Numerical and theoretical study of a Dual Mesh Method using finite volume schemes for two phase flow problems in porous media

S. Verdière[1] and M.H. Vignal[2]

Abstract In this paper we are interested in two phase flow problems in porous media. We use a Dual Mesh Method to discretize this problem with finite volume schemes. In a simplified case (elliptic - hyperbolic system) we prove the convergence of approximate solutions to the exact solutions. We use the Dual Mesh Method in physically complex problems (heterogeneous cases with non constant total mobility). We validate numerically the Dual Mesh Method on practical examples by computing error estimates for different test-cases.

1 Introduction

We are interested in a two phase flow problem in porous media. The capillary pressure and gravity are neglected. We suppose also that there are a water phase, denoted by w, and an oil phase, denoted by o. Let Ω be an open bounded polygonal subset of $\mathbb{R}^2$, then the goal is to determine the saturation S of the water phase and the pressure P of the fluid, solutions to the following system :

$$\text{div}\Big(K(x)\, m(S(x,t))\, \nabla P(x,t)\Big) = 0, \quad x \in \Omega,\ t \in \mathbb{R}_+, \tag{1}$$

$$\frac{\partial S(x,t)}{\partial t} - \text{div}\left(K(x)\ \frac{K_{rw}(S(x,t))}{\mu_w} \nabla P(x,t)\right) = 0, \quad x \in \Omega,\ t \in \mathbb{R}_+, \tag{2}$$

with some boundary and initial conditions which yield a well posed problem. In (1), (2) K is the absolute permeability tensor, $K_{r\varphi}$ (respectively μ_φ) is the relative permeability (respectively viscosity) of the phase φ, for $\varphi = o$ or w. Furthermore, m is the total mobility such that :

$$m(S) = \frac{K_{rw}(S)}{\mu_w} + \frac{K_{ro}(S)}{\mu_o}.$$

Petrophysical parameters, we mean absolute permeability and relative permeabilities, are given by geophysicists as constant functions over each cell of a very high resolution grid (HR grid), which can be composed of millions cells. However, it is necessary to reduce the number of cells in order to run fluid flow simulations. In classical methods, these parameters are homogenized in order to obtain information over a low resolution grid (LR grid) by performing static upscaling. Indeed, before the fluid flow simulation is done, the petrophysical parameters must be upscaled. So, in a classical way, the pressure equation (1) and the saturation equation (2) are solved over the same grid, the LR grid. But this method is impossible to implement when the upscaling step depends on the saturation profile (see [6]).

[1]Institut Français du Pétrole, Université de Pau et des Pays de l'Adour
[2]Ecole Normale Supérieure de Lyon

The Dual Mesh Method, already proposed in [13] and [14], allows us to overcome this drawback, by solving the pressure equation over the LR grid and the saturation equation over the HR grid. The main step of the algorithm of the Dual Mesh Method is to reconstruct the Darcy's velocity ($K\nabla P$) over the HR grid with the values known over the LR grid. In this paper two methods of reconstruction are given. The first one, proposed by T. Gallouët, consists in an interpolation of the flow-rate known over each interface of the LR grid. The idea of the second method, proposed by D. Guérillot and S. Verdière (see [13] and [14]), is to solve a local problem over each cell belonging to the LR grid. This last idea allows us to reconstruct the flow-rate by keeping, when a heterogeneous case is considered, the information given over the geological grid, i.e. the HR grid. The first part of this paper deals with results for a simplified problem. An homogeneous case with a total mobility equal to one is considered, yielding an elliptic - hyperbolic system. Since the total mobility is constant there is no upscaling step. Finite Volumes schemes are used to discretize the system with a five point scheme for the pressure equation and an upwind explicit scheme for the saturation equation. We present the two reconstruction methods of the flow rate, and then we prove the convergence of the approximate solutions, given by the numerical schemes, to the exact solutions to the problem. In the second part, numerical results are presented. To discretize the flow problem, we use the same finite volume schemes as in the theoretical section with an implicit version for the pressure equation and with an upscaling step. We use the two previous reconstruction methods for the simplified problem. The first method is cheaper in computing cost for the simplified problem, but we have not been able to generalize it yet to more complex problems. Then a more physical problem with heterogeneity and non constant total mobility is also tested with the second method (by solving local problems) but not with the first method because of the previous remarks. These results confirm the validity of the Dual Mesh Method, even for heterogeneous porous media and non linear problems.

2 Convergence results

The problem in which we are interested in this section is the following. Let Ω denote a bounded polygonal open subset of $\mathbb{R}^2$ which is an union of rectangles. We set $\Gamma = \partial\Omega$ the boundary of Ω. Then let us consider the following elliptic - hyperbolic problem :

$$\Delta P(x) = 0, \qquad x \in \Omega, \tag{3}$$

$$\frac{\partial S(x,t)}{\partial t} - \operatorname{div}\Big(\nabla P(x)\, S(x,t)\Big) = 0, \qquad x \in \Omega,\ t \in \mathbb{R}_+, \tag{4}$$

$$\nabla P(\tau).n(\tau) = g(\tau), \qquad \tau \in \Gamma, \tag{5}$$

$$S(\tau,t) = \overline{S}(\tau,t), \quad (\tau,t) \in \Gamma^+ \times \mathbb{R}_+, \tag{6}$$

$$S(x,0) = S_0(x), \qquad x \in \Omega, \tag{7}$$

where $\Gamma^+ = \{\tau \in \Gamma\ ;\ g(\tau) > 0\}$ and where n is the unit normal vector to Γ outward to Ω. We assume that $S_0 \in L^\infty(\Omega)$ and $\overline{S} \in L^\infty(\Gamma^+ \times \mathbb{R}_+)$, $g \in H^{1/2}(\Gamma)$ such that $P \in H^2(\Omega)$ and such that $\int_\Gamma g(\tau)\, d\tau = 0$.

We search $P \in H^2(\Omega)$ solution to (3), (5) in a variational sense, i.e. :

$$\int_\Omega \nabla P(x).\nabla\varphi(x)\,dx - \int_\Gamma g(\tau)\,\varphi(\tau)\,d\tau = 0, \qquad \forall\ \varphi \in H^1(\Omega),$$

and S in $L^\infty(\Omega \times \mathbb{R}_+)$ solution to (4), (6), (7) in a weak sense, i.e. :

$$\begin{aligned}\int_\Omega\int_{\mathbb{R}_+} S(x,t)\varphi_t(x,t)\,dx\,dt - \int_\Omega\int_{\mathbb{R}_+} S(x,t)\,\nabla P(x).\nabla\varphi(x,t)\,dx\,dt \\ + \int_\Omega S_0(x)\,\varphi(x,0)\,dx + \int_\Gamma\int_{\mathbb{R}_+} \overline{S}(\tau,t)\,\varphi(\tau,t)\,g(\tau)\,d\tau\,dt = 0,\end{aligned} \tag{8}$$

$\forall\ \varphi \in C_c^1(\Omega^+ \times \mathbb{R}_+, \mathbb{R}_+)$ where $\Omega^+ = \Omega \cup \Gamma^+$.

2.1 Discretization of the elliptic equation

2.1.1 Assumptions on the low resolution grid

One considers therefore a LR grid (lower resolution grid), denoted by Ω_H. It is a family of rectangles with different sizes and we assume the following hypotheses :

$$\left\{\begin{array}{l}\bullet \text{ The intersection between two cells of } \Omega_H \text{ is either a point or a} \\ \text{line segment, and this line segment is an edge of each of both cells.} \\ \bullet \text{ There exist } \alpha > 0 \text{ and } H > 0 \text{ such that for all edge } \sigma \text{ of the grid } \Omega_H \\ \text{ one has } \alpha\,H \le l(\sigma) \le H,\end{array}\right. \tag{9}$$

where $l(\sigma)$ is the length of σ.

Some notation will be useful for the numerical scheme description :

$\forall\ M \in \Omega_H$, let denote by $N(M)$ the set of the neighbors of M, i.e. the set of the cells of Ω_H which have a common interface with M, by $\mathcal{A}_{\partial\Omega}(M)$ the set of the edges of M included in Γ, by x_M the center of M and by $V(M)$ the area of M. Furthermore for all $M_v \in N(M)$, let denote by σ_{MM_v} the interface between M and M_v, by n_{MM_v} the unit normal vector to σ_{MM_v} outward to M and by d_{MM_v} the distance between x_M and x_{M_v} .

2.1.2 Discretized equation for the elliptic problem

One defines the approximate solution on the LR grid by $P_{\Omega_H}(x) = P_M$ for almost every $x \in M$ and all $M \in \Omega_H$. Then one discretizes (3) on Ω_H. For that, a five point finite volume scheme is used. The principle of finite volume schemes, see [3], is to integrate the equation on each control volume, here the cells. Then we approximate the pressure flux through an edge σ_{MM_v} by : $(P_{M_v} - P_M)/d_{MM_v}$.

The discretized equation is given by :

$$\sum_{M_v \in N(M)} l(\sigma_{MM_v})\ \frac{(P_{M_v} - P_M)}{d_{MM_v}} + \sum_{\sigma \in \mathcal{A}_{\partial\Omega}(M)} l(\sigma)\,g_\sigma = 0 \quad \forall\ M \in \Omega_H, \tag{10}$$

where $g_\sigma = \frac{1}{l(\sigma)}\int_\sigma g(\tau)\,d\tau$. Observe that here we call flux the mean value of the pressure which is not really the physical flux.

2.2 Discretization of the hyperbolic equation

2.2.1 Assumptions on the high resolution grid

We want to discretize (4) on a higher resolution grid than Ω_H, so one defines Ω_h a high resolution grid (HR grid) of Ω which is a rectangular grid satisfying the following regularity hypotheses :

$$(11) \quad \begin{cases} \bullet \text{ The intersection between two cells of } \Omega_h \text{ is either a point or a} \\ \text{line segment, and this line segment is an edge of each of both cells.} \\ \bullet \text{ There exist } \beta > 0 \text{ and } h > 0 \text{ such that for all edge } c \text{ of the grid } \Omega_h \\ \ \text{one has } \beta\, h \leq l(c) \leq h. \\ \bullet \text{ For all } m \in \Omega_h, \text{ there exists } M \in \Omega_H \text{ such that } m \subset M. \end{cases}$$

Then we have to reconstruct the pressure flux through each edge of the HR grid Ω_h using the known values on the LR grid Ω_H. We present here two methods. The first one, proposed by T. Gallouët, gives the approximate fluxes through "small" edges, using a weighted mean value of the approximate fluxes through the edges of Ω_H located on each side of a given edge of Ω_h. The second method, proposed by D. Guérillot and S. Verdière (see [6]), consists in a resolution of local problems in the cells of Ω_H.

Some notation will be useful for the description of the reconstruction flux methods as for the one of the numerical scheme :

$\forall\, m \in \Omega_h$, let denote by x_m the center of m, by M_m the element of Ω_H such that $m \subset M_m$, by $N(m)$ the set of the neighbors of m, i.e. the set of the cells of Ω_H which have a common interface with m, by $N_{int}(m)$ the set of the neighbors of m located in the interior of M_m, by $N_{ext}(m)$ the set of the neighbors of m located in the exterior of M_m and by $\mathcal{A}_{\partial\Omega}(m)$ the set of edges of m included in Γ. Furthermore, $\forall\, m_v \in N(m)$ let denote by d_{mm_v} the distance between x_m and x_{m_v}, by c_{mm_v} the interface between m and m_v, by n_{mm_v} the unit normal vector to c_{mm_v} outward to m, by Q_{mm_v} the approximate pressure flux through the edge c_{mm_v} from m to m_v and by $\overline{Q}_{mm_v}$ the exact pressure flux through c_{mm_v} from m to m_v, i.e.

$$\overline{Q}_{mm_v} = \frac{1}{l(c_{mm_v})} \int_{c_{mm_v}} \nabla P(\tau).n_{mm_v}(\tau)\, d\tau.$$

Finally $\forall\, m \in \Omega_h$ and $\forall\, c \in \mathcal{A}_{\partial\Omega}(m)$ let denote by σ_c the edge of M_m which contains c, by Q_c the approximate pressure flux through the edge c outward to Ω and by $\overline{Q}_c$ the exact pressure flux through c outward to Ω, i.e.

$$\overline{Q}_c = \frac{1}{l(c)} \int_c g(\tau)\, d\tau.$$

2.2.2 Reconstruction of the approximate pressure flux by interpolation

Let $m \in \Omega_h$ and $m_v \in N(m)$, then we approximate the pressure flux through the interface between m and m_v by a weighted mean value of the approximate pressure fluxes through the edges of Ω_H located on each side of c_{mm_v}, i.e. :

$$Q_{mm_v} = \sum_{M_v \in N(M_m)} n_{mm_v}.n_{M_mM_v} \frac{P_{M_v} - P_{M_m}}{d_{M_mM_v}} \left(\frac{H_{mm_v} - d(c_{mm_v}, \sigma_{M_mM_v})}{H_{mm_v}} \right)$$

$$(12) \qquad + \sum_{\sigma \in \mathcal{A}_{\partial\Omega}(M_m)} n_{mm_v}.n_\sigma \; g_\sigma \left(\frac{H_{mm_v} - d(c_{mm_v}, \sigma)}{H_{mm_v}} \right),$$

where $d(c_{mm_v}, \sigma_{M_mM_v}) = \inf\Big\{d(x,y)\,;\, x \in c_{mm_v} \text{ and } y \in \sigma_{M_mM_v}\Big\}$, $d(\cdot,\cdot)$ is the euclidean distance, n_σ is the unit normal vector to σ outward to Ω and where :

$H_{mm_v} = \begin{cases} L_{M_m} & \text{if } \vec{\sigma}_{mm_v}.\vec{L}_{M_m} = 0 \\ l_{M_m} & \text{otherwise} \end{cases}$, L_{M_m} is the length of M_m and l_{M_m} is its width.

