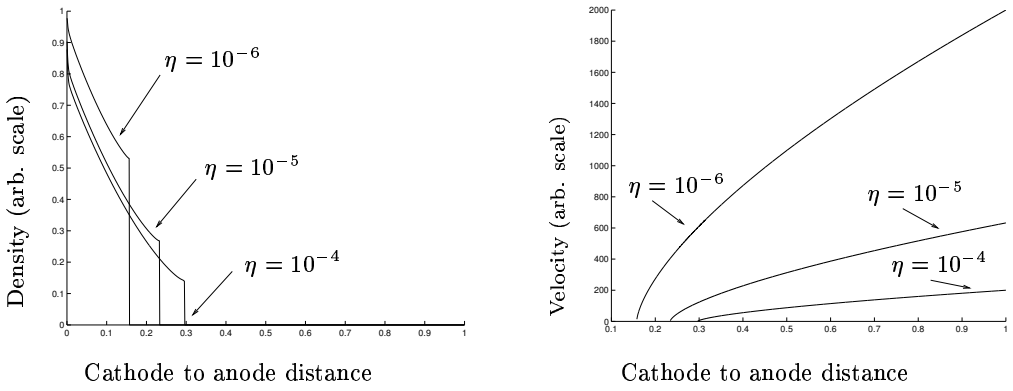

FIG. 4.10. *Density in the plasma region and velocity in the beam part both given by the asymptotic model for* $\eta = 10^{-4}$, $\eta = 10^{-5}$, $\eta = 10^{-6}$ *at time* $t = 0.11\tau$

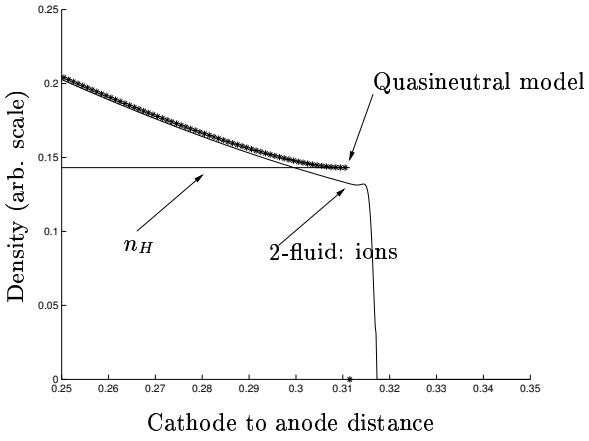

FIG. 4.11. *Plasma density at time* $t = 0.116\,\tau$ *when the hyperbolic condition breaks down in the quasineutral model with parameters:* $\eta = 10^{-4}$, $\lambda = 10^{-3}$ *and* $\varepsilon = 0.5$.

$0.116\,\tau$, we compare the density given by the quasineutral model (3.1)-(3.2) with the analytic value of n_H (3.4) computed from the plasma-vacuum position. In the last cell before the interface, at the plasma front, the hyperbolicity condition is not satisfied by the quasineutral model and the numerical program is stopped. This problem does not depend on the mesh size or the stability condition. Moreover, as shown on Figure 4.11, the solution of the original two-fluid system is not in the domain of hyperbolicity of the quasineutral model. Thus, for greater times, we are not able to compute the solution of the asymptotic model. Future works dealing with new models for the quasineutral region are in progress in [9] in order to improve these results.

5. Proofs.

5.1. Proof of proposition 3.1. For $\eta > 0$, let us denote by $X_i^\eta(t)$ the position of the ion-vacuum interface at time t. Then $X_i^\eta(t) = \sup\{x \in [0,1]\,;\, n_i^\eta(y,t) > 0,\ \forall y \in [0,x]\}$. Of course, $X_i^\eta(t) \to X(t)$ as $\eta \to 0$. We suppose that in $[0, X(t) - \delta]$, where $\delta > 0$ is an arbitrary small number, the unknowns of the system (2.7)-(2.11) can be

expanded according to the following (Hilbert) expansion:

$$n_\alpha^\eta(x,t) = n_\alpha(x,t) + \eta n_{\alpha,1}(x,t) + \dots ,$$

and similarly for u_α^η and ϕ^η. Note that we exclude the occurrence of oscillations at any period tending to zero with η. This is a strong restriction, since it is believed that the quasineutral limit might be an oscillatory one. However, our numerical comparisons (see section 4) do not indicate the presence of such oscillations and we discard them here. Inserting the Hilbert expansion into (2.7)-(2.11) we obtain, at leading order:

$$(n_i)_t + (n_i u_i)_x = 0\,, \quad (n_e)_t + (n_e u_e)_x = 0, \tag{5.1}$$

$$n_i\,\phi_x = 0\,, \quad n_e\,\phi_x = 0\,, \quad -\lambda\,\phi_{xx} = n_i - n_e \tag{5.2}$$

and at order 1:

$$(n_i u_i)_t + (n_i u_i^2 + p_i(n_i))_x = -n_{i,1}\,\phi_x - n_i(\phi_1)_x\,, \tag{5.3}$$

$$\varepsilon((n_e u_e)_t + (n_e u_e^2)_x) + (p_e(n_e))_x = n_{e,1}\,\phi_x - n_e(\phi_1)_x\,. \tag{5.4}$$

