Convergence of a Finite Volume scheme for an elliptic-hyperbolic system

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Abstract

We study here the convergence of a finite volume scheme for a coupled system of an hyperbolic and an elliptic equations defined on an open bounded set of $I\!R^2$.
On the elliptic equation, a four points finite volume scheme is used then an error estimate on a discrete H^1 norm of order h is proved, where h defines the size of the triangulation.
On the hyperbolic equation, one uses an upstream scheme with respect to the flow, then using an estimate on the variation of the approximate solution, the convergence of the approximate solution toward a solution of the coupled system is shown, under a stability condition.

Résumé

On étudie ici la convergence d'un schéma de type volumes finis pour un système formé d'une équation elliptique et d'une équation hyperbolique linéaires, définies sur un ouvert borné de $I\!R^2$.
Pour l'équation elliptique un schéma volumes finis à quatre points est utilisé, une inégalité discrète de Poincaré pour les fonctions à moyenne nulle, est établie afin de montrer une estimation d'erreurs en norme H^1 discrète, de l'ordre de la taille des mailles.
Sur l'équation hyperbolique, on utilise un schéma décentré vers l'amont de l'écoulement, à l'aide d'une estimation faible sur la variation de la solution approchée, on montre, sous une condition de stabilité, la convergence de cette solution vers la solution faible du problème.

1 Introduction

One considers a problem coming from the modelization of a diphasic flow in a porous medium. In a simplified case it leads to the determination of the velocity u of one of the two phases and of the pressure P.

Let Ω be an open bounded polygonal connected set of $I\!R^2$, one notes $\Gamma = \partial\Omega$.
Let $g \in L^\infty(\Gamma)$, $u_0 \in L^\infty(\Omega)$ and $\overline{u} \in L^\infty(\Gamma^+ \times I\!R_+)$, be given,

with $\Gamma^+ = \{\gamma \in \Gamma \ ; \ g(\gamma) \geq 0\}$, one assumes $\displaystyle\int_\Gamma g(\gamma)\ d\gamma = 0$;

then one considers the problem defined by :

$$\Delta P(x) = 0, \quad x \in \Omega \tag{1}$$

$$u_t(x,t) - div(u(x,t)\ \nabla P(x)) = 0, \quad x \in \Omega, t \in I\!R_+ \tag{2}$$

with following boundary conditions and initial condition :

$$\nabla P(\gamma).n(\gamma) = g(\gamma), \quad \gamma \in \Gamma \tag{3}$$

$$u(\gamma,t) = \overline{u}(\gamma,t), \quad \gamma \in \Gamma^+, t \in I\!R_+ \tag{4}$$

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{5}$$

where n is the outward unit normal to Γ.

More precisely, one searches u in $L^\infty(\Omega \times I\!R_+)$ and P in $H^1(\Omega)$ solutions of (1)-(5) in the following weak sense :

$$\int_\Omega \nabla P(x).\nabla\Phi(x)\,dx - \int_\Gamma g(\gamma)\,\Phi(\gamma)\,d\gamma = 0 \quad \text{for all } \Phi \in H^1(\Omega)$$

and

$$\int_\Omega \int_{I\!R_+} u(x,t)\frac{\partial \varphi}{\partial t}(x,t)\,dx\,dt - \int_\Omega \int_{I\!R_+} u(x,t)\,\nabla P(x).\nabla\varphi(x,t)\,dx\,dt + \int_\Omega u_0(x)\,\varphi(x,0)\,dx + \int_\Gamma \int_{I\!R_+} \overline{u}(\gamma,t)\,\varphi(\gamma,t)\,g^+(\gamma)\,d\gamma\,dt = 0$$

for all $\varphi \in C_c^\infty(\Omega^+ \times I\!R_+)$ with $\Omega^+ = \Omega \cup \Gamma^+$

Note that the test functions φ are equal to zero on $\Gamma^- = \{\gamma \in \Gamma\ ;\ g(\gamma) < 0\}$ but not necessarily on Γ^+.