For the edges $c \in \mathcal{A}_{\partial\Omega}(m)$ located in the boundary of Ω, we choose the following approximation :

$$Q_c = n_{\sigma_c}.n_c \; g_{\sigma_c}$$

Remark 1 *One can observe that the approximate pressure fluxes satisfy the conservativity principle and a conservation equation, since one has :*

$$(13) \qquad Q_{mm_v} = -Q_{m_vm} \qquad \forall\, m \in \Omega_h \text{ and } \forall\, m_v \in N(m),$$

$$(14) \qquad \sum_{m_v \in N(m)} l(c_{mm_v})\, Q_{mm_v} + \sum_{c \in \mathcal{A}_{\partial\Omega}(m)} l(c)\, Q_c = 0 \qquad \forall\, m \in \Omega_h.$$

2.2.3 Reconstruction of the approximate pressure fluxes by solving local problems

This method consists in searching an approximate pressure on the HR grid to reconstruct the pressure flux. To do it, we solve local problems for each $M \in \Omega_H$. At first, we need the pressure flux reconstructions on the edges of the HR grid, included in the edges of the LR grid Ω_H. These reconstructions correspond to the boundary conditions of the previous problems.

Let $m \in \Omega_h$ and $m_v \in N_{ext}(m)$, we recall that $M_m \neq M_{m_v}$. One approximates the pressure flux Q_{mm_v} through c_{mm_v} by the pressure flux $Q_{M_mM_{m_v}}$ through $\sigma_{M_mM_{m_v}}$, so :

$$Q_{mm_v} = Q_{M_mM_{m_v}} = \frac{P_{M_{m_v}} - P_{M_m}}{d_{M_mM_{m_v}}}.$$

For the edges c included in the boundary of Ω, the exact pressure flux is given by :

$$Q_c = g_c = \frac{1}{l(c)} \int_c g(\tau)\, d\tau.$$

Now we can reconstruct the pressure fluxes through the edges of Ω_h located in the interior of a cell of Ω_H. For that, we search an approximate pressure in the interior of each "big" cell, which is supposed to be constant on each cell of Ω_h, that is to say $p_{\Omega_h}(x) = p_m$, for almost every $x \in m$ and all $m \in \Omega_h$. We want p_{Ω_h} to be a "good" approximation of the exact solution to problem (3), (5). Besides, we will see in the sequel that P_{Ω_H} is a "good aproximation" of this one. So we construct p_{Ω_h} as follows : we consider each $M \in \Omega_H$ as a subset of Ω, and the set of the elements of Ω_h, which are in M, as a grid of M. Then we discretize (3) on M as we have already done it in the previous section for Ω. First, we integrate (3), on $m \in \Omega_h$, $m \subset M = M_m$. Using Green's formula we obtain :

$$\sum_{m_v \in N_{int}(m)} \int_{c_{mm_v}} \nabla P(\tau)\, n_{mm_v}\, d\tau + \sum_{m_v \in N_{ext}(m)} \int_{c_{mm_v}} \nabla P(\tau)\, n_{mm_v}\, d\tau \sum_{c \in \mathcal{A}_{\partial\Omega}(m)} \int_c g(\tau)\, d\tau = 0$$

We assume the pressure fluxes to be known on the boundary of M, thus on each c_{mm_v} such that $m_v \in N_{ext}(m)$. One has : $Q_{mm_v} = Q_{M_m M_{m_v}} = (P_{M_{m_v}} - P_{M_m})/d_{M_m M_{m_v}}$. For c_{mm_v} such that $m_v \in N_{int}(m)$, we use the same discretization as the one used in the previous section, so one has $Q_{mm_v} = (p_{m_v} - p_m)/d_{mm_v}$.
Thus the $(p_m)_{m \in \Omega_h}$ are solutions to the following problem :

$$\sum_{m_v \in N_{int}(m)} l(c_{mm_v})\, \frac{p_{m_v} - p_m}{d_{mm_v}} + \sum_{m_v \in N_{ext}(m)} l(c_{mm_v})\, \frac{P_{M_{m_v}} - P_{M_m}}{d_{M_m M_{m_v}}} + \sum_{c \in \mathcal{A}_{\partial\Omega}(m)} l(c)\, g_c = 0, \quad \forall\, m \in \Omega_h. \tag{15}$$

This equation allows to construct approximate pressure fluxes through each interface between two neighboring cells m and m_v such that $M_m = M_{m_v}$; one sets :

$$Q_{mm_v} = \frac{p_{m_v} - p_m}{d_{mm_v}}.$$

Remark 2 *This reconstruction satisfies the flux conservativity principle (13). It also satisfies the conservation equation (14).*

2.2.4 Discretized equation associated to the hyperbolic equation

Before discretizing (4), one has to define a time step δ. So, let Ω_H and Ω_h be two rectangular grids of Ω, satisfying respectively (9) and (11), and let $\eta \in (0,1)$, then we choose $\delta \in \mathbb{R}_+^*$ satisfying the following stability condition :

$$\frac{\delta}{V(m)} \left(\sum_{\substack{m_v \in N(m) \\ Q_{mm_v} > 0}} Q_{mm_v}\, l(c_{mm_v}) + \sum_{c \in \mathcal{A}_{\partial\Omega}(m)} l(c)\, Q_c^+ \right) \le 1 - \eta, \qquad \forall\, m \in \Omega_h. \tag{16}$$

One sets $t^n = n\, \delta\ \forall\, n \in \mathbb{N}$.
First we discretize the initial and boundary conditions ; one defines $\forall\, m \in \Omega_h$:

$$S_m^0 = \frac{1}{V(m)} \int_m S_0(x)\; dx,$$

and $\forall\ m \in \Omega_h$, $\forall\ c \in \mathcal{A}_{\partial\Omega}(m)$ and $\forall\ n \in \mathbb{N}$:

$$\overline{S}_c^n = \frac{1}{\delta\, l(c)} \int_{t^n}^{t^{n+1}} \int_c \overline{S}(\tau, t)\, d\tau\, dt.$$

To discretize the saturation equation we use an explicit Euler scheme in time and an upwind finite volume scheme in space . One approximates S by $S_{\Omega_h,\delta}(x,t) = S_m^n$ if $x \in m$ and $t \in [t^n, t^{n+1}[$. Then the discretized equation is the following :

$$V(m)\, \frac{(S_m^{n+1} - S_m^n)}{\delta} - \sum_{m_v \in N(m)} S_{mm_v}^n\, l(c_{mm_v})\, Q_{mm_v} - \sum_{c \in \mathcal{A}_{\partial\Omega}(m)} l(c)\, \left(\overline{S}_c^n\, Q_c^+ - S_m^n\, Q_c^-\right) = 0,$$

$\forall\ m \in \Omega_h$ and $\forall\ n \in \mathbb{N}$, where $S_{mm_v}^n = \begin{cases} S_{m_v}^n & \text{if } Q_{mm_v} > 0 \\ S_m^n & \text{otherwise} \end{cases}$ and where Q_c^+ and Q_c^-, are defined by :

- if fluxes are reconstructed by interpolation :

$$Q_c^+ = n_{\sigma_c}.n_c\, \frac{1}{l(\sigma_c)} \int_{\sigma_c} \max(g(\tau), 0)\, d\tau \quad \text{and} \quad Q_c^- = n_{\sigma_c}.n_c\, \frac{1}{l(\sigma_c)} \int_{\sigma_c} \max(-g(\tau), 0)\, d\tau\,;$$

- if fluxes are reconstructed by solving local problems :

$$Q_c^+ = g_c^+ = \frac{1}{l(c)} \int_c \max(g(\tau), 0)\, d\tau \quad \text{and} \quad Q_c^- = g_c^- = \frac{1}{l(c)} \int_c \max(-g(\tau), 0)\, d\tau.$$

Using (14) and (15), this scheme can be rewritten as :

$$V(m)\, \frac{(S_m^{n+1} - S_m^n)}{\delta} - \sum_{\substack{m_v \in N(m) \\ Q_{mm_v} > 0}} Q_{mm_v}\, (S_{m_v}^n - S_m^n)\, l(c_{mm_v})$$

$$- \sum_{c \in \mathcal{A}_{\partial\Omega}(m)} l(c)\, \left(\overline{S}_c^n - S_m^n\right)\, Q_c^+ = 0. \tag{17}$$

2.3 Error estimates for the discretized elliptic problem on the LR grid

In this section the existence and the uniqueness, up to a constant, of solutions to (10) is proved. Afterwards, one proves the convergence of approximate solution to the exact solution of (3), (5) proving error estimates in a discrete H^1–norm. This proof generalizes the results given in [7] and [15]. Indeed in [7], R. Herbin considers a diffusion convection problem with a Dirichlet boundary condition, and in [15] the elliptic problem is the same as the one discribed here. But in these two papers the exact solution is assumed to be smooth (C^2), whereas here, one only assumes H^2 regularity. Some error estimates in discrete H^1–norm, are also proved in [8] for an exact solution in H^m $(3/2 < m \leq 2)$ and for a diffusion convection problem with an homogeneous Dirichlet boundary condition on a square grid.

Here the results are established on a non regular rectangular grid and can be generalized to more complex grids (see Remark 3). In [11] the authors prove the convergence of mixed finite volume schemes for diffusion equations with an homogeneous Dirichlet boundary condition on a rectangular mesh. They establish error estimates assuming the exact solution is H^2. In [2] the authors are interested in the convergence of a diamond-path scheme for a diffusion - convection problem. They prove error estimates assuming the exact solution is H^2.

Proposition 1 *Let Ω_H be a grid of Ω satisfying the regularity hypotheses (9), and g be in $H^{1/2}(\Gamma)$, such that $\int_\Gamma g(\tau)\, d\tau = 0$. Then, there exists a unique solution up to a constant, $(P_M)_{M\in\Omega_H}$, to problem (10).*

We do not give the proof of this classical result close to those, for instance, in [10], [9] or [15].
Then we show the following :

Theorem 1 *Let Ω_H be a grid of Ω satisfying the regularity hypotheses (9), and g be in $H^{1/2}(\Gamma)$, such that $\int_\Gamma g(\tau)\, d\tau = 0$. One denotes by $P(.)$ the exact solution to problem (3), (5) such that $\int_\Omega P(x)\, dx = 0$. One assumes that g is such that P is in $H^2(\Omega)$. Let $(P_M)_{M\in\Omega_H}$ satisfying (10) and $\sum_{M\in\Omega_H} V(M)\, P_M = \sum_{M\in\Omega_H} V(M)\, P(x_M)$. The error on the cell M, for all $M \in \Omega_H$, is defined by $E_M = P_M - P(x_M)$.*
Then there exist C_1 and C_2, depending only on Ω, α and on the H^2–norm of P, such that

$$\left(\sum_{M\in\Omega_H}\sum_{M_v\in N(M)} \frac{(E_{M_v}-E_M)^2}{d_{MM_v}}\, l(\sigma_{MM_v})\right)^{1/2} \leq C_1\, H$$

$$\text{and} \qquad \left(\sum_{M\in\Omega_H} V(M)\,|E_M|^2\right)^{1/2} \leq C_2\, H$$

Proof :
At first, let observe that $P(x_M)$ is defined $\forall\ M \in \Omega_H$, because $H^2(\Omega)$ is continuously imbedded in $C(\overline{\Omega})$. So let $M \in \Omega_H$ and $M_v \in N(M)$. One defines the consistency error $R_{MM_v}(P)$ through the interface between M and M_v by :

$$R_{MM_v}(P) = \frac{P(x_{M_v}) - P(x_M)}{d_{MM_v}} - \frac{1}{l(\sigma_{MM_v})}\int_{\sigma_{MM_v}} \nabla P(\tau).n_M(\tau)\, d\tau.$$

On the boundary the consistency error equals zero since we know the exact flux g.
One defines, $\forall\ \sigma_{MM_v} \subset \Omega$, $\mathcal{V}_{MM_v}$ the quadrangle with x_M, x_{M_v} and the two vertices of σ_{MM_v} for vertices (see figure 1).
Then, we can consider the consistency error as a function $R_H(P)(.)$ constant on each dual cell $\mathcal{V}_{MM_v}$, and so :

$$\|R_H(P)\|_{L^2(\Omega)} = \left(\frac{1}{4}\sum_{M\in\Omega_H}\sum_{M_v\in N(M)} d_{MM_v}\, l(\sigma_{MM_v}) R^2_{MM_v}(P)\right)^{1/2}.$$

We are going to show that the scheme is consistent in a finite volumes sense (see [3]).