Using the first two eqs. of (5.2), we obtain $\phi_x(x,t) = 0$, which, with the boundary condition (2.12) at $x = 0$, leads to $\phi \equiv 0$ on $[0, X(t) - \delta]$. Then, from the third eq. (5.2), we deduce that $n_i = n_e = n$ on $[0, X(t) - \delta]$.

Thus, subtracting the two density conservation eqs in (5.1) and defining $j = n_i u_i - n_e u_e = n(u_i - u_e)$, we get $j_x = 0$ and $u_e = u_i - j/n$. Adding up (5.3) and (5.4) and inserting this relation into the resulting equation, we get (3.2). This concludes the proof of proposition 3.1 since any of the two equation (5.1) leads to the first equation (3.1). □

5.2. Proof of proposition 3.3. We show that the travelling-wave problem (3.20)-(3.24) can be reduced to a nonlinear Poisson equation. Then, this Poisson equation is analyzed via phase-portrait techniques.

From the ion mass conservation equation in the plasma (3.20), we have $n_i(u_i - \sigma) = \text{Constant}$. But $n_i \equiv 0$ for $\xi > 0$, so that $\sigma = u_i = \text{Constant}$. Then, with the boundary conditions (3.25), we deduce that $\sigma = u_i = u_-$. Next, from the electron mass conservation equation (3.22) and the boundary conditions (3.25) and (3.26), we obtain that $n_e(u_e - \sigma) = \text{Constant}$ and $n_e(u_e - \sigma) = n_e(u_e - u_i) \to -j$ as $\xi \to -\infty$ and $n_e(u_e - \sigma) \to j_e$ as $\xi \to \infty$. Therefore, $j_e = -j$. We deduce that $u_e' = j_e(1/n_e)'$.

The remaining equations (3.21), (3.23), (3.24) then reduce to

$$(p_i(n_i))' = -n_i\phi'\,, \forall \xi < 0 \quad ; \quad n_i = 0\,, \forall \xi > 0\,. \tag{5.5}$$

$$\varepsilon j_e^2\,(1/n_e)' + p_e(n_e)' = n_e\phi'\,, \quad \text{on} \quad \mathbb{R} \tag{5.6}$$

$$-\lambda\phi'' = n_i - n_e\,. \tag{5.7}$$

Let us define $h_\alpha(n)$ for $\alpha = e, i$ such that $\partial_n h_\alpha = (1/n)\partial_n p_\alpha$, i.e. $h_\alpha(n) = c_\alpha\,\gamma\,(\gamma-1)^{-1}\,n^{\gamma-1}$. $h_\alpha(n)$ is an increasing function of n. From (5.5) and (3.25), we get $h_i(n_i(\xi)) - h_i(n_-) = -\phi(\xi)$ for all $\xi < 0$, or, since h_i is invertible:

$$n_i(\xi) = n_i[\phi(\xi)]\,, \quad \xi < 0 \quad ; \quad n_i(\xi) = 0\,, \quad \xi > 0\,, \tag{5.8}$$

with

$$n_i[\phi] = h_i^{-1}\,(h_i(n_-) - \phi)\,. \tag{5.9}$$

Note that n_i is a decreasing function of ϕ.

Next, introducing the function $k_e(n)$ defined by

$$k_e(n) = \frac{\varepsilon j_e^2}{2n^2} + h_e(n)\,, \quad n \geq 0\,,$$

we have $n\partial_n k_e = -\varepsilon\,(j_e^2/n^2) + \partial_n p_e$, and using again (3.25), eq. (5.5) can be written

$$k_e(n_e(\xi)) - k_e(n_-) = \phi(\xi) \quad \text{for} \quad \xi \in \mathbb{R}\,. \tag{5.10}$$

In contrast with h_i, k_e is a non monotonous function: it is decreasing on $[0, n_{min}]$ and increasing on $[n_{min}, +\infty)$ where n_{min} is given by

$$n_{min}(j) = \left(\frac{\varepsilon j^2}{c_e\,\gamma}\right)^{\frac{1}{\gamma+1}} > n_P(j)\,. \tag{5.11}$$

Let us denote by $k_{e,+}$ (resp. $k_{e,-}$) the restriction of k_e to $[n_{min}, +\infty)$ (resp. to $[0, n_{min}]$). Then $k_{e,+}$ is an increasing function and $k_{e,-}$ is a decreasing one. Note that the electron gas is subsonic on the interval $[n_{min}, +\infty)$ while it is supersonic on the interval $(0, n_{min}]$ (indeed, the condition $k_e' > 0$ is equivalent to $p_e'(n_e) > \varepsilon j_e^2/n_e^2 = \varepsilon u_e^2$, which is exactly saying that the sound speed is larger than the fluid velocity).