To discretize these equations, a finite volume scheme is used, then the results presented by R. Eymard and T. Gallouët in [5] and R. Herbin in [8] are generalized. Indeed the system considered in [5] is the same as the one presented here but the authors use a coupled finite element-finite volume scheme, then discrete unknowns are localized at the vertices of the meshes whereas in this note they are localized at the cells centers. For this scheme they prove a convergence property toward a weak solution of system (1)-(5). The scheme used on the pressure is a finite element scheme, hence the convergence of the approximate solution of the elliptic equation follows from the finite element framework. In this note one uses a four points finite volume scheme for the pressure, then a convergence proof is given by generalizing the results of R. Herbin in [8]. One of the two essential differences comes from boundary condition, since in [8] the boundary condition is a Dirichlet condition whereas here a Neumann condition is considered, then estimates are changed by boundary terms. In particular, for our estimate it is necessary to count the number of triangles which are "after" a given triangle in all directions, while in [8] only one direction is sufficient. In a same way, to prove the error estimate in discrete H^1 norm, the discrete L^2 norm of the error is majorized by its discrete H_0^1 norm, assuming the solution's mean value equal to zero since problem's solutions differ from a constant, whereas in [8] the discrete L^2 norm of the error is majorized by the sum of its discrete H_0^1 norm and of its discrete L^2 norm on the boundary.
The second difference comes from the assumptions on the meshes. In [8] all the meshes must have a measure of same order, whereas here deformations of the meshes are authorized (see section 2.1).
On the velocity an upstream finite volume scheme with respect to the flow is used, then, under a stability condition, the convergence of the approximate solution toward a solution of the hyperbolic equation is shown. To prove this result, one needs an estimate on the variation of the approximate solution, it uses an estimate on discrete H^1 norm of the approximate solution of the elliptic equation, this result in [5] is given by the finite element framework. Other results on the existence and the uniqueness of solutions of hyperbolic equations are given in [7], [3], [1], [10].
Numerical experiments on the comparison between finite element scheme and finite volume scheme, done by J.M. Fiard and R. Herbin in [6] for a conduction problem and by R. Herbin and O. Labergerie in [9] for a diffusion-convection problem, have shown that the approximation of fluxes is better for the finite volume scheme. Furthermore, comparison between the scheme presented here and the weighted finite volume scheme of R. Eymard and T. Gallouët have also been done in [6] on a system more general than (1)-(5), where (1) is changed by $div\Big(f(u(x,t))\,\nabla P(x)\Big) = 0,\ x \in \Omega$. This numerical test shows that the scheme presented here gives better results than those given by the weighted finite volume scheme. Other authors have been interested by finite volume scheme on triangular meshes, see for instance [11]. In [11], results are restricted to particular meshes, whereas, here, as it has been already remark, the assumptions are most general.

2 Discretization

Before discretizing the elliptic and the hyperbolic equations, one first gives assumptions on the triangulation.

2.1 Assumptions on the triangulation

Let $\mathcal{T} = (K_j)_{1\leq j\leq L}$ be a triangulation of Ω which satisfies :

there exists η such that for any θ angle of an element of $\mathcal{T}$, one has :

$$\eta < \theta < \frac{\pi}{2} - \eta \tag{6}$$

One defines $h_j = \sqrt{S(K_j)}$, where $S(K_j)$ denotes the 2D Lebesgue measure of K_j, then the assumption (6) gives the following result :

there exists $\alpha_1 > 0$ and $\alpha_2 > 0$, depending only on η, such that for all side a of the triangulation, the length $l(a)$ of a verifies :

$$\alpha_1.h_j \leq l(a) \leq \alpha_2.h_j \tag{7}$$

if a is an edge of the cell K_j.

Then one defines $h \in I\!R_+^*$ by $h = \max\limits_{j=1}^{L} h_j$.

Some notations will be useful to describe the numerical scheme :

NOTATIONS

$\mathcal{T}_{ext} = \Big\{ j \in \{1,\ldots,L\}\ ;\ K_j \in \mathcal{T},\ K_j \cap \Gamma \neq \emptyset \Big\}$

$\mathcal{A}_{ext}$ the set of the edges of the triangulation which are on the boundary Γ of Ω

$\mathcal{A}_{int}$ the set of the edges of the triangulation which are in Ω

$g^+(\gamma) = max(g(\gamma), 0)$ and $g^-(\gamma) = (-g)^+$

For all $K_j \in \mathcal{T},\ 1 \leq j \leq L$ one notes :

$c_i(j)$ the edges of K_j, $i = 1, 2$ or 3

x_j the intersection of the orthogonal bisectors of the edges of K_j

$g_{ij}^+ = \displaystyle\int_{c_i(j)} g^+(\gamma)\ d\gamma$ and $g_{ij}^- = \displaystyle\int_{c_i(j)} g^-(\gamma)\ d\gamma$, $i = 1, 2$ or 3, if $c_i(j) \in \mathcal{A}_{ext}$

$g_{ij} = g_{ij}^+ - g_{ij}^-$, $i = 1, 2$ or 3, if $c_i(j) \in \mathcal{A}_{ext}$

$\tau_{ext}(j)$ the set of the suffix $i = 1, 2$ or 3 such that $c_i(j) \in \mathcal{A}_{ext}$

τ_j the set of the suffix of the neighbours of K_j

$c_{jk} = \partial K_j \cap \partial K_k$ for all $k \in \tau_j$

$d_{jk} = d(x_j, c_{jk}) + d(x_k, c_{jk})$, for all $k \in \tau_j$, where d is the euclidian distance of $I\!R^2$

x_{jk} the center of the side c_{jk}

2.2 Discretization of the elliptic equation

To discretize (1), a four points finite volume scheme is used ; the principle of the finite volume schemes, (see [4]), is to integrate equations on each control volume (here $K_j \in \mathcal{T}$), so one has :

$$\int_{\partial K_j} \nabla P(\gamma).n_{K_j}(\gamma)\ d\gamma = 0$$

where n_{K_j} denotes the outward unit normal to ∂K_j.