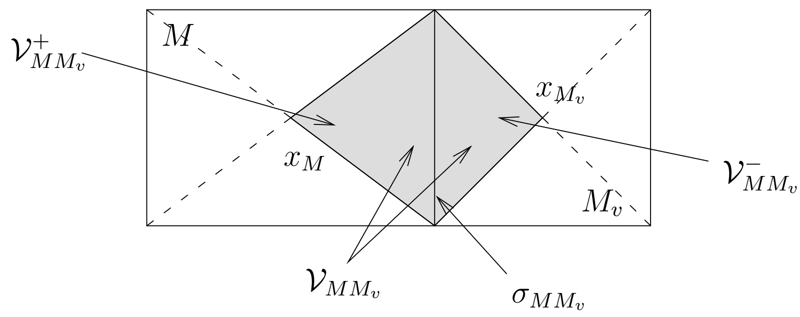

Figure 1: Dual mesh

Lemma 1 *Under assumptions of Theorem 1, one proves the existence of C, depending only on α, such that :*

$$\|R_H(P)\|_{L^2(\Omega)} \le C\,\|P\|_{H^2(\Omega)}\,H.$$

Proof of Lemma 1 :

To show this result, we first assume that $P \in C^\infty(\overline{\Omega})$. Let $M \in \Omega_H$ and $M_v \in N(M)$. We choose for coordinate system (denoted by $\mathcal{R}^{MM_v}$) the one with x_M for origin and with $(x_M x_{M_v})$ for axis of abscissae. Let denote by a and b ($a < b$) the ordinate of the two vertices of σ_{MM_v}, by d_M the distance between x_M and σ_{MM_v}, by d_{M_v} the distance between x_{M_v} and σ_{MM_v} and set :

$$\overline{P}_{MM_v} = \frac{1}{l(\sigma_{MM_v})}\int_{\sigma_{MM_v}} P(\tau)\,d\tau \quad \text{ and } \quad |D^2P| = \sum_{i=1}^{2}\sum_{j=1}^{2}\left|\frac{\partial^2 P}{\partial x_i \partial x_j}\right|;$$

then $R_{MM_v}(P)$ can be rewritten as :

$$R_{MM_v}(P) = \frac{1}{l(\sigma_{MM_v})}\int_a^b \frac{\partial P}{\partial x_1}(d_M, s)\,ds - \frac{P(d_{MM_v},0) - \overline{P}_{MM_v} + \overline{P}_{MM_v} - P(0,0)}{d_{MM_v}}.$$

Using Taylor's expansion and a change of variables, one obtains :

$$|R_{MM_v}(P)| \le \frac{H^2}{l(\sigma_{MM_v})\,d_{MM_v}}\left(\frac{1}{d_M}\int_{\mathcal{V}^+_{MM_v}} |D^2P(z)|\,dz + \frac{1}{d_{M_v}}\int_{\mathcal{V}^-_{MM_v}} |D^2P(z)|\,dz\right),$$

where $\mathcal{V}^+_{MM_v} = \mathcal{V}_{MM_v} \cap M$ and $\mathcal{V}^-_{MM_v} = \mathcal{V}_{MM_v} \cap M_v$.
Cauchy Schwarz' inequality yields :

$$\|R_H(P)\|^2_{L^2(\Omega)} \le \frac{8\,H^2}{\alpha^2}\,\|P\|^2_{H^2(\Omega)} \tag{18}$$

which ends the proof of Lemma 1 when $P \in C^\infty(\overline{\Omega})$. So let $P \in H^2(\Omega)$ then there exists $(P_j)_{j\in\mathbb{N}}$, $P_j \in C^\infty(\overline{\Omega})$ $\forall\ j \in \mathbb{N}$, such that $\lim_{j\to\infty} \|P_j - P\|_{H^2(\Omega)} = 0$. Furthermore, since

$H^2(\Omega)$ is continuously imbedded in $C(\overline{\Omega})$, there exists C_Ω, depending only on Ω, such that :

$$\|P_j - P\|_{L^\infty(\Omega)} \leq C_\Omega \, \|P_j - P\|_{H^2(\Omega)}. \tag{19}$$

Thanks to (18), one has :

$$\|R_H(P_j)\|_{L^2(\Omega)} \leq \frac{2\sqrt{2}\,H}{\alpha} \, \|P_j\|_{H^2(\Omega)}. \tag{20}$$

We are going to show that $R_H(P_j)$ converges to $R_H(P)$ in $L^2(\Omega)$ when j goes to infinity. Indeed :

$$|R_{MM_v}(P) - R_{MM_v}(P_j)| \leq \frac{1}{\sqrt{l(\sigma_{MM_v})}} \left(\int_{\sigma_{MM_v}} \Big(\nabla P(\tau) - \nabla P_j(\tau) \Big)^2 \right)^{1/2} + \\ + \frac{2}{d_{MM_v}} \, \|P - P_j\|_{L^\infty(\Omega)}.$$

Then one uses the following lemma :

Lemma 2 *Let Ω_H be a rectangular grid of Ω satisfying the regularity hypotheses (9). Let $M \in \Omega_H$ and $M_v \in N(M)$. Let $u \in H^1(\mathcal{T}_{MM_v})$, where $\mathcal{T}_{MM_v}$ is defined in figure 2. Then, the trace of u on σ_{MM_v} exists ; it is an element of $L^2(\sigma_{MM_v})$. Moreover :*

$$\|u\|_{L^2(\sigma_{MM_v})} \leq \frac{2}{\sqrt{\alpha\, H}} \|u\|_{H^1(\mathcal{T}_{MM_v})}. \tag{21}$$

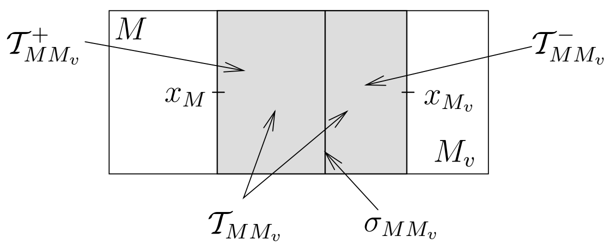

Figure 2:

Proof of Lemma 2 :

The proof of this lemma is well known on a square of side one. Then, (21) is proved using a change of variables.

Thus Lemma 2 and inequality (19) give the existence of C, depending only on Ω, such that :

$$\|R_H(P_j) - R_H(P)\|_{L^2(\Omega)} \leq \left(\frac{2}{\alpha} + \frac{2\,C}{\alpha\,H} \right) \|P_j - P\|_{H^2(\Omega)}.$$

Passing to the limit in (20), one concludes the proof of Lemma 1. This concludes the proof of the scheme's consistency ; let show now the error estimates in Theorem 1. Using (3), (5) and (10), one proves the following result (for details see [7]) :

$$\sum_{M\in\Omega_H}\sum_{M_v\in N(M)} l(\sigma_{MM_v})\,\frac{\Big(E_{M_v}-E_M\Big)^2}{d_{MM_v}} = \sum_{M\in\Omega_H}\sum_{M_v\in N(M)} l(\sigma_{MM_v})\,R_{MM_v}(P)\,\Big(E_M-E_{M_v}\Big);$$

thanks to Cauchy Schwarz' inequality and to Lemma 1, there exists C, depending only on α, such that :

$$\begin{aligned}\left(\sum_{M\in\Omega_H}\sum_{M_v\in N(M)} l(\sigma_{MM_v})\,\frac{\Big(E_{M_v}-E_M\Big)^2}{d_{MM_v}}\right)^{1/2} &\leq \left(\sum_{M\in\Omega_H}\sum_{M_v\in N(M)} l(\sigma_{MM_v})\,d_{MM_v}\,R^2_{MM_v}(P)\right)^{1/2}\\ &\leq C\,\|P\|_{H^2(\Omega)}\,H.\end{aligned}$$

One concludes the proof of Theorem 1, using a discrete Poincaré-Wirtinger's inequality (see [15]).

Remark 3 *This proof can be generalized to more complex grids, which satisfy (9), and such that the orthogonal bisectors of a cell are concurrent. For all cell M, x_M is the intersection between the orthogonal bisectors of M, one assumes the distances between x_M and the edges of M to be minorized by $\alpha_1\,H$ ($\alpha_1>0$). In the case of a triangular grid this property is satisfied if all the angles of the grid are majorized by $\pi/2-\eta,\ \eta>0$.*
Furthermore, one can prove error estimate in the L^q–norm for all $q<+\infty$ of order h and an error estimate in the L^∞–norm, of order $h\,ln(h)$.

2.4 Error estimates for reconstructed pressure fluxes

To show the convergence of the approximate saturation one needs error estimates on the approximate pressure fluxes reconstructed on the edges of the HR grid Ω_h. Furthermore, when fluxes are reconstructed by solving local problems, we must prove that the reconstruction always exists.

Proposition 2 *Let Ω_H and Ω_h be two rectangular grids of Ω satisfying, respectively, (9) and (11), and g in $H^{1/2}(\Gamma)$ such that $\int_\Gamma g(\tau)\,d\tau=0$. Let denote by $P(.)$ the exact solution to problem (3), (5), such that $\int_\Omega P(x)\,dx=0$. Let $(P_M)_{M\in\Omega_H}$ satisfying (10) and $\sum_{M\in\Omega_H} V(M)\,P_M=\sum_{M\in\Omega_H} V(M)\,P(x_M)$.*
Then there exists a solution to (15). Furthermore let $(p^{(1)}_m)_{m\in\Omega_h}$ and $(p^{(2)}_m)_{m\in\Omega_h}$ be two solutions to this problem ; then $\forall\ M\in\Omega_H$, $\exists\ C_M$, depending only on M, such that :

$$p^{(2)}_m=p^{(1)}_m+C_M \qquad \forall\ m\in\Omega_h,\ m\subset M$$

Proof :

We observe that (15) defines n_H linearly independant problems, where n_H is the number of elements in Ω_H. Then let $M \in \Omega_H$; we are going to prove that there exists a solution $(p_m)_{\substack{m\in\Omega_h \\ m\subset M}}$ to problem (15) unique up to a constant. This proof is close to the one given in [15] ; one shows that : if, $\forall\ m \in \Omega_h,\ m \subset M$,

$$\sum_{m_v \in N_{ext}(m)} l(c_{mm_v})\ \frac{P_{M_{m_v}} - P_{M_m}}{d_{M_m M_{m_v}}} + \sum_{c \in \mathcal{A}_{\partial\Omega}(m)} \int_c g(\tau)\ d\tau = 0,$$

then $p_m = p_{m_v}\ \forall\ m \in \Omega_h,\ m \subset M$ and $\forall\ m_v \in N_{int}(m)$. So the dimension of the kernel, of the linear system defined by (15) for $m \in M$, is 1.
It remains to establish the existence of solutions. So, we assume that there exist solutions and we sum the equations (15), one obtains :

$$\sum_{M_v \in N(M)} \frac{P_{M_v} - P_M}{d_{MM_v}}\ l(\sigma_{MM_v}) + \sum_{\sigma \in \mathcal{A}_{\partial\Omega}(M)} \int_\sigma g(\tau)\ d\tau = 0.$$

Let denote by L_M the number of elements of Ω_h included in M. Then the image, of the linear system defined by (15) for $m \in M$, is included in

$$\mathcal{B}_{L_M} = \Big\{ b =^t (b_1, \cdots, b_{L_M}) \in \mathbb{R}^{\ L_M} \text{ such that } \sum_{i=1}^{L_M} b_i = 0 \Big\}$$

Since the dimension of the kernel is 1 the image is exactly $\mathcal{B}_{L_M}$.
Since $(P_M)_{M\in\Omega_H}$ satisfies (10), there exist solutions to (15).
Then one proves the following result which gives estimates on reconstructed fluxes.

Proposition 3 *Let Ω_H and Ω_h be two rectangular grids of Ω satisfying respectively (9) and (11), and g in $H^{1/2}(\Gamma)$ such that $\int_\Gamma g(\tau)\, d\tau = 0$. One denotes by $P(.)$ the exact solution to problem (3), (5) such that $\int_\Omega P(x)\, dx = 0$. Let $(P_M)_{M\in\Omega_H}$ satisfying (10) and $\sum_{M\in\Omega_H} V(M)\, P_M = \sum_{M\in\Omega_H} V(M)\, P(x_M)$. Furthermore one assumes that g is such that P is in $H^2(\Omega)$.*
Let $(p_m)_{m\in\Omega_h}$ satisfying (15) and such that $\sum_{\substack{m\in\Omega_h \\ m\subset M}} V(m)\, p_m = \sum_{\substack{m\in\Omega_h \\ m\subset M}} V(m)\, P(x_m)$ for all $M \in \Omega_H$. One sets

$$A_h = \left(\sum_{m\in\Omega_h} \sum_{m_v \in N(m)} l(c_{mm_v})\, d_{mm_v} \left(Q_{mm_v} - \overline{Q}_{mm_v} \right)^2 \right)^{1/2}.$$

Then there exist C_1 and C_2, depending only on Ω, β, α and P, such that

$$A_h \le C_1\, H, \quad \textit{if fluxes are reconstructed by interpolation ;}$$

$$A_h \le C_2\, \sqrt{H} \quad \textit{if fluxes are reconstructed by solving local problems.}$$

Proof :

Fluxes reconstructed by interpolation :

Thanks to the interpolated fluxes definition, one has :

$$A_h \le A_{1h} + A_{2h},$$

$$A_{1h} \le \left(4\,h \sum_{M\in\Omega_H}\ \sum_{M_v\in N(M)}\ \sum_{c\in\mathcal{A}_h(M)} l(c) \left| n_c.n_{MM_v}\,\frac{P_{M_v}-P_M}{d_{MM_v}} - |n_c.n_{MM_v}|\,\overline{Q}_c\right|^2\right)^{1/2},$$

$$A_{2h} \le \left(4\,h \sum_{M\in\Omega_H}\ \sum_{\sigma\in\mathcal{A}_{\partial\Omega}(M)}\ \sum_{c\in\mathcal{A}_h(M)} |n_c.n_\sigma|\,l(c)\left|\overline{Q}_\sigma - \overline{Q}_c\left(n_c.n_\sigma\right)\right|^2\right)^{1/2}.$$