Therefore, the inversion of (5.10) involves two solutions, using either branches $k_{e,+}^{-1}$ or $k_{e,-}^{-1}$. For a smooth solution, the passage from one branch to the other one implies that n takes the critical value n_{min} at a given point. For an unsmooth solution, the entropy condition only allows jumps from the supersonic branch $k_{e,-}$ to the subsonic one $k_{e,+}$. When ξ is close enough to $-\infty$, by continuity, the branch will be determined by the position of n_- with respect to n_{min}. Furthermore, n_- being the limit value of a solution to the quasineutral model at the interface, it belongs to its domain of hyperbolicity and consequently satisfies $n_- > n_H$.

So, we must discuss according to the position of n_- relative to the value n_{min}.

(i) Case $n_- > n_{min}$: Then, in the neighborhood of $\xi = -\infty$, we can invert (5.10) using $k_{e,+}^{-1}$ and get:

$$n_e(\xi) = n_e[\phi(\xi)]\,, \quad n_e[\phi] = k_{e,+}^{-1}\,(\phi + k_e(n_-))\ . \tag{5.12}$$

Note that n_e is an increasing function of ϕ. Then, in the neighborhood of $\xi = -\infty$, the Poisson equation is written:

$$\phi_{\xi\xi} = \lambda^{-1}(n_e[\phi] - n_i[\phi])\,. \tag{5.13}$$

with the condition that ϕ and ϕ_ξ tend to zero as $\xi \to -\infty$. For (5.13) written as a first order system for the two-dimensional variable (ϕ, E):

$$\phi_\xi = E\,, \quad E_\xi = \lambda^{-1}(n_e[\phi] - n_i[\phi])\,, \tag{5.14}$$

the point $(0,0)$ is a stationary point. Moreover, the right-hand side of (5.14) is a strictly increasing function of ϕ. Therefore, the point $(0,0)$ is a hyperbolic point.

There are two branches of solutions leaving $(0,0)$ (see Figure 5.1): the first one for which both ϕ and E are increasing as ξ increases, the second one for which both ϕ and E are decreasing. None of these branches correspond to the requested solution.

Indeed, let us consider the first branch. Then E and ϕ increase w.r.t. ξ, and so does n_e while n_i decreases. Therefore, $n_e[\phi] - n_i[\phi]$ continuously increases. No

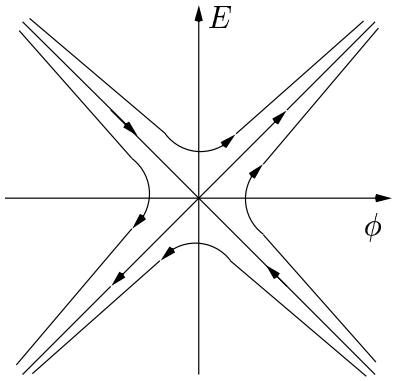

FIG. 5.1. *Phase-portrait of the solution of system (5.14) near the stationary point* $(0,0)$ *in the cases* $n_- > n_{min}$ *and* $n_- < n_P$.

jump to the other k_e^{-1} branch is allowed since n_e is in the subsonic zone and jumps are only allowed from the supersonic to the subsonic zone by the entropy condition. Furthermore, when ξ crosses 0, n_i vanishes identically, but this does not change the sign of $n_e[\phi] - n_i[\phi]$ which further increases. n_e cannot be bounded because ϕ is strictly convex and thus, strictly increasing. Then n_e is also strictly increasing and tends to ∞ when $\xi \to \infty$.

Now, let us consider the second branch. Then, as long as n_i is positive (i.e. $\xi < 0$), E and ϕ decrease w.r.t. ξ, and so does n_e while n_i increases. Suppose that at some point $\xi_0 < 0$, $n_e[\phi]$ reaches the value n_{min}. Then, ϕ cannot decrease any longer since otherwise, (5.10) could not be satisfied. However, $n_{min} - n_i(\xi_0) < 0$. Therefore, this point is not a stationary point and the solution cannot be extended any further. Consequently, the point $\xi = 0$ must be crossed before n_e reaches the value n_{min}. But, at this point, $n_i(\xi - 0) > n_- > 0$. Therefore, the so-obtained solution, if it exists, does not satisfy the constraint that n_i should be continuous across $\xi = 0$.

We conclude that in the case $n_- > n_{min}$, there exists no solution of the transition problem satisfying the requirements of proposition (3.3). We now turn to the other case.