One approximates P by $P_\mathcal{T}$ with $P_\mathcal{T}(x) = P_j$ if $x \in K_j$, so the discretized equation can be given by approximating the flux of P through one edge $c_i(j)$ of K_j by :

$$\begin{cases} l(c_{jk})\ \dfrac{(P_k - P_j)}{d_{jk}} & \text{if } \exists\ k \in \{1,\dots,L\} \text{ such that } c_i(j) = c_{jk} \\ \displaystyle\int_{c_i(j)} g(\gamma)\ d\gamma & \text{if } c_i(j) \in \mathcal{A}_{ext} \end{cases}$$

Then one has :

(8) $$\sum_{k \in \tau_j} l(c_{jk})\ \frac{(P_k - P_j)}{d_{jk}} + \sum_{i \in \tau_{ext}(j)} g_{ij} = 0 \quad \text{for all } j \in \{1,\dots,L\}$$

with the convention $\sum_{\emptyset} = 0.$

One can remark that on the domain's boundary the approximation of the flux is exact.

2.3 Discretization of the hyperbolic equation

Before discretizing (2), one defines the time step δ, so let $\mathcal{T}$ be a triangulation of Ω which satisfies the assumption (6) and $\alpha \in]0,1[$, then one chooses $\delta \in I\!R_+^*$ which satisfies the following conditions :

(9) $$\begin{cases} 1 - \delta\ \dfrac{5}{S(K_k)}\ l(c_{jk})\ \dfrac{(P_j - P_k)}{d_{jk}} > \alpha & \forall\ (j,k) \in \mathcal{S} \\ \\ 1 - \delta\ \dfrac{5}{S(K_j)}\ g_{ij}^+ > \alpha & \forall\ j \in \mathcal{T}_{ext} \end{cases}$$

where $\mathcal{S} = \{\ (j,k) \in \{1,\dots,L\}^2\ ;\ (K_j,K_k) \in \mathcal{T} \times \mathcal{T},\ k \in \tau_j \text{ and } P_j > P_k\}$
One notes $t^n = n\ \delta$ for all $n \in I\!N$.

To discretize (2), first an Euler scheme explicit in time is used, and as for the elliptic equation, one integrates (2) on each control volume :

$$\int_{K_j} \frac{u(x,t^{n+1}) - u(x,t^n)}{\delta}\ dx - \int_{\partial K_j} u(\gamma,t^n)\ \nabla P(\gamma).n_{K_j}\ d\gamma = 0$$

One approximates u by $u_{\mathcal{T},\delta}$ with $u_{\mathcal{T},\delta}(x,t) = u_j^n$ if $x \in K_j$ and $t \in [t^n, t^{n+1}[$. Then an upstream discrete value with respect to the flow is chosen for u at the interfaces of meshes, and at the boundary.

One defines u_j^0 for all $j \in \{1,\dots,L\}$ by $u_j^0 = \dfrac{1}{S(K_j)} \displaystyle\int_{K_j} u_0(x)\ dx$

and $\overline{u}_{ji}^n$ for all $j \in \mathcal{T}_{ext}$ and for $i \in \tau_{ext}(j)$ by $\overline{u}_{ji}^n = \dfrac{1}{\delta\ l(c_i(j))} \displaystyle\int_{t^n}^{t^{n+1}} \int_{c_i(j)} \overline{u}(\gamma,t)\, d\gamma\, dt$

So, the discretized equation is given by :

$$S(K_j)\ (u_j^{n+1} - u_j^n) - \delta \sum_{k\in\tau_j} u_{jk}^n\ l(c_{jk})\ \frac{(P_k - P_j)}{d_{jk}} - \delta \sum_{i\in\tau_{ext}(j)} (\overline{u}_{ji}^n\ g_{ij}^+ - u_j^n\ g_{ij}^-) = 0$$

(10) for all $j \in \{1,\ldots,L\}$ and all $n \in I\!N$

where $u_{jk}^n = \begin{cases} u_j^n & \text{if } P_j > P_k \\ u_k^n & \text{else} \end{cases}$

3 Convergence of the four points finite volume scheme for the elliptic equation

Let $\mathcal{T}$ be a triangulation of Ω which satisfies the property (6).
One proves in this section the existence of solutions $(P_j)_{1\le j\le L}$ of (8) and that these solutions differ only from a constant, proving the following result :

Proposition 1 *Let $g \in L^\infty(\Gamma)$, one defines for all $j \in \{1,\ldots,L\}$: $g_{ij} = \int_{c_i(j)} g(\gamma)\, d\gamma$ if $c_i(j) \in \mathcal{A}_{ext}$. Then :*