First we focus on A_{1h}. One denotes by $c^{\perp}_{MM_v}$ the orthogonal projection of c on σ_{MM_v} $\forall\ M \in \Omega_H,\ M_v \in N(M)$ and $c \in \mathcal{A}_h(M)$, then :

$$\begin{aligned} A_{1h} \le \Bigg[12\,h \sum_{M\in\Omega_H}\ \sum_{M_v\in N(M)}\ \sum_{c\in\mathcal{A}_h(M)} l(c) \Bigg(&|n_c.n_{MM_v}| \left|\frac{E_{M_v}-E_M}{d_{MM_v}}\right|^2 + \\ &+ |n_c.n_{MM_v}| \left|\frac{P(x_{M_v})-P(x_M)}{d_{MM_v}} - \frac{1}{l(c^{\perp}_{MM_v})}\int_{c^{\perp}_{MM_v}} \nabla P(\tau).n_{MM_v}\,d\tau\right|^2 + \\ &+ |n_c.n_{MM_v}|\ \left|n_c.n_{c^{\perp}_{MM_v}}\overline{Q}_{c^{\perp}_{MM_v}} - \overline{Q}_c\right|^2\Bigg)\Bigg]^{1/2}. \end{aligned}$$

Thus :

$$\begin{aligned} A_{1h} \le \Bigg[\frac{12\,H}{\beta} \sum_{M\in\Omega_H}\ \sum_{M_v\in N(M)}\ \sum_{c\subset\sigma_{MM_v}} l(c) \Bigg(&\left|\frac{E_{M_v}-E_M}{d_{MM_v}}\right|^2 + \left|n_c.n_{c^M_\perp}\overline{Q}_{c^M_\perp} - \overline{Q}_c\right|^2 + \\ &+ \left|\frac{P(x_{M_v})-P(x_M)}{d_{MM_v}} - \frac{1}{l(c)}\int_c \nabla P(\tau).n_{MM_v}\,d\tau\right|^2\Bigg)\Bigg]^{1/2}, \end{aligned}$$

where $\forall\ c \subset \sigma_{MM_v}$ ($M \in \Omega_H,\ M_v \in N(M)$) $c^M_\perp$ is the orthogonal projection of c on the edge of M which is the opposite edge to σ_{MM_v} (i.e. the edge σ of M such that $|n_\sigma.n_{\sigma_{MM_v}}| = 1$). So :

$$\begin{aligned} A_{1h} \le \sqrt{\frac{12}{\beta\,\alpha}} \Bigg(&\sum_{M\in\Omega_H}\ \sum_{M_v\in N(M)} l(\sigma_{MM_v})\frac{|E_{M_v}-E_M|^2}{d_{MM_v}} \\ &+ \sum_{m\in\Omega_h}\ \sum_{m_v\in N_{ext}(m)} l(c_{mm_v})\,d_{MM_v}\,|B_{mm_v}|^2 + \\ &+ H \sum_{M\in\Omega_H}\ \sum_{M_v\in N(M)}\ \sum_{c\subset\sigma_{MM_v}} l(c)\left|n_c.n_{c^M_\perp}\overline{Q}_{c^M_\perp} - \overline{Q}_c\right|^2\Bigg)^{1/2}. \end{aligned}$$

One uses for coordinate system the one which has one of the two vertices of c for origin and such that its axis of ordinate is parallel to c. Then denoting by l_M the distance between c and c_M, using Taylor's expansion and Cauchy Schwarz' inequality, one has :

$$\left|n_c.n_{c_\perp^M}\overline{Q}_{c_\perp^M}-\overline{Q}_c\right|\leq\frac{\sqrt{l_M}}{\sqrt{l(c)}}\left(\int_0^{l(c)}\int_0^{l_M}\left|\frac{\partial^2 P}{\partial {x_1}^2}(r,s)\right|^2 dr\,ds\right)^{1/2}. \tag{22}$$

Thus, the previous inequality, Theorem 1 and Lemma 5 yield :

$$A_{1h}\leq C\,H,$$

where C depends only on P, β, α and on Ω.
Now, we focus on A_{2h}. Let denote by σ_M the edge of M which is the opposite edge to σ (i.e the edge σ_M of M such that $|n_\sigma.n_{\sigma_M}|=1$) $\forall\ \sigma$ which is an edge of M located in the boundary of Ω ; one denotes by M_σ the neighbor of M such that $\sigma_M=\partial M\cap\partial M_\sigma$. Then as previously, one denotes by $c^\perp_{MM_\sigma}$ the orthogonal projection of c on σ_M. So one can write :

$$\begin{aligned}A_{2h}\leq\Bigg[16\,h\sum_{M\in\Omega_H}\sum_{\sigma\in\mathcal{A}_{\partial\Omega}(M)}\sum_{c\in\mathcal{A}_h(M)} l(c)\left|n_c.n_\sigma\right|\Bigg(\left|\overline{Q}_\sigma-\overline{Q}_{\sigma_M}\,(n_\sigma.n_{\sigma_M})\right|^2+\\
+\left|\overline{Q}_{\sigma_M}\,(n_\sigma.n_{\sigma_M})-\frac{P(x_{M_\sigma})-P(x_{M_\sigma})}{d_{MM_\sigma}}\right|^2+\left|\frac{P(x_{M_\sigma})-P(x_{M_\sigma})}{d_{MM_\sigma}}-\overline{Q}_{c^\perp_{MM_\sigma}}\,(n_\sigma.n_{c^\perp_{MM_\sigma}})\right|^2+\\
+\left|\overline{Q}_{c^\perp_{MM_\sigma}}\,(n_\sigma.n_{c^\perp_{MM_\sigma}})-\overline{Q}_c\,(n_\sigma.n_c)\right|^2\Bigg)\Bigg]^{1/2}.\end{aligned}$$

Using a method close to the one used for proving (22), and Lemmas 1 and 5, one obtains :

$$A_{2h}\leq C\,H,$$

where C depends only on α, β, Ω and on the H^2–norm of P.

Fluxes reconstructed by solving local problems :
First we show the following lemma which gives an estimate on the approximate fluxes through edges of Ω_h which are located in the interior of the cells of Ω_H. It is an estimate in a discrete H^1_0–norm, that is to say an estimate in the L^2–norm of the discrete gradient of the error.

Lemma 3 *We assume that the assumptions of Proposition 3 hold. The error on the cell m, $\forall\ m\in\Omega_h$, is defined by $e_m=p_m-P(x_m)$.*
Then $\exists\ C$, depending only on Ω, β, α and P, such that :

$$\left(\sum_{m\in\Omega_h}\sum_{m_v\in N_{int}(m)}\frac{(e_{m_v}-e_m)^2}{d_{mm_v}}\,l(c_{mm_v})\right)^{1/2}\leq C\sqrt{H}. \tag{23}$$

Proof of Lemma 3 :
At first one proves the consistency of approximate fluxes on edges of Ω_h located in the interior of "coarse" cells (those of Ω_H).

Lemma 4 *Under the hypotheses of the Proposition 3, $\exists\ C$, depending only on β, such that :*

$$\left(\sum_{m \in \Omega_h} \sum_{m_v \in N_{int}(m)} l(c_{mm_v})\, d_{mm_v}\, R^2_{mm_v}(P) \right)^{1/2} \leq C\, \|P\|_{H^2(\Omega)}\, h,$$

where

$$R_{mm_v}(P) = \frac{P(x_{m_v}) - P(x_m)}{d_{mm_v}} - \overline{Q}_{mm_v}. \tag{24}$$

The proof of this lemma is close to the one of Lemma 1. In particular, one shows the following inequality $\forall\ m \in \Omega_h$ and $\forall\ m_v \in N(M)$:

$$\|\nabla P\|_{L^2(c_{mm_v})} \leq \frac{2}{\sqrt{\beta\, h}}\, \|P\|_{H^2(\mathcal{T}_{mm_v})}, \tag{25}$$

where $\mathcal{T}_{mm_v}$ is defined as for the LR grid (see figure 2).
Now, one can prove Lemma 3. Let $m \in \Omega_h$; (15), (3) and (5) yield :

$$\sum_{m_v \in N_{int}(m)} \left(\frac{p_{m_v} - p_m}{d_{mm_v}} - \overline{Q}_{mm_v} \right) l(c_{mm_v}) + \sum_{m_v \in N_{ext}(m)} \left(\frac{P_{M_{m_v}} - P_{M_m}}{d_{M_m M_{m_v}}} - \overline{Q}_{mm_v} \right) l(c_{mm_v}) = 0.$$

We multiply this equation by e_m and we sum over $m \in \Omega_h$. Using (24), as for the conservativity of the exact and approximate fluxes, we obtain :

$$\sum_{m \in \Omega_h} \sum_{m_v \in N_{int}(m)} l(c_{mm_v})\, \frac{(e_{m_v} - e_m)^2}{d_{mm_v}} = \sum_{m \in \Omega_h} \sum_{m_v \in N_{int}(m)} R_{mm_v}(P)\, l(c_{mm_v})\, (e_m - e_{m_v}) + 2\, B_1,$$

where

$$B_1 = \sum_{m \in \Omega_h} \sum_{m_v \in N_{ext}(m)} \left(\frac{P_{M_{m_v}} - P_{M_m}}{d_{M_m M_{m_v}}} - \overline{Q}_{mm_v} \right) l(c_{mm_v})\, e_m.$$

Thanks to Cauchy Schwarz' inequality and to Lemma 4, there exists C, depending only on β, such that :

$$\sum_{m \in \Omega_h} \sum_{m_v \in N_{int}(m)} l(c_{mm_v})\, \frac{(e_{m_v} - e_m)^2}{d_{mm_v}} \leq 2\, B_1 +$$

$$C\, \|P\|_{H^2(\Omega)}\, h \left(\sum_{m \in \Omega_h} \sum_{m_v \in N_{int}(m)} l(c_{mm_v})\, \frac{(e_{m_v} - e_m)^2}{d_{mm_v}} \right)^{1/2}. \tag{26}$$

Now, we remark that B_1 can be rewritten as :

$$B_1 = \sum_{m \in \Omega_h} \sum_{m_v \in N_{ext}(m)} \left(\frac{E_{M_{m_v}} - E_{M_m}}{d_{M_m M_{m_v}}} + \frac{P(x_{M_{m_v}}) - P(x_{M_m})}{d_{M_m M_{m_v}}} - \overline{Q}_{mm_v} \right) l(c_{mm_v})\, e_m.$$

Using, one more time, Cauchy Schwarz' inequality, one has :

$$B_1 \leq \frac{1}{\sqrt{\alpha\, H}} \left(\sum_{m\in\Omega_h} \sum_{m_v \in N_{ext}(m)} l(c_{mm_v})\, |e_m|^2 \right)^{1/2} \times$$

$$\times \left[\left(\sum_{m\in\Omega_h} \sum_{m_v \in N_{ext}(m)} d_{M_m M_{m_v}} l(c_{mm_v}) \left(\frac{P(x_{M_{m_v}}) - P(x_{M_m})}{d_{M_m M_{m_v}}} - \overline{Q}_{mm_v} \right)^2 \right)^{1/2} + \right.$$

$$\left. + \left(\sum_{M\in\Omega_H} \sum_{M_v \in N(M)} \frac{(E_{M_v} - E_M)^2}{d_{MM_v}}\, l(\sigma_{MM_v}) \right)^{1/2} \right].$$

Then we show the following result :

Lemma 5 *Under the assumptions of Proposition 3, $\exists\, C$, depending only on α, such that :*

$$\left(\sum_{m\in\Omega_h} \sum_{m_v \in N_{ext}(m)} d_{M_m M_{m_v}} l(c_{mm_v})\, B^2_{mm_v} \right)^{1/2} \leq C\, H,$$

where :

$$B_{mm_v} = \frac{P(x_{M_{m_v}}) - P(x_{M_m})}{d_{M_m M_{m_v}}} - \overline{Q}_{mm_v}.$$

Proof of Lemma 5 :
First one assumes $P \in C^\infty(\overline{\Omega})$. The notation of the figure 3 is used.

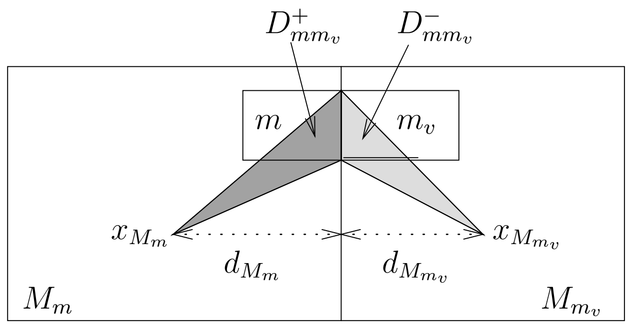

Figure 3:

One sets $\overline{P}_{mm_v} = \frac{1}{l(c_{mm_v})} \int_{c_{mm_v}} P(\tau)\, d\tau$ and one uses for coordinate system the one with x_{M_m} for origin and $(x_{M_m} x_{M_{m_v}})$ for axis of abscissae. We denote by a and b ($a < b$) the ordinate of the vertices of c_{mm_v}. Then :

$$B_{mm_v} = \frac{P(d_{M_m M_{m_v}}, 0) - \overline{P}_{mm_v} + \overline{P}_{mm_v} - P(0,0)}{d_{M_m M_{m_v}}} - \frac{1}{l(c_{mm_v})} \int_a^b \frac{\partial P}{\partial x_1}(d_{M_m}, s)\, ds.$$

Using Taylor's expansion, a change of variables and Cauchy Schwarz' inequality, one obtains :

$$\left(\sum_{m\in\Omega_h}\sum_{m_v\in N_{ext}(m)} d_{M_mM_{m_v}}\, l(c_{mm_v})\, B^2_{mm_v}\right)^{1/2} \le \frac{\|P\|_{H^2(\Omega)}}{\alpha^2\,\beta}\, H.$$

To conclude, we use, as in the proof of Lemma 1, the density of $C^\infty(\overline{\Omega})$ in $H^2(\Omega)$. This ends the proof of Lemma 5. Then this result and Theorem 1 give :

$$B_1 \le C\sqrt{H}\left(\sum_{M\in\Omega_H}\sum_{\substack{m\in\Omega_h\\ m\subset M}} l(\partial m\cap\partial M)\,|e_m|^2\right)^{1/2},$$

where C depends only on Ω, α, β and on the H^2–norm of P.
Furthermore, as in [15] (see Section 4.2), one can prove the existence of C, depending only on β, such that $\forall\ M\in\Omega_H$:

$$\sum_{\substack{m\in\Omega_h\\ m\subset M}} l(\partial m\cap\partial M)\,|e_m|^2 \le C\left(\sum_{\substack{m\in\Omega_h\\ m\subset M}}\sum_{m_v\in N_{int}(m)} l(c_{mm_v})\,\frac{(e_{m_v}-e_m)^2}{d_{mm_v}} + \sum_{\substack{m\in\Omega_h\\ m\subset M}} V(m)|e_m|^2\right).$$

Using this result and the discrete Poincaré-Wirtinger's inequality (see [3] or [15]), one shows the existence of C, depending only on Ω, α, β and on the H^2–norm of P, such that :

$$B_1 \le C\sqrt{H}\left(\sum_{m\in\Omega_h}\sum_{m_v\in N_{int}(m)} l(c_{mm_v})\,\frac{(e_{m_v}-e_m)^2}{d_{mm_v}}\right)^{1/2}.$$

This result and (26) lead to (23).