(ii) Case $n_H \leq n_- < n_{min}$: In this case, we invert (5.10) in the neighborhood of $\xi = -\infty$ using $k_{e,-}^{-1}$ and get:

$$n_e(\xi) = n_e[\phi(\xi)], \quad n_e[\phi] = k_{e,-}^{-1}\left(\phi + k_e(n_-)\right) . \tag{5.15}$$

Now, n_e is a decreasing function of ϕ. Therefore, $n_e[\phi] - n_i[\phi]$ is the difference of two decreasing functions and its monotony requires further investigations. We compute:

$$(d/d\phi)(n_e[\phi] - n_i[\phi])|_{\phi=0} = (\partial_n k_{e,-}(n_-))^{-1} + (\partial_n h_i(n_-))^{-1} ,$$

and deduce that

$$(d/d\phi)(n_e[\phi] - n_i[\phi])|_{\phi=0} > 0 \quad \Longleftrightarrow \quad n_- < n_P . \tag{5.16}$$

Therefore, $(0,0)$ is a hyperbolic point of system (5.14) if $n_- < n_P$ and an elliptic point if $n_P < n_- \leq n_{min}$. If $(0,0)$ is an elliptic point, there is no solution to (5.14) with $(0,0)$ as initial condition other than the constant solution $(\phi, E) = (0,0)$ itself. So, we consider the case $n_- < n_P$.

In this case, starting from the hyperbolic point $(0,0)$, there are two branches of solutions (see again figure 5.1); the first branch has increasing ϕ and E and the second one has decreasing ϕ and E in the neighborhood of $\xi = -\infty$. Let us consider the solution with increasing ϕ. We want to show that this branch has the required properties.

For this branch, both $n_e[\phi]$ and $n_i[\phi]$ are decreasing functions of ξ and we have $n_e[\phi] - n_i[\phi] > 0$ for ξ in the neighborhood of $\xi = -\infty$, because of (5.16). We want to show that $n_e[\phi] - n_i[\phi] > 0$ for all $\xi \in \mathbb{R}$. For that purpose, it suffices to show that $(d/d\phi)(n_e[\phi] - n_i[\phi]) > 0$ along the trajectory ϕ. With the hypothesis $\gamma \geq 2$, $\partial_n h_i(n) = c_i \gamma n^{\gamma-2}$ is a nondecreasing function of n. Therefore, we have

$$\begin{aligned}(d/d\phi)(n_e[\phi] - n_i[\phi]) &= (\partial_n k_{e,-}(n_e[\phi]))^{-1} + (\partial_n h_i(n_i[\phi]))^{-1} \\ &\geq (\partial_n k_{e,-}(n_e[\phi]))^{-1} + (\partial_n h_i(n_e[\phi]))^{-1} > 0\,,\end{aligned}$$

because $n_e[\phi] \leq n_- < n_P$ and using (5.16) with $n_e[\phi]$. It follows that E and ϕ continuously increase w.r.t. ξ and simultaneously, n_e and n_i decrease. At some point (say $\xi = 0$, without loss of generality) n_i crosses the value 0. Beyond $\xi = 0$, the solution continues with $n_i = 0$ and $n_e > 0$. Then, ϕ and E continue to increase and $n_e[\phi]$ decreases to the limit value 0 at $\xi = +\infty$.

If we consider an unsmooth solution with a jump of n_e to the subsonic branch $k_{e,+}$ at some point, then ϕ and E shall continuously increase, but now, n_e is constrained to follow the subsonic branch and tends to ∞ as $\xi \to \infty$, which does not correspond to the requested solutions.

The second branch of solutions, starting with decreasing ϕ and E in the neighborhood of $\xi = -\infty$ may also correspond to an admissible solution, ... or not. We are not able to decide about this point right now. If yes, this solution has a non monotonous behaviour, since the potential must eventually increase to positive values for n_i to match the value 0 at some point. These non monotonous solutions may correspond to non physical solutions of the transition problem, with weaker stability properties. Their investigation is deferred to future work.

In any case, we have shown that the necessary and sufficient condition for the existence of at least one solution to the transition problem with the right conditions at infinity is $n_- < n_P$. This concludes the proof of the proposition. □.

6. Conclusion. In this paper, we have presented an asymptotic model for an expanding plasma in the vacuum in the presence of a large applied electric field. This model is obtained through an asymptotic analysis of the two-fluid Euler-Poisson model under convenient scaling hypotheses. The resulting model consists of a quasineutral fluid model for the plasma region and a Child-Langmuir type model for the emitted electron beam. The two models are connected through conditions derived from the analysis of a travelling-wave model for the transition region. The travelling-wave problem has been rigorously investigated and the overall model is validated by numerical experiments. Two-dimensional extensions of this approach are under investigation.

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