1. *if $\sum_{i\in\tau_{ext}(j)} g_{ij} = 0 \quad \forall\ j \in \{1,\ldots,L\}$ and $(P_j)_{1\le j\le L}$ satisfy (8) then*

$$P_j = P_k \quad \forall\ j,k \in \{1,\ldots,L\}$$

2. *if $(P_j)_{1\le j\le L}$ satisfy (8) then $\sum_{j=1}^{L} \sum_{i\in\tau_{ext}(j)} g_{ij} = 0$*

3. *if $\int_\Gamma g(\gamma)\, d\gamma = 0$ then there exists $(P_j)_{1\le j\le L}$ solutions of (8) and solutions differ only from a constant*

Furthermore one proves the numerical scheme's convergence proving an error estimate on discrete H^1 norm of order h :

Theorem 1 *Let $g \in L^\infty(\Gamma)$, one denotes by P the weak solution of (1), (3) such that $\sum_{j=1}^{L} S(K_j)\, P(x_j) = 0$, where x_j is the intersection of the orthogonal bisectors of the edges of K_j, one supposes g such that P is in $C^2(\overline{\Omega})$.*
Let $(P_j)_{1\le j\le L}$ satisfy (8) and $\sum_{j=1}^{L} S(K_j)\, P_j = 0$, one defines the error by $e_j = P_j - P(x_j)$ for all $j \in \{1,\ldots,L\}$.
Then there exists C_1 and C_2 positive, independent of $\mathcal{T}$ such that :

$$\left(\sum_{j=1}^{L} \sum_{k\in\tau_j} \frac{(e_k - e_j)^2}{d_{jk}}\, l(c_{jk}) \right)^{1/2} \le C_1.h \qquad \textit{and} \qquad \left(\sum_{j=1}^{L} S(K_j)\, |e_j|^2 \right)^{1/2} \le C_2.h$$

3.1 Proof of Proposition 1

Proof of the first part of the Proposition 1

One supposes that $\forall\, j \in \{1,\dots,L\} \displaystyle\sum_{i\in\tau_{ext}(j)} g_{ij} = 0$ and $(P_j)_{1\leq j\leq L}$ satisfy (8).

Let $j_0 \in \{1,\dots,L\}$ such that $P_{j_0} = \min\Big\{P_k \ ;\ k \in \{1,\dots,L\}\Big\}$

Thanks to (8), one has :

$$\sum_{k\in\tau_{j_0}} l(c_{j_0k})\,\frac{P_k - P_{j_0}}{d_{j_0k}} + \sum_{i\in\tau_{ext}(j_0)} g_{ij_0} = 0$$

But according to assumptions $\displaystyle\sum_{i\in\tau_{ext}(j_0)} g_{ij_0} = 0$; and $l(c_{j_0k}) > 0$, $d_{j_0k} > 0$ and $P_k - P_{j_0} \geq 0$ $\forall\, k \in \tau_{j_0}$, so one has :

$$P_k = P_{j_0} \quad \forall\, k \in \tau_{j_0}$$

Using the fact that Ω is connected, one obtains by induction : $P_j = P_k \quad \forall\, j,k \in \{1,\dots,L\}$

Proof of the second part of the Proposition 1

Supposing that $(P_j)_{1\leq j\leq L}$ satisfy (8) and summing these equations one gets :

$$\sum_{j=1}^{L}\sum_{i\in\tau_{ext}(j)} g_{ij} = 0$$

Proof of the third part of the Proposition 1

One supposes that $\displaystyle\int_\Gamma g(\gamma)\,d\gamma = 0.$

Then thanks to the first part of this Proposition, (8) is a linear system which has a kernel of dimension 1.

So, thanks to the second part of this Proposition, the image space of this linear system is the set of the $B \in I\!R^L$, $B = {}^t(b_1, b_2, \dots, b_L)$ such that $\displaystyle\sum_{j=1}^{L} b_j = 0.$

Then, as $\displaystyle\sum_{j=1}^{L} b_j = \int_\Gamma g(\gamma)\,d\gamma = 0$, there exists $(P_j)_{1\leq j\leq L}$ solutions of (8) and these solutions differ only from a constant.

3.2 Proof of Theorem 1

Definition of the consistency error on the fluxes

As it has been remark in section 2.2, fluxes are exact on the domain's boundary, then one defines the consistency error only at the interfaces of meshes.