Remark 4 *Using the discrete Poincaré-Wirtinger's inequality, one can prove, as in Theorem 1 (see Section 2.3), that the L^2–norm of the error is in $O(\sqrt{H})$.*

One ends the proof of Proposition 3 for reconstructed fluxes by solving local problems. First we remark that :

(27) $$A_h \le A_{3h} + A_{4h}$$

$$A_{3h} = \left(\sum_{m\in\Omega_h}\sum_{m_v\in N_{ext}(m)} d_{mm_v}\, l(c_{mm_v})\left|\frac{P_{M_{m_v}}-P_{M_m}}{d_{M_mM_{m_v}}} - \overline{Q}_{mm_v}\right|^2\right)^{1/2}$$

$$A_{4h} = \left(\sum_{m\in\Omega_h}\sum_{m_v\in N_{int}(m)} d_{mm_v}\, l(c_{mm_v})\left|\frac{p_{m_v}-p_m}{d_{mm_v}} - \overline{Q}_{mm_v}\right|^2\right)^{1/2}$$

so one has :

$$A_{3h} \leq \left(2 \sum_{m \in \Omega_h} \sum_{m_v \in N_{ext}(m)} l(c_{mm_v})\, d_{mm_v} \left| \frac{P(x_{M_{m_v}}) - P(x_{M_m})}{d_{M_m M_{m_v}}} - \overline{Q}_{mm_v} \right|^2 \right.$$

$$\left. +2 \sum_{M \in \Omega_H} \sum_{M_v \in N(M)} \left(\sum_{c \subset \sigma_{MM_v}} l(c)\, d_c \right) \left| \frac{E_{M_v} - E_M}{d_{MM_v}} \right|^2 \right)^{1/2}.$$

Then Theorem 1 and Lemma 5 give the existence of C, depending only on α, Ω and on the H^2–norm of P, such that :

$$A_{3h} \leq C\, H. \tag{28}$$

In a same way, one has :

$$A_{4h} \leq \left(2 \sum_{m \in \Omega_h} \sum_{m_v \in N_{int}(m)} l(c_{mm_v}) d_{mm_v} R_{mm_v}(P)^2 \right)^{1/2} +$$

$$+ \left(2 \sum_{m \in \Omega_h} \sum_{m_v \in N_{int}(m)} l(c_{mm_v}) \frac{(e_{m_v} - e_m)^2}{d_{mm_v}} \right)^{1/2}.$$

Thus, thanks to Lemmas 3 and 4, one has :

$$A_{4h} \leq C \sqrt{H}, \tag{29}$$

where C depends only on α, β, Ω and on the H^2–norm of P. Using (28) and (29) in (27), one obtains the result.

2.5 Convergence of the approximate saturation to the weak solution to hyperbolic problem

The aim of this section is to establish the convergence of the approximate saturation to the weak solution to (4), (6) and (7). We use a technique introduced by R. Eymard and T. Gallouët in [4] for the same problem as the one considered here, but in their paper the authors of [4] use a coupled finite element – finite volume scheme. These results have been extended to a finite volume scheme in [15]. In this section, one proves the following result :

Theorem 2 *Let Ω_H and Ω_h be two rectangular grids of Ω satisfying respectively (9) and (11), and g in $H^{1/2}(\Gamma)$ such that $\int_\Gamma g(\tau)\, d\tau = 0$. Let $(P_M)_{M \in \Omega_H}$ verifying (10) and $\sum_{M \in \Omega_H} V(M)\, P_M = \sum_{M \in \Omega_H} V(M)\, P(x_M)$. One denotes by $P(.)$ the exact solution to (3), (5) such that $\int_\Omega P(x)\, dx = 0$. One assumes that g is such that P is in $H^2(\Omega)$. Let $(p_m)_{m \in \Omega_h}$ satisfying (15) and such that for all $M \in \Omega_H$:*

$$\sum_{\substack{m \in \Omega_h \\ m \subset M}} V(m)\, p_m = \sum_{\substack{m \in \Omega_h \\ m \subset M}} V(m)\, P(x_m).$$

Let $\delta \in \mathbb{R}_+^*$ satisfying the stability condition (16). One denotes by $S_{\Omega_h,\delta}$ the approximate solution to (17).
Then :

$$\lim_{h\to 0} S_{\Omega_h\delta} = S$$

in $L^\infty(\Omega \times \mathbb{R}_+)$, for the weak $\star$ topology, where $S \in L^\infty(\Omega \times \mathbb{R}_+)$ is the weak solution to problem (4), (6), (7), i.e. S verifies (8).

Remark 5 *Since the hyperbolic equation is linear, the weak solution is unique so we do not need to use the notion of entropy solution.*

To prove this theorem, one first proves the two following lemmas. The first one gives an L^∞ estimate on the approximate solution, and the second one gives a weak estimate on the variations of $S_{\Omega_h,\delta}$.

Lemma 6 *Under the assumptions of Theorem 2, one has :*

$$\|S_{\Omega_h,\delta}\|_\infty \leq U = \max\Big(\|S_0\|_\infty, \|\overline{S}\|_\infty\Big).$$

Proof :
One uses (17) and the stability condition (16) to show that S_m^{n+1} is a convex combination of S_m^n, $S_{m_v}^n$ ($m_v \in N(m)$), and $\overline{S}_c^n$ ($c \in \mathcal{A}_{\partial\Omega}(m)$). So by induction, one obtains :

$$|S_m^{n+1}| \leq \max\Big(\|S_0\|_\infty, \|\overline{S}\|_\infty\Big)$$

$\forall\, m \in \Omega_h$ (for more details see [4] or [15]).

Lemma 7 *Suppose that the assumptions of Theorem 2 still hold. Let $T \in (0, +\infty)$; one sets $N_T = \max\Big\{n \in \mathbb{N}\ ;\ (n-1)\delta \leq T\Big\}$, and one defines EF_{1h} and EF_{2h} by :*

$$EF_{1h} = \sum_{n=0}^{N_T} \sum_{m\in\Omega_h} \delta \left(\sum_{\substack{m_v\in N(m)\\ Q_{mm_v}>0}} h\, Q_{mm_v}\, |S_{m_v}^n - S_m^n|\, l(c_{mm_v}) + \sum_{c\in\mathcal{A}_{\partial\Omega}(m)} H\, \left|\overline{S}_c^n - S_m^n\right|\, l(c)\, Q_c^+ \right),$$

$$EF_{2h} = \sum_{n=0}^{N_T} \sum_{m\in\Omega_h} \delta \left(\sum_{\substack{m_v\in N(m)\\ Q_{mm_v}>0}} h\, Q_{mm_v}\, |S_{m_v}^n - S_m^n|\, l(c_{mm_v}) + \sum_{c\in\mathcal{A}_{\partial\Omega}(m)} h\, \left|\overline{S}_c^n - S_m^n\right|\, l(c)\, Q_c^+ \right).$$

Then there exist C_1 and C_2, depending only on S_0, $\overline{S}$, Ω, α, β, g, T, η and on the H^2–norm of P, such that :

(30) $EF_{1h} \leq C_2 \left(\sqrt{h}+H\right)$ *if fluxes are reconstructed with interpolation,*

(31) $EF_{2h} \leq C_1 \sqrt{h}$ *if fluxes are reconstructed by solving local problems.*

Proof :
As in [4] or [15], one proves the following result :

$$\sum_{n=0}^{N_T}\sum_{m\in\Omega_h}\delta\left(\sum_{\substack{m_v\in N(m)\\ Q_{mm_v}>0}} Q_{mm_v}\,|S^n_{m_v}-S^n_m|^2\,l(c_{mm_v})+\sum_{c\in\mathcal{A}_{\partial\Omega}(m)}\left|\overline{S}^n_c-S^n_m\right|^2\,l(c)\,Q^+_c\right)\le C_B,$$

where $C_B=\dfrac{1}{\eta}\Big(\|S_0\|^2_\infty\,V(\Omega)+T\,\|g\|_{L^2(\Gamma)}\,\|\overline{S}\|^2_\infty\,l(\Gamma)^{1/2}\Big)$.
It yields :

$$EF_{1h}\le\sqrt{2\,C_B\,T}\left(h\,BV_1^{1/2}+H\,\sqrt{\|g\|_{L^2(\Gamma)}\,l(\Gamma)^{1/2}}\right), \tag{32}$$

where :

$$BV_1=\sum_{m\in\Omega_h}\sum_{m_v\in N(m)}|Q_{mm_v}|\,l(c_{mm_v});$$

similarly :

$$EF_{2h}\le\sqrt{2\,C_B\,T}\,h\left(BV_1^{1/2}+\sqrt{\|g\|_{L^2(\Gamma)}\,l(\Gamma)^{1/2}}\right). \tag{33}$$

Then it is sufficient to show that $BV_1\le C/h$. One can observe that :

$$BV_1\le\frac{\sqrt{V(\Omega)}}{\beta\,h}A_h+\frac{\sqrt{V(\Omega)}}{\sqrt{\beta\,h}}\left(\sum_{m\in\Omega_h}\sum_{m_v\in N(m)}\int_{c_{mm_v}}|\nabla P(\tau)|^2\,d\tau\right)^{1/2}.$$

Thus using inequality (25), Proposition 3 and inequalities (33) and (32), one has :

$$EF_{1h}\le C\left(\sqrt{h}+H\right),$$

$$EF_{2h}\le C\,\sqrt{h},$$

where C depends only on Ω, β, α, on the H^2–norm of P and on the L^2–norm of g.

Proof of Theorem 2 :
Since $S_{\Omega_h,\delta}$ is bounded in $L^\infty(\Omega\times\mathbb{R}_+)$, there exist a subsequence, still denoted by $S_{\Omega_h,\delta}$, and $S\in L^\infty(\Omega\times\mathbb{R}_+)$, such that $\lim_{h\to0}S_{\Omega_h\delta}=S$ in $L^\infty(\Omega\times\mathbb{R}_+)$ for the weak $\star$ topology. Then, we are going to prove that S is the weak solution of (4), (6) and (7).
Let $\varphi\in C^\infty_c(\Omega^+\times\mathbb{R}_+,\mathbb{R}_+)$, and $T\in\mathbb{R}^\star_+$ such that $\forall\ x\in\Omega^+$, $supp\,(\varphi(x,\cdot))\subset[0,T]$.
One sets $N_T=\max\{n\in\mathbb{N}\,;\,(n-1)\,\delta\le T\}$.
Then one multiplies (17) by $\frac{\delta}{V(m)}\int_m\varphi(x,t^n)\,dx$ and one sums over $n\in\mathbb{N}$ and $m\in\Omega_h$; one obtains :

$$E_{1h}+E_{2h}=0,$$

with :

$$E_{1h}=\sum_{n=0}^{N_T}\sum_{m\in\Omega_h}(S^{n+1}_m-S^n_m)\int_m\varphi(x,t^n)\,dx,$$

$$E_{2h} = -\sum_{n=0}^{N_T} \delta \sum_{m\in\Omega_h} \left(\sum_{\substack{m_v\in N(m)\\ Q_{mm_v}>0}} Q_{mm_v} \left(S^n_{m_v} - S^n_m\right) l(c_{mm_v}) \, \frac{1}{V(m)} \int_m \varphi(x,t^n)\,dx \right.$$
$$\left. + \sum_{c\in\mathcal{A}_{\partial\Omega}(m)} \left(\overline{S}^n_c - S^n_m\right) l(c)\, Q^+_c \, \frac{1}{V(m)} \int_m \varphi(x,t^n)\,dx \right).$$

By transfering the differences on φ and passing to the limit, one proves :

$$\lim_{h\to 0} E_{1h} = -\iint_{\Omega\times\mathbb{R}_+} S(x,t)\,\varphi_t(x,t)\,dx\,dt - \int_\Omega S_0(x)\,\varphi(x,0)\,dx.$$

So it just remains to show that :

$$\lim_{h\to 0} E_{2h} = E_2 = \int_\Omega \int_{\mathbb{R}_+} S(x,t)\,\nabla P(x).\nabla\varphi(x,t)\,dx\,dt - \int_\Gamma \int_{\mathbb{R}_+} \overline{S}(\tau,t)\,\varphi(\tau,t)\,g^+(\tau)\,d\tau\,dt.$$

We begin to establish this result when fluxes are reconstructed by solving local problems.