The exact flux on the side c_{jk} in the direction of K_j to K_k is :

$$\overline{F}_{c_{jk}}(K_j) = \frac{1}{l(c_{jk})}\int_{c_{jk}} \nabla P(\gamma).n_{K_j}(\gamma)\ d\gamma$$

and the approximate flux on the side c_{jk} in the same direction is :

$$F_{c_{jk}}(K_j) = \frac{P_k - P_j}{d_{jk}}$$

One defines the consistency error, denoted by $R_{c_{jk}}(K_j)$, by :

$$R_{c_{jk}}(K_j) = \overline{F}_{c_{jk}}(K_j) - \frac{P(x_k) - P(x_j)}{d_{jk}}$$

One can remark that the conservativity of exact and approximate fluxes implies :

$$R_{c_{jk}}(K_j) = -R_{c_{jk}}(K_k)$$

Now the following result is proved :

Lemma 1 *Under the assumptions of Theorem 1, there exists a constant $C_R \geq 0$ independent of $\mathcal{T}$ such that :*

$$|R_{c_{jk}}(K_j)| \leq C_R.h \quad \forall j, k \in \{1, \dots, L\}$$

Proof of Lemma 1
Using a first order Taylor expansion one proves that there exists $C_1 \geq 0$ and $C_2 \geq 0$ depending only on α_1, α_2 and on the second order derivative of P such that :

$$\begin{aligned}
|R_{c_{jk}}(K_j)| &\leq \left| \frac{1}{l(c_{jk})} \int_{c_{jk}} \nabla P(\gamma).n_{K_j} \, d\gamma - \nabla P(x_{jk}).n_{K_j} \right| + \left| \nabla P(x_{jk}).n_{K_j} - \frac{P(x_k) - P(x_j)}{d_{jk}} \right| \\
&\leq C_1.h + C_2.h
\end{aligned}$$

So the proof of Lemma 1 is completed.

Let $e_k = P_k - P(x_k)$ be the error on the cell K_k for all $k \in \{1, \dots, L\}$, then one proves the first result of Theorem 1, i.e. the following property :

There exists $C \geq 0$ independent of $\mathcal{T}$ such that :

$$\left(\sum_{j=1}^{L} \sum_{k \in \tau_j} \frac{(e_k - e_j)^2}{d_{jk}} \, l(c_{jk}) \right)^{1/2} \leq C.h \tag{11}$$

Proof of the inequality (11)
Thanks to (1) one has :

$$\sum_{k \in \tau_j} l(c_{jk}) \, \overline{F}_{c_{jk}}(K_j) + \sum_{i \in \tau_{ext}(j)} g_{ij} = 0 \tag{12}$$

One subtracts (8) from (12), one multiplies by e_j and one sums over j, then using the conservativity of the exact and approximate fluxes, the properties (6) and (7), the Lemma 1 and the Young inequality (for more details see [8]), one gets :

$$\sum_{j=1}^{L} \sum_{k \in \tau_j} \frac{(e_k - e_j)^2}{d_{jk}} \, l(c_{jk}) \leq \frac{6 \, (\alpha_2)^2}{\alpha_1} \; C_R^2 \; S(\Omega) \, h^2$$

where $S(\Omega)$ is the 2D Lebesgue measure of the domain Ω.
Then the proof of the inequality (11) is completed. Let's complete the proof of Theorem 1.

Proof of Theorem 1
The L^2 discrete error estimate is shown by using a discrete Poincaré-Wirtinger inequality, i.e. the following result :

There exists $C > 0$, independent of $\mathcal{T}$, such that :

$$\sum_{j=1}^{L} S(K_j)\ |e_j - m|^2 \leq C \sum_{a \in \mathcal{A}_{int}} \frac{|e_{K_a^+} - e_{K_a^-}|^2}{d_a}\ l(a) \tag{13}$$

with $m = \dfrac{1}{S(\Omega)} \displaystyle\sum_{k=1}^{L} S(K_k)\, e_k$, where $e_{K_a^+}$ and $e_{K_a^-}$ are the errors on the both elements of $\mathcal{T}$ for which a is an edge and d_a is defined as follows :
for all $a \in \mathcal{A}_{int}$ there exists j and k in $\{1, \ldots, L\}$ such that $a = c_{jk}$ then $d_a = d_{jk}$.

This result will be proved in four steps :

Step 1 :
Let $\mathcal{P}$ be a square included in $\overline{\Omega}$ and the both directions $\mathcal{D}_1$ and $\mathcal{D}_2$ defined by $\mathcal{P}$, see figure 1.

(24,18)(190,18) (182.000,16.000)(190.000,18.000)(182.000,20.000) 4.000(71,49)(71,18) 4.000(

Figure 1:

One notes $d = |b - c|$ and one chooses for coordinate system the coordinate system defined by any point of $I\!R^2$ and the both directions $\mathcal{D}_1$ and $\mathcal{D}_2$.
Let $\mathcal{A}_{int,\mathcal{P}}$ be the set of the sides a of $\mathcal{A}_{int}$ such that $a \cap \mathcal{P} \neq \emptyset$.