Approximate pressure fluxes reconstructed by solving local problems :
We define E_{3h} by :

$$E_{3h} = -\sum_{n=0}^{N_T} \delta \sum_{m\in\Omega_h} \left(\sum_{\substack{m_v\in N(m)\\ Q_{mm_v}>0}} Q_{mm_v} \left(S^n_{m_v} - S^n_m\right) \int_{c_{mm_v}} \varphi(\tau,t^n)\,d\tau \right.$$
$$\left. + \sum_{c\in\mathcal{A}_{\partial\Omega}(m)} \left(\overline{S}^n_c - S^n_m\right) \int_c g^+(\tau)\,\varphi(\tau,t^n)\,d\tau \right).$$

First we show that the difference between E_{2h} and E_{3h} tends to 0 when h goes to 0. Indeed :

$$|E_{2h}-E_{3h}| \le \sum_{n=0}^{N_T} \delta \sum_{m\in\Omega_h} \left[\sum_{c\in\mathcal{A}_{\partial\Omega}(m)} \left| \int_c g^+(\tau) \left(\varphi(\tau,t^n) - \frac{1}{V(m)}\int_m \varphi(x,t^n)\,dx\right) d\tau \right| \left(\overline{S}^n_c - S^n_m\right) \right.$$
$$\left. + \sum_{\substack{m_v\in N(m)\\ Q_{mm_v}>0}} Q_{mm_v} \left|S^n_{m_v} - S^n_m\right| \left| \int_{c_{mm_v}} \left(\varphi(\tau,t^n) - \frac{1}{V(m)} \int_m \varphi(x,t^n)\,dx\right) d\tau \right| \right].$$

Thanks to the regularity of φ and to Lemma 7, one obtains :

$$|E_{2h} - E_{3h}| \le C_1\, EF_{2h} \le C\,\sqrt{h},$$

where C_1 depends only on the first order derivatives of φ and C depends only on φ, Ω, β, α, S_0, $\overline{S}$, T, η, on the H^2–norm of P and on the L^2–norm of g.

To determine the limit of E_{3h}, one defines E_{4h} by :

$$E_{4h} = \sum_{n=0}^{N_T} \delta \left(\int_\Omega S_{\Omega_h,\delta}(x,t^n)\, \nabla P(x).\nabla\varphi(x,t^n)\, dx - \int_\Gamma \int_{\mathbb{R}_+} \overline{S}_{\Omega_h}(\tau,t^n)\, \varphi(\tau,t^n)\, g^+(\tau)\, d\tau \right),$$

where $\overline{S}_{\Omega_h}(\tau,t) = \overline{S}_a^n$ if $\tau \in a$ and $t \in [t^n, t^{n+1}[$.
Thanks to the sequential weak $\star$ compactness of $S_{\mathcal{T},\delta}$ in L^∞, the limit of E_{4h} is E_2 when h goes to 0. Then we prove that the difference between E_{3h} and E_{4h} goes to 0 when h goes to 0.
Thanks to the conservativity of exact fluxes and since $P \in H^2(\Omega)$ is solution to (3), (5), one has :

$$|E_{3h} - E_{4h}| \le \left| \sum_{n=0}^{N_T} \delta \sum_{m\in\Omega_h} \sum_{m_v\in N(m)} S_m^n \int_{c_{mm_v}} (Q_{mm_v} - \nabla P(\tau).n_m(\tau))\, \varphi(\tau,t^n)\, d\tau \right| ;$$

which, combined with (3), (5) and (15), implies :

$$|E_{3h} - E_{4h}| \le \left| \sum_{n=0}^{N_T} \delta \sum_{m\in\Omega_h} \sum_{m_v\in N(m)} S_m^n \int_{c_{mm_v}} \Big(Q_{mm_v} - \nabla P(\tau).n_m(\tau) \Big) \times \right.$$
$$\left. \times \Big(\varphi(\tau,t^n) - \varphi(x_m,t^n) \Big)\, d\tau \right|,$$

then

$$(34)\quad |E_{3h} - E_{4h}| \le M_\varphi\, h\, U\, T \sum_{m\in\Omega_h} \sum_{m_v\in N(m)} l(c_{mm_v}) \left| Q_{mm_v} - \overline{Q}_{mm_v} \right| \le \frac{M_\varphi\, U\, T\, \sqrt{V(\Omega)}}{\beta}\, A_h,$$

where $M_\varphi = \sup_{(x,t)\in\overline{\Omega}\times[0,T]} |\nabla\varphi(x,t)|$ and where we recall that $U = \max(\|\overline{S}\|_\infty, \|S_0\|_\infty)$.
Using Proposition 3, one gets :

$$|E_{3h} - E_{4h}| \le C\sqrt{H},$$

where C depends only on α, β, Ω, T, φ, S_0, $\overline{S}$ and on the H^2–norm of P. This ends the proof of Theorem 2 when fluxes are reconstructed by solving local problems.

Fluxes reconstructed by interpolation
We define E'_{3h} by :

$$E'_{3h} = -\sum_{n=0}^{N_T} \delta \sum_{m\in\Omega_h} \left(\sum_{\substack{m_v\in N(m)\\ Q_{mm_v}>0}} Q_{mm_v}\, (S_{m_v}^n - S_m^n) \int_{c_{mm_v}} \varphi(\tau,t^n)\, d\tau \right.$$
$$\left. + \sum_{c\in\mathcal{A}_{\partial\Omega}(m)} \Big(\overline{S}_c^n - S_m^n \Big) \frac{l(c)}{l(\sigma_c)} \int_{\sigma_c} g^+(\tau)\, \varphi(\tau,t^n)\, d\tau \right).$$

Then, as for the difference between E_{2h} and E_{3h} in the previous section, using the regularity of φ and Lemma 7, one gets :

$$|E_{2h} - E'_{3h}| \le C_\varphi \, EF_{1h} \le C\,(\sqrt{h} + H),$$

where C_φ depends only on the first derivatives of φ and C depends only on φ, Ω, β, α, S_0, $\overline{S}$, T, η, on the H^2–norm of P and on the L^2–norm of g.

We are going to prove that the difference between E_{4h} and E'_{3h} converges to zero when h goes to 0. Fisrt, as in the previous section, using the conservativity of exact fluxes and since $P \in H^2(\Omega)$ is solution of (3), (5), one has :

$$\begin{aligned}|E_{4h} - E'_{3h}| \le & \left| \sum_{n=0}^{N_T} \delta \sum_{m\in\Omega_h} \sum_{m_v \in N(m)} S_m^n \int_{c_{mm_v}} \Big(Q_{mm_v} - \nabla P(\tau).n_m(\tau)\Big) \varphi(\tau, t^n)\, d\tau \right| \\ & + \sum_{n=0}^{N_T} \delta \sum_{m\in\Omega_h} \sum_{c\in\mathcal{A}_{\partial\Omega}(m)} \left|\overline{S}_c^n - S_m^n\right| \frac{1}{l(\sigma_c)} \left| \int_c \int_{\sigma_c} g(\tau)\,\varphi(\tau, t^n) - g(\gamma)\,\varphi(\gamma, t^n)\, d\gamma\, d\tau \right|.\end{aligned}$$

Let E_{8h} be defined by :

$$E_{8h} = \sum_{n=0}^{N_T} \delta \sum_{m\in\Omega_h} \sum_{c\in\mathcal{A}_{\partial\Omega}(m)} \left|\overline{S}_c^n - S_m^n\right| \frac{1}{l(\sigma_c)} \left| \int_c \int_{\sigma_c} g(\tau)\,\varphi(\tau, t^n) - g(\gamma)\,\varphi(\gamma, t^n)\, d\gamma\, d\tau \right| ;$$

then thanks to the regularity of φ, there exists C, depending only on φ, such that :

$$|E_{8h}| \le 2\,U \sqrt{T\, l(\Gamma)} \left[C\,\sqrt{T}\, \|g\|_{L^2(\Gamma)} \sqrt{H} + E_{9h} \right],$$

where

$$E_{9h} = \left(\sum_{n=0}^{N_T} \delta \sum_{m\in\Omega_h} \sum_{c\in\mathcal{A}_{\partial\Omega}(m)} \frac{1}{l(c)\, l(\sigma_c)^2} \left(\int_c \int_{\sigma_c} \Big(g(\tau)\,\varphi(\gamma, t^n) - g(\gamma)\,\varphi(\tau, t^n) \Big)\, d\gamma\, d\tau \right)^2 \right)^{1/2}.$$

Using $g(\tau) = \nabla P(\tau).n(\tau)$ for a.e. $\tau \in \partial\Omega$, this can be written in the following way :

$$\begin{aligned}E_{9h} = \Bigg(\sum_{n=0}^{N_T} \delta \sum_{M\in\Omega_H} \sum_{\sigma\in\mathcal{A}_{\partial\Omega}(M)} \sum_{c\subset\sigma} l(c) \Bigg| & \frac{1}{l(c)} \int_c \nabla P(\tau).n(\tau)\, d\tau \times \frac{1}{l(\sigma)} \int_\sigma \varphi(\gamma, t^n)\, d\gamma - \\ & \frac{1}{l(c)} \int_c \varphi(\tau, t^n)\, d\tau \times \frac{1}{l(\sigma)} \int_\sigma \nabla P(\gamma).n(\gamma)\, d\gamma \Bigg|^2 \Bigg)^{1/2}.\end{aligned}$$

Then as in proof of Proposition 3 for fluxes reconstructed with interpolation (see (22)), one denotes by σ_M the edge of M which is the opposite edge to σ (i.e. the edge of M such that $|n_\sigma.n_{\sigma_M}| = 1$) and M_σ the neighbor of M such that $\partial M_\sigma \cap \partial M = \sigma_M$ (see figure 4). One has :

$$E_{9h} = \left(5 \sum_{n=0}^{N_T} \delta \sum_{M\in\Omega_H} \sum_{\sigma\in\mathcal{A}_{\partial\Omega}(M)} \sum_{c\subset\sigma} l(c) \Big({E_{1,\sigma,c}}^2 + {E_{2,\sigma,c}}^2 + {E_{3,\sigma,c}}^2 + {E_{4,\sigma,c}}^2 + {E_{5,\sigma,c}}^2 \Big) \right)^{1/2},$$

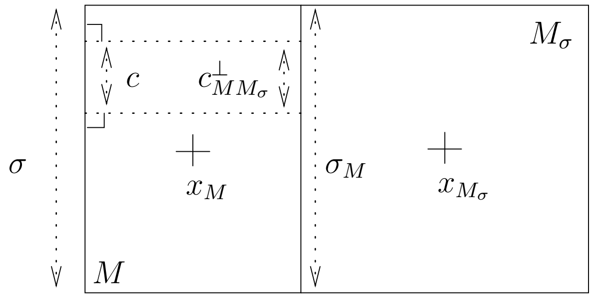

Figure 4:

where :

$$E_{1,\sigma,c} = \left|\frac{1}{l(\sigma)}\int_\sigma \varphi(\gamma,t^n)\,d\gamma\left(\frac{1}{l(c)}\int_c \nabla P(\tau).n(\tau)\,d\tau - \frac{1}{l(c^{\perp}_{MM_\sigma})}\int_{c^{\perp}_{MM_\sigma}} \nabla P(\tau).n(\tau)\,d\tau\right)\right|,$$

$$E_{2,\sigma,c} = \left|\frac{1}{l(\sigma)}\int_\sigma \varphi(\gamma,t^n)\,d\gamma\left(\frac{1}{l(c^{\perp}_{MM_\sigma})}\int_{c^{\perp}_{MM_\sigma}} \nabla P(\tau).n(\tau)\,d\tau - \frac{P(x_{M_\sigma})-P(x_M)}{d_{MM_\sigma}}\right)\right|,$$

$$E_{3,\sigma,c} = \left|\frac{P(x_{M_\sigma})-P(x_M)}{d_{MM_\sigma}}\left(\frac{1}{l(\sigma)}\int_\sigma \varphi(\gamma,t^n)\,d\gamma - \frac{1}{l(c)}\int_c \varphi(\tau,t^n)\,d\tau\right)\right|,$$

$$E_{4,\sigma,c} = \left|\frac{1}{l(c)}\int_c \varphi(\tau,t^n)\,d\tau\left(\frac{P(x_{M_\sigma})-P(x_M)}{d_{MM_\sigma}} - \frac{1}{l(\sigma_M)}\int_{\sigma_M} \nabla P(\gamma).n(\gamma)\,d\gamma\right)\right|,$$

$$E_{5,\sigma,c} = \left|\frac{1}{l(c)}\int_c \varphi(\tau,t^n)\,d\tau\left(\frac{1}{l(\sigma_M)}\int_{\sigma_M} \nabla P(\gamma).n(\gamma)\,d\gamma - \frac{1}{l(\sigma)}\int_\sigma \nabla P(\gamma).n(\gamma)\,d\gamma\right)\right|.$$

Then using the regularity of P, Lemma 5, the regularity of φ, and Lemma 1, one proves :

$$E_{8h} \le C_1\left(\sqrt{H} + E_{9h}\right) \le C_2\sqrt{H},$$

where C_1 depends only on α, β, T, φ and on the H^2–norm of P and C_2 depends only on α, β, T, U, φ, Γ, on the H^2–norm of P and on the L^2–norm of g.
So one obtains :

$$|E_{4h} - E'_{3h}| \le \left|\sum_{n=0}^{N_T}\delta\sum_{m\in\Omega_h}\sum_{m_v\in N(m)} S^n_m \int_{c_{mm_v}}\Big(Q_{mm_v} - \nabla P(\tau).n_m(\tau)\Big)\varphi(\tau,t^n)\,d\tau\right| + C\sqrt{H}.$$

But thanks to (14), (3) and (5), one has :

$$|E_{4h} - E'_{3h}| \le C\sqrt{H} + \sum_{c\in\mathcal{A}_{\partial\Omega}(m)} \frac{|S^n_m|\,\varphi(x_m,t^n)}{l(\sigma_c)}\left|\int_c\int_{\sigma_c}\Big(g(\tau)-g(\gamma)\Big)\,d\gamma\,d\tau\right| +$$

$$+\left|\sum_{n=0}^{N_T}\delta\sum_{m\in\Omega_h}\sum_{m_v\in N(m)}S_m^n\int_{c_{mm_v}}\Big(Q_{mm_v}-\nabla P(\tau).n_m(\tau)\Big)\Big(\varphi(\tau,t^n)-\varphi(x_m,t^n)\Big)\,d\tau\right|.$$

Using the regularity of φ and an argument similar to the one used for E_{8h}, one shows that :

$$|E_{4h}-E'_{3h}|\leq C\sqrt{H}+T\,U\,C_\varphi\,h\sum_{m\in\Omega_h}\sum_{m_v\in N(m)}l(c_{mm_v})\left|Q_{mm_v}-\overline{Q}_{mm_v}\right|, \tag{35}$$

where C depends only on α, β, T, U, φ, Γ, P and on g and where C_φ depends only on φ. Using Cauchy Schwarz' inequality and proposition 3, one gets :

$$|E_{4h}-E'_{3h}|\leq C\left(H+\sqrt{H}\right)\leq C\sqrt{H},$$

where C depends only on α, β, T, U, $\overline{S}$, φ, Γ, Ω, P and on g.
This ends the proof of Theorem 2.

3 Numerical results

Two simulators (for a 1 grid resolution and for the Dual Mesh Method) have been built allowing to solve a 2 phase flow problem in heterogeneous porous media corresponding to the coupled problem (1), (2). For both simulators, we have used the same Finite Volume schemes, those described in Section 2. They are based on an IMPES scheme (implicit in pressure and explicit in saturation). The algorithm of the simulator of the Dual Mesh Method is as follows :

Step 1 - Calculation of the parameters necessary to solve the pressure equation thanks to an adaptive homogenization from the HR to the LR grid.

Step 2 - Calculation of the pressure over the LR grid.

Step 3 - Reconstruction of the flow-rate over the HR grid by using the pressure over the LR grid.

Step 4 - Resolution of the saturation equation over the HR grid.

The Darcy's velocity ($K\,\nabla P$) is reconstructed over each interface of the mesh. The flow-rate is defined by the product of the Darcy's velocity and the length of the considered interface.
It is possible and even adviced to have also different time steps. For more details, see [14]. The upscaling step (step 1) is integrated in the Dual Mesh Method simulator to homogenize the product of the absolute permeability and the total mobility (see [6]). Indeed, as the parameters are given over the HR grid, this step allows to determine the discrete coefficients of the pressure equation when the medium is heterogeneous (see Test-Cases 2 and 3) and the total mobility depends on the saturation (see Test-Case 3). In the

homogeneous case with constant total mobility (see Test-Case 1), there is no upscaling step. This test corresponds to the theoretical case. Neumann and Dirichlet boundary conditions are used.
Let us now describe the different test cases used. From a physical point of view, each one corresponds to a secondary recovery process of oil by injection of water in the field.

3.1 Description of the test cases

A classical test in petroleum engineering is considered : the so called quarter of five spot geometry. The figure 5 represents the LR grid. This one is drawn in 3D to justify the m^3, but numerical tests are done in 2D. Over Γ_1, respectively over Γ_2, we impose the pressure, respectively the flow-rate (see figure 5). Over $\Gamma_3 = \Gamma/(\Gamma_1 \cup \Gamma_2)$, an homogeneous Neumann condition is used. In order to avoid difference between numerical results due to Productivity Indices (PI) problems, source terms are in the boundary conditions rather than in wells.
Several grids have been taken which are all rectangular and regular. These ones are different according to the simulations. The number of cells in x equals the number of

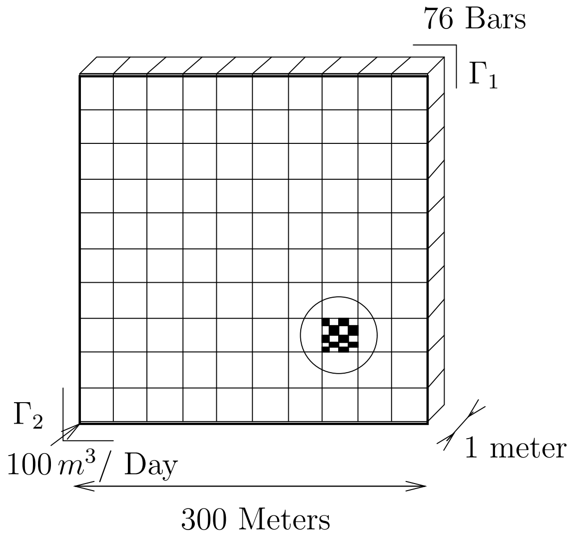

Figure 5: Geometrical Characteristics of the Test-Cases

cells in y. Thus, we introduce the notation (n_H, n_h) which corresponds to the numerical solution of the problem with a $n_H \times n_H$ cells for the pressure equation and $n_h \times n_h$ cells for the saturation equation. When $n_H \neq n_h$, the Dual Mesh Method is used. So, we denote by mi (resp. mlpb) the reconstruction by interpolation (resp. by solving local problems). When the medium is heterogeneous, the method mlpb described in Section 2 becomes widespread by sharing each discrete flow-rate at the interface of the LR Grid

in proportion to the value of the absolute permeability and the total mobility of the LR Grid (for more details, see [14]).
For each test-case, the solution over the finest grid is considered as the reference.
For a given time, an error estimate on the saturation in L^1–norm and an error estimate on the flow-rate in L^2–norm are calculated for each simulation by considering the finest grid as the reference. So, one defines : $\|e(S)\|_{L^1}$ (resp. $\|e(Q)\|_{L^2}$), the L^1 (resp. L^2) norm of the difference between the reference case and the considered resolution for the saturation (resp. flow-rate).
For every test case, and every simulation, these error estimates are calculated at the time $t = 2000$ days.

3.2 Validation over the Simplified Case - Test Case 1

Let us consider a simulation over an homogeneous case ($K = 100\, mD$) and with a constant total mobility ($m = 1$). This is the case studied in the theoretical approach.
Three differents grids are chosen, giving 8 simulations: $(16,16)$, $(16,48)_{mi}$, $(16,48)_{mlpb}$, $(16,144)_{mi}$, $(16,144)_{mlpb}$, $(48,144)_{mi}$, $(48,144)_{mlpb}$ and $(144,144)$. The last one is the reference. For the first five simulations, we fix the grid ($n_H \times n_H$) where the pressure equation is solved, and we try to determine the influence of n_h (i.e. h) on the flow-rate (see Table 1) and on the saturation (see figure 6 page 28). For the last four simulations, the grid ($n_h \times n_h$) where the saturation equation is solved is fixed, and we try to determine the influence of n_H (i.e. H) on the flow-rate (see Table 1) and on the saturation (see figure 6 page 28). Unfortunately, the number of simulations did not allow us to determine

Simulations	Reconstruction Method	$\|e(Q)\|_{L^2}$
(16,16)		$3,19\ \ 10^{-1}$
(16,48)	mi	$2,12\ \ 10^{-2}$
(16,48)	mlpb	$1,94\ \ 10^{-2}$
(16,144)	mi	$2,80\ \ 10^{-4}$
(16,144)	mlpb	$2,67\ \ 10^{-4}$
(48,144)	mi	$4,82\ \ 10^{-5}$
(48,144)	mlpb	$4,70\ \ 10^{-5}$

Table 1: Influence of n_H and n_h over $\|e(Q)\|_{L^2}$ - Test-Case 1

the influence of n_H and n_h. We could not make more simulations because between two different simulations, we must multiply n_h at least by 3; then, the difference between errors are too important to conclude on the influence of n_h. Furthermore, it is not possible to have a very high n_h and n_h must be a multiple of n_H. Having said that, numerical results show the convergence of the approximate solutions as in the theoretical approach.

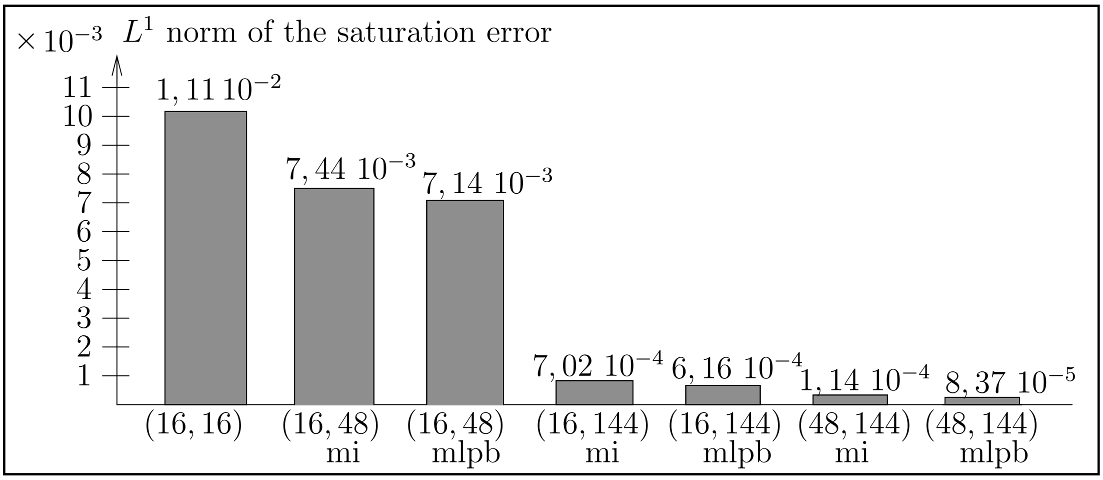

Figure 6: Influence of n_H and n_h over $\|e(S)\|_{L^1}$ - Test-Case 1

3.3 An heterogeneous Case with Constant Total Mobility - Test Case 2

The simulator is now applied over an heterogeneous case with total mobility equal to 1 everywhere. The absolute permeability map is generated by a lognormal distribution with a correlation length equal to 3 meters in the x and y directions (see figure 7 page 29). With these choices, an algebraic estimator (see [5]) is used to generate permeability maps over lower grids. The method *mi* based on interpolation is not used anymore. Indeed, the interpolator doesn't take into account the heterogeneity of the porous media over the HR Grid.
As in the Test-Case 1, Table 2 and figure 8 page 30 give error estimates.
The observed results for the method mlpb are of the same order as the results of the Test-Case 1. However, the errors are a little bit more important in this case, due to the presence of the heterogeneities. So, even if the reconstruction of the flow-rate is more delicate, the accuracy of the results is reasonable.

Simulations	$\|e(Q)\|_{L^2}$
$(16, 16)$	$3,94\ \ 10^{-1}$
$(16, 48)_{mlpb}$	$2,69\ \ 10^{-2}$
$(16, 144)_{mlpb}$	$5,58\ \ 10^{-4}$

Table 2: Influence of n_h over $\|e(Q)\|_{L^2}$ - Test-Case 2

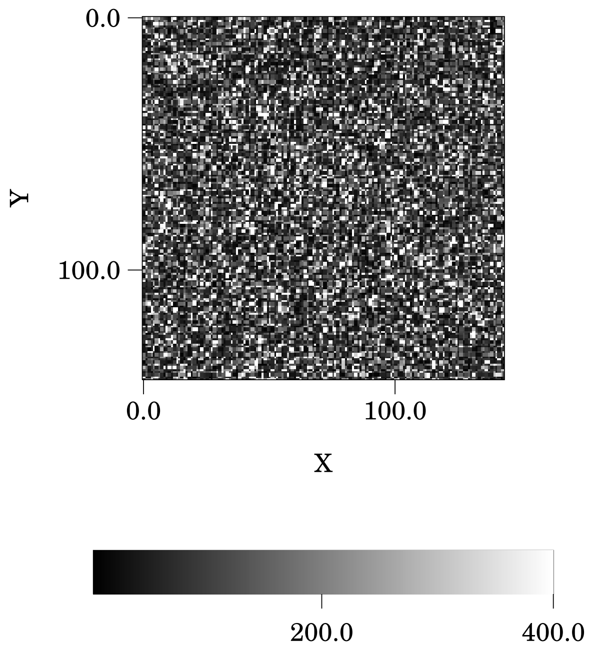

Figure 7: Permeability Map (in MD) of the Test-Case 2

3.4 An heterogeneous Case with no Constant Total Mobility - Test Case 3

Let us show the extension of the Dual Mesh Method with a non constant total mobility ; it means that the total mobility depends on the saturation. In this test-case, we have extended the Dual Mesh Method to a real coupled system. So, the pressure equation must be solved for each time step and this test shows the interest of the Dual Mesh Method in term of efficiency and CPU time in comparison with the resolutions over a single grid.
An heterogeneous porous medium is generated by a lognormal distribution with a correlation length equal to 25 meters in the x and y directions (see figure 9 page 31).
The law of relative permeabilities used are of the Corey type (see [1]), i.e.

$$k_{rw}(S) = k_{wm}\, S^{*n_w} \text{ and } k_{ro}(S) = k_{om}\,(1 - S^*)^{n_o} \text{ with } S^* = \frac{S - S_{wi}}{1 - S_{wi} - S_{or}}$$

An unfavourable mobility ratio is chosen for the Test-Case 3 ($M = \dfrac{k_{rw}}{\mu_w} \Big/ \dfrac{k_{ro}}{\mu_o} = 2$).
The Table 3 summarizes the different fluid properties of this Test-Case.
Six simulations are considered: $(10, 10)$, $(10, 30)_{mlpb}$, $(30, 30)$, $(10, 90)_{mlpb}$, $(30, 90)_{mlpb}$ and $(90, 90)$.