Let K_j and K_k in $\mathcal{T}$ such that $K_j \cap \mathcal{P} \neq \emptyset$ and $K_k \cap \mathcal{P} \neq \emptyset$, $x = (x_1, x_2) \in K_j \cap \mathcal{P}$ and $y = (y_1, y_2) \in K_k \cap \mathcal{P}$, one denotes by [x,y] the line segment delimited by x and y, and one defines :

$\mathcal{A}_{x_1,x_2,y_2}$ (respectively $\mathcal{A}_{x_1,y_1,y_2}$) the set of the sides of $\mathcal{A}_{int}$ such that the intersection with the line segment $[(x_1,x_2),(x_1,y_2)]$ (respectively $[(x_1,y_2),(y_1,y_2)]$) is a point.

$\mathcal{A}_{x_1}^{(2)}$ (respectively $\mathcal{A}_{y_2}^{(1)}$) the set of the sides of $\mathcal{A}_{int,\mathcal{P}}$ such that the intersection with the line defined by $(x_1, 0)$ (respectively $(0, y_2)$) and parallel to $\mathcal{D}_2$ (respectively $\mathcal{D}_1$) is a point.

Then one has :

$$e_j - e_k \leq \sum_{a \in \mathcal{A}_{x_1,x_2,y_2}} \left| e_{K_a^+} - e_{K_a^-} \right| + \sum_{a \in \mathcal{A}_{x_1,y_1,y_2}} \left| e_{K_a^+} - e_{K_a^-} \right| \qquad a.e.\ y \in \mathcal{P} \cap \Omega, \quad \forall\, x \in \mathcal{P} \cap \Omega$$

Integrating over $K_k \cap \mathcal{P}$ and summing over k, one obtains for all $x \in \mathcal{P} \cap \Omega$:

$$d^2 \, |e_j - m_{\mathcal{P}}| \leq \int_b^c \int_b^c \sum_{a \in \mathcal{A}^{(2)}_{x_1}} \left| e_{K_a^+} - e_{K_a^-} \right| \, dy_1 \, dy_2 + \int_b^c \int_b^c \sum_{a \in \mathcal{A}^{(1)}_{y_2}} \left| e_{K_a^+} - e_{K_a^-} \right| \, dy_1 \, dy_2$$

with $m_{\mathcal{P}} = \dfrac{1}{S(\mathcal{P})} \displaystyle\sum_{k=1}^{L} S(K_k \cap \mathcal{P}) \, e_k$.

For all $a \in \mathcal{A}_{int}$, let θ_a be the angle between a and $\mathcal{D}_1$, then swapping the summation and the integral in the second terms, one has :

$$d \, |e_j - m_{\mathcal{P}}| \leq d \sum_{a \in \mathcal{A}^{(2)}_{x_1}} \left| e_{K_a^+} - e_{K_a^-} \right| + \sum_{a \in \mathcal{A}_{int,\mathcal{P}}} \left| e_{K_a^+} - e_{K_a^-} \right| \, l(a) \, \cos \theta_a$$

Using the Cauchy Schwarz inequality, one gets :

$$\begin{aligned} d^2 \, |e_j - m_{\mathcal{P}}|^2 \leq d^2 & \left(\sum_{a \in \mathcal{A}^{(2)}_{x_1}} \frac{\left| e_{K_a^+} - e_{K_a^-} \right|^2}{\sin \theta_a \, d_a} \right) \times \left(\sum_{a \in \mathcal{A}^{(2)}_{x_1}} \sin \theta_a \, d_a \right) \\ & + \left(\sum_{a \in \mathcal{A}_{int,\mathcal{P}}} \left| e_{K_a^+} - e_{K_a^-} \right|^2 \right) \times \left(\sum_{a \in \mathcal{A}_{int,\mathcal{P}}} l(a)^2 \right) \end{aligned}$$

and then :

$$|e_j - m_{\mathcal{P}}|^2 \leq \Big(d + 2 \, \alpha_2 \, h \Big) \sum_{a \in \mathcal{A}^{(2)}_{x_1}} \frac{\left| e_{K_a^+} - e_{K_a^-} \right|^2}{\sin \theta_a \, d_a} + 3 \, \alpha_2 \, \left(1 + \frac{4 \, \alpha_2 \, h}{d} \right) \sum_{a \in \mathcal{A}_{int,\mathcal{P}}} \left| e_{K_a^+} - e_{K_a^-} \right|^2$$

Integrating over $[b, c]$ with respect to x_1 and swapping the summation and the integral, one has for all $x_2 \in [b, c]$:

$$\sum_{K_j \in \mathcal{R}^{(1)}_{x_2}} l \left(K_j \cap \mathcal{D}^{(1)}_{x_2} \right) \, |e_j - m_{\mathcal{P}}|^2 \leq C_1 \; d \sum_{a \in \mathcal{A}_{int,\mathcal{P}}} \frac{\left| e_{K_a^+} - e_{K_a^-} \right|^2}{d_a} \; l(a) \tag{14}$$

where $\mathcal{D}^{(1)}_{x_2}$ is the line defined by the point $(0, x_2)$ and the direction $\mathcal{D}_1$, $\mathcal{R}^{(1)}_{x_2}$ is the set of the triangles which are cut by $\mathcal{D}^{(1)}_{x_2}$ and $C_1 = d + 2 \, \alpha_2 \, h + \dfrac{6 \, (\alpha_2)^2}{\alpha_1} \Big(d + 4 \, \alpha_2 \, h \Big)$.