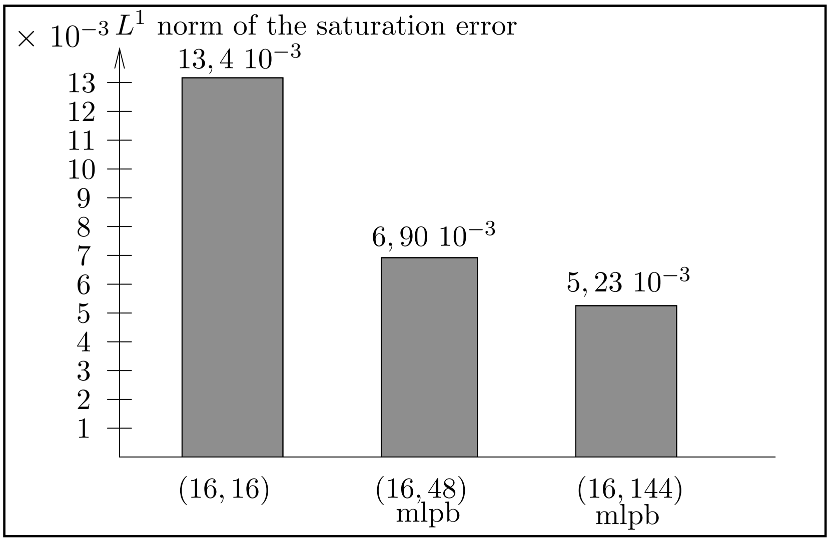

Figure 8: Influence of n_h over $\|e(S)\|_{L^1}$ - Test-Case 2

Mobility Ratio	S_{wi}	S_{or}	n_w	n_o	k_{wm}	k_{om}	μ_w (c P)	mu_o (c P)
2	0.2	0.2	2	1.5	0.4	1	1	0.8

Table 3: Fluid Properties of the Test-Case 3

The error estimates on the flow-rate and on the saturation are shown in the Table 4 and figure 10 page 32. This last one reveals that the convergence of the approximate solutions is again obtained although the problem is complex because of the heterogeneities and the non linearity.
The watercut curves (see figure 11 page 33) are very interesting for the reservoir engineering. It corresponds to the ratio between the water flow-rate and the total flow-rate at the producer well (on Γ_1). It is very important to have a good evaluation of the breakthrough of the water in the productor. The comparison of watercut (see figure 11 page 33) for each simulation shows that the results obtained with a fully fine simulation are similar to those obtained with the Dual Mesh Method (mplb). We have considered the solution of the pressure equation over the 10×10 grid and the solution of the saturation equation over the finer grid (30×30 or 90×90 depending on the case considered). Table 5 shows the CPU times for the different simulations. We notice again that the Dual Mesh Method used here has also two different time-steps. Indeed, the time-step in pressure is calculated as if the pressure and saturation equations were both calculated over the LR Grid. So, the pressure time-step is calculated thanks to a ratio between the CFL Conditions for the saturation equation over the HR and LR Grid (see [6]). Computing cost using (n_H, n_h) Dual Mesh Methods by using this two different time-steps is quite of the same order as the (n_H, n_H) simulation, while the (n_h, n_h) simulation requires a very long CPU time. These

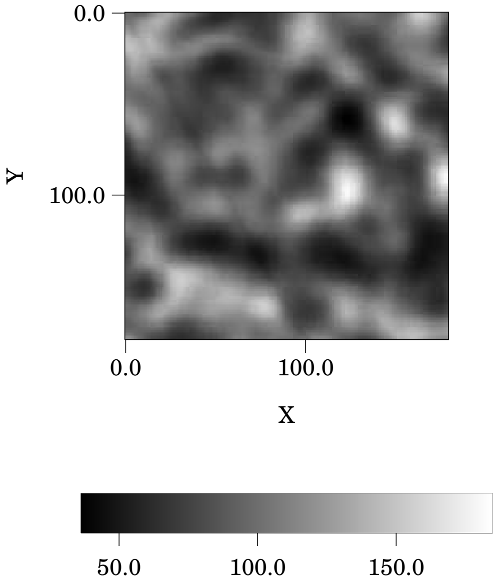

Figure 9: Permeability Map (in MD) of the Test Case 3

Simulations	$\|e(Q)\|_{L^2}$
$(10, 10)$	$2,15 \ \ 10^{-1}$
$(10, 30)_{mlpb}$	$1,27 \ \ 10^{-2}$
$(30, 30)$	$1,27 \ \ 10^{-2}$
$(10, 90)_{mlpb}$	$2,18 \ \ 10^{-4}$
$(30, 90)_{mlpb}$	$3,38 \ \ 10^{-5}$

Table 4: Comparison of $\|e(Q)\|_{L^2}$ - Test-Case 3

results are obvious to the extent that instead of solving the linear system over $n_h \times n_h$ cells as in the HR Grid resolution, the (n_H, n_h) Dual Mesh Method requires only to solve the pressure equation over $n_H \times n_H$ cells for each time-step in pressure and to solve n_h local problems. A parallel calculation could be a very good way to solve the saturation equation, using an explicit scheme in order to improve the CPU time.

Conclusion :

In this paper, in the homogeneous case with constant total mobility, we have proved the convergence of the Dual Mesh Method with two different reconstruction methods of fluxes. The first one has a cheaper computing cost but the second can be extended to more complex problems (heterogeneous cases with non constant total mobility). For both methods, we gave for the reconstructed fluxes an error estimate, in H for the interpolation method

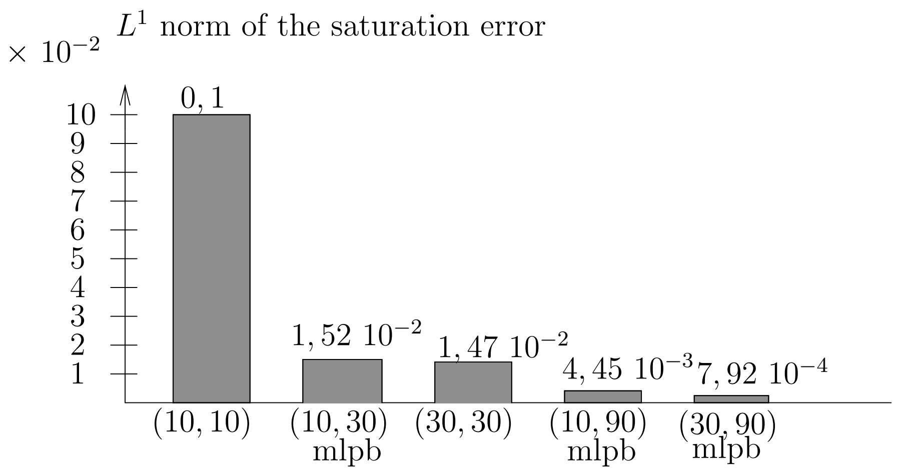

Figure 10: Influence of n_h over $\|e(S)\|_{L^1}$ - Test-Case 2

Pressure Grid → Saturation Grid ↓	10×10	30×30	90×90
10×10	11	XXXX	XXXX
30×30	52	293	XXXX
90×90	797	1350	18052

Table 5: Comparison of the CPU-Time (s) - Test-Case 3

and in $h + \sqrt{H}$ for the local problems method. We think that the second result is not optimal.
These results are sufficient to pass to the limit in the discretized equation associated to the saturation equation and thus we proved the convergence of the approximate saturation to the exact saturation with the Dual Mesh Method.
We used the Dual Mesh Method with local problem reconstruction in heterogeneous cases with non constant total mobility. The calculation of the error estimates for different test-cases showed the numerical convergence of the Dual Mesh Method algorithm.
Eventually the Dual Mesh Method was validated by its efficiency and its computing cost in comparison with classical methods (only one grid).
So, it is possible to apply the Dual Mesh Method to full field simulations.

Aknowledgments :
The authors wish to thank T. Gallouët, D. Guérillot, and J.-M. Thomas for their attention

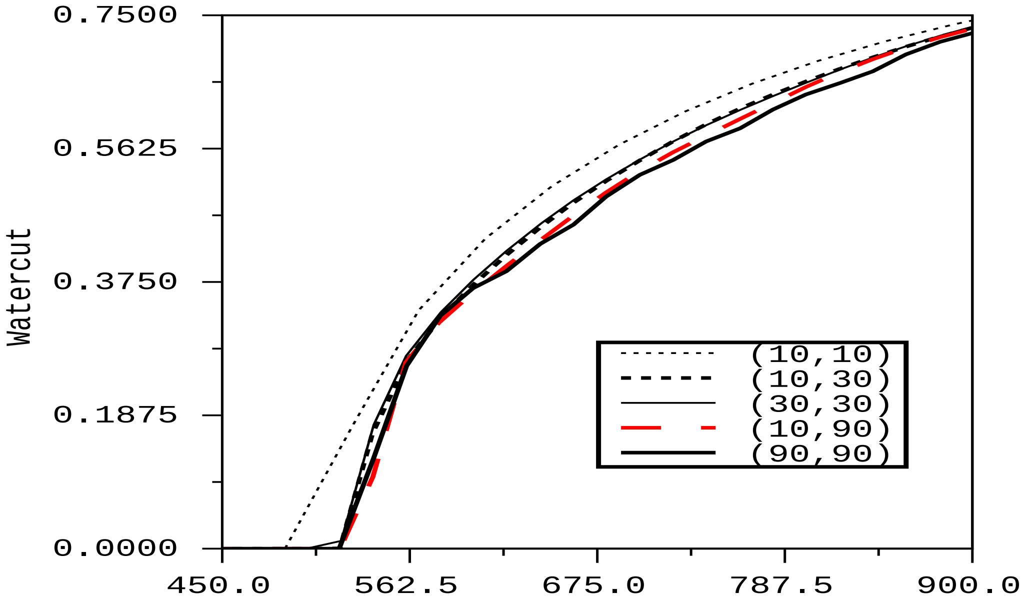

Figure 11: Comparison of Watercut - Test-Case 3

to this work and their worthwhile remarks. They also thank the Institut Français du Pétrole for its permission to publish this work.

References

[1] Brooks R.H., Corey A.T., 1966, *Properties of Porous Media Affecting Fluid Flow*, Journal of the Irrigation and Drainage Division, Proceeding of ASCE, Vol. 92, No I R2.

[2] Coudière Y., Vila J.P., Villedieu P., *Convergence of a finite volume scheme for a diffusion-convection problem*, submitted.

[3] Eymard R., Gallouët T., Herbin R., *The finite volume method*, in preparation for the "Handbook of Numerical Analysis", Ph. Ciarlet and J.L. Lions eds.

[4] Eymard R., Gallouët T., 1993, *Convergence d'un schéma de type Eléments Finis-Volumes Finis pour un système formé d'une équation elliptique et d'une équation hyperbolique*, M^2AN, 27, No 7, 843–861.

[5] Guérillot D., Rudkiewicz J.L., Ravenne C., Renard G., 1989, *An Integrated Model for Computer Aided Reservoir Description from Outcrop Study to Fluid Flow Simulation*, Rev IFP, Vol 45–1, 71–77.

[6] Guérillot D., Verdière S., 1997, *Adaptive Upscaling with the Dual Mesh Method*, the Proceedings of Niper II, San Antonio.

[7] Herbin R., 1995, *An error estimate for a finite volume scheme for a diffusion convection problem on a triangular mesh*, Num. Meth. in P.D.E., 165–173.

[8] Lazarov R. D., Mishev I. D., Vassilevski P. S., 1996, *Finite Volume Method for Converction-Diffusion Problems*, SIAM J. Numer. Anal., 33, 31–55.

[9] Rabier P., Thomas J.-M., 1985, *Exercices d'analyse numérique des équations aux dérivées partielles*, Masson, Paris.

[10] Raviart P.-A., Thomas J.-M., 1983, *Introduction à l'analyse des équations aux dérivées partielles*, Masson, Paris.

[11] Thomas J.-M., Trujillo D., 1995, *Analysis of Finite Volumes Methods*, Proceedings of Mathematical Modeling through Porous Media, St Etienne, Ed. World Scientific, A. Bourgeat, C. Carasso, S. Luckaus, A. Michelic, 318–336.

[12] Vassilevski P. S., Petrova S. I., Lazarov R. D., 1992, *Finite Difference Schemes on Triangular Cell-Centered Grids With Local Refinement*, SIAM J. Sci. Stat. Comput., 13, No 6, 1287–1313.

[13] Verdière S., Guérillot D., 1995, *Different pressure and Saturation Grids in Heterogeneous Porous Media*, Proceedings of Mathematical Modeling through Porous Media, St Etienne, Ed. World Scientific, A. Bourgeat, C. Carasso, S. Luckaus, A. Michelic, 506–511.

[14] Verdière S., Guérillot D., Thomas J.-M., 1996, Dual Mesh Method in Heterogeneous Porous Media, Proceedings of the 5th European Conference on the Mathematics of Oil Recovery, Leoben (Austria).

[15] Vignal M. H., 1996, Convergence of a finite Volume scheme for an elliptic hyperbolic system, M^2AN, 30, No 7, 841–872.