With the same arguments, one can prove the following result for all $x_1 \in [b, c]$:

$$\sum_{K_j \in \mathcal{R}^{(2)}_{x_1}} l \left(K_j \cap \mathcal{D}^{(2)}_{x_1} \right) \, |e_j - m_{\mathcal{P}}|^2 \leq C_1 \; d \sum_{a \in \mathcal{A}_{int,\mathcal{P}}} \frac{\left| e_{K_a^+} - e_{K_a^-} \right|^2}{d_a} \; l(a) \tag{15}$$

One concludes by integrating (14) with respect to x_2 so :

$$\sum_{j=1}^{L} S(K_j \cap \mathcal{P}) \, |e_j - m_{\mathcal{P}}|^2 \leq C_1 \; d^2 \sum_{a \in \mathcal{A}_{int,\mathcal{P}}} \frac{\left| e_{K_a^+} - e_{K_a^-} \right|^2}{d_a} \; l(a) \tag{16}$$

Step 2 :

In this step the inequalities (15), (14) and (16) are proved over half of the square $\mathcal{P}$ denoted T, i.e the triangle (A, B, D) (see figure 1).
Using the orthogonal symmetry in relation to $[BD]$, the problem is the same as the one of the step 1, then with the same notations, one has the following results :

$$\sum_{K_j \in \mathcal{R}_b^{(1)}} l\left(K_j \cap \mathcal{D}_b^{(1)}\right) \left|e_j - m_T\right|^2 \leq 2\, C_1\, d \sum_{a \in \mathcal{A}_{int,T}} \frac{\left|e_{K_a^+} - e_{K_a^-}\right|^2}{d_a}\, l(a)$$

$$\sum_{K_j \in \mathcal{R}_b^{(2)}} l\left(K_j \cap \mathcal{D}_b^{(2)}\right) \left|e_j - m_T\right|^2 \leq 2\, C_1\, d \sum_{a \in \mathcal{A}_{int,T}} \frac{\left|e_{K_a^+} - e_{K_a^-}\right|^2}{d_a}\, l(a)$$

and :

$$\sum_{j=1}^{L} S(K_j \cap T) \left|e_j - m_T\right|^2 \leq C_1\, d^2 \sum_{a \in \mathcal{A}_{int,T}} \frac{\left|e_{K_a^+} - e_{K_a^-}\right|^2}{d_a}\, l(a)$$

Step 3 :

One proves the result over an ordinary triangle of Ω.
So let an ordinary triangle T of Ω and a right-angle triangle $\overline{T}$, see figure 2.

(44.000,117.000)(42.000,125.000)(40.000,117.000) (42,125)(42,18)(187,18) (179.000,16.000)(187.000,18.000)(179.000,20.00

Figure 2:

Then there exists a linear application F which transforms T in $\overline{T}$ defined as follow :

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \dfrac{x}{l_1} - \dfrac{y}{l_1\, \tan\theta} \\ \\ \dfrac{y}{l_2\, \sin\theta} \end{pmatrix}$$

As $T = \bigcup_{j=1}^{L} \left(K_j \cap T\right)$ then $\overline{T} = \bigcup_{j=1}^{L} \left(F\left(K_j \cap T\right)\right)$
Let the function $\overline{e}$ defined from $\overline{T}$ to $I\!R$ by $\overline{e}(x, y) = e_j$ if $(\overline{x}, \overline{y}) \in F\left(K_j \cap T\right)$.

One has the following result :
there exists η such that for all triangulation $\mathcal{T} = (K_j)_{1 \leq j \leq L}$ of Ω one has, for all angle of an element of $\mathcal{T}$:

$$\eta < \theta < \frac{\pi}{2} - \eta$$

Then there exists $\overline{\eta}(\eta, F)$ such that for all angle of an element of $\Big(F\left(K_j\right)\Big)_{1\leq j\leq L}$:

$$\theta > \overline{\eta}(\eta, F)$$

but one can have : $\theta \geq \dfrac{\pi}{2}$.
So one defines $(T_k)_{1\leq k\leq L'}$ a triangulation of $F(\Omega)$ such that :

1. $\forall\ k \in \{1,\dots,L'\}$, there exists $j \in \{1,\dots,L\}$ such that :

$$T_k \subset F\left(K_j\right)$$

2. for all θ angle of an element of $(T_k)_{1\leq k\leq L'}$, one has :

$$\overline{\eta}(\eta, F) < \theta < \frac{\pi}{2} - \overline{\eta}(\eta, F)$$

According to the step 2, one has:

$$\sum_{j=1}^{L} S\Big(F\Big(K_j \cap T\Big)\Big)\ \left|e_j - m_{\overline{T}}\right|^2 = \sum_{k=1}^{L'} S\left(T_k \cap \overline{T}\right)\ \left|e_j - m_{\overline{T}}\right|^2 \leq C \sum_{a\in\mathcal{A}_{int,\overline{T}}} \frac{\left|e_{K_a^+} - e_{K_a^-}\right|^2}{d_a}\ l(a)$$

where $m_{\overline{T}}$ is the mean value of $\overline{e}$ over $\overline{T}$.

Remarking that :

$$\sum_{a\in\mathcal{A}_{int,\overline{T}}} \frac{\left|e_{K_a^+} - e_{K_a^-}\right|^2}{d_a}\ l(a) \leq C_{\alpha_1,\alpha_2,\eta,F} \sum_{a\in\mathcal{A}_{int,\overline{T}}} \left|e_{K_a^+} - e_{K_a^-}\right|^2 = C_{\alpha_1,\alpha_2,\eta,F} \sum_{a\in\mathcal{A}_{int,T}} \left|e_{K_a^+} - e_{K_a^-}\right|^2$$

and that
$m_{\overline{T}} = m_T = \dfrac{1}{S(T)} \displaystyle\sum_{j=1}^{L} S(K_j \cap T)\, e_j$, one gets :

$$\sum_{j=1}^{L} S\Big(F\Big(K_j \cap T\Big)\Big)\ |e_j - m_T|^2 \leq C_{\alpha_1,\alpha_2,\eta,F} \sum_{a\in\mathcal{A}_{int,T}} \left|e_{K_a^+} - e_{K_a^-}\right|^2$$

Thus :

$$S\Big(F\Big(K_j \cap T\Big)\Big) = \int_{F\left(K_j\cap T\right)} d\overline{x}\, d\overline{y} = \frac{1}{l_1\, l_2\, \sin\theta} \int_{K_j\cap T} dx\, dy = \frac{1}{l_1\, l_2\, \sin\theta}\ S(K_j \cap T)$$

So, one obtains :

$$\sum_{j=1}^{L} S(K_j \cap T)\ |e_j - m_T|^2 \leq l_1\, l_2\, \sin\theta\ C_{\alpha_1,\alpha_2,\eta,F} \sum_{a\in\mathcal{A}_{int,T}} \left|e_{K_a^+} - e_{K_a^-}\right|^2$$

With the same arguments, one proves the following inequalities :

$$\sum_{K_j\in\mathcal{R}_I} l\left(K_j \cap I\right)\ |e_j - m_T|^2 \leq l_1\, C_{\alpha_1,\alpha_2,\eta,F} \sum_{a\in\mathcal{A}_{int,T}} \left|e_{K_a^+} - e_{K_a^-}\right|^2 \tag{17}$$

$$\text{(18)} \qquad \sum_{K_j \in \mathcal{R}_J} l\,(K_j \cap J)\ |e_j - m_T|^2 \leq l_2 \sin\theta\, C_{\alpha_1,\alpha_2,\eta,F} \sum_{a \in \mathcal{A}_{int,T}} \left| e_{K_a^+} - e_{K_a^-} \right|^2$$

where $\mathcal{R}_I$ (respectively $\mathcal{R}_J$) is the set of the triangles K of $\mathcal{T}$ such that $K \cap I$ (respectively $K \cap J$) is not emptyset.

Step 4 :

Let a subset T of Ω which is the union of two triangles T_1 and T_2, one denotes $\partial T_1 \cap \partial T_2$ by I.
As $m = \dfrac{S(T_1)\, m_1 + S(T_2)\, m_2}{S(\Omega)}$, one can write :

$$\sum_{j=1}^{L} S(K_j \cap T_1)\ |e_j - m_T|^2 \leq 2 \left(\sum_{j=1}^{L} S(K_j \cap T_1)\ |e_j - m_1|^2 + S(T_1)\ \frac{S(T_2)}{S(\Omega)}\, |m_2 - m_1|^2 \right)$$

According to the step 3 one has :

$$\sum_{j=1}^{L} S(K_j \cap T_1)\ |e_j - m_T|^2 \leq 2 \left(C\ S(T_1) \sum_{a \in \mathcal{A}_{int,T_1}} \left| e_{K_a^+} - e_{K_a^-} \right|^2 + S(T_1)\ \frac{S(T_2)}{S(\Omega)}\, |m_2 - m_1|^2 \right)$$

Then it just remains to estimate the difference between m_2 and m_1.

Let $x = (x_1, x_2) \in I$, so :

$$|m_2 - m_1|^2 \leq 2 \left(|e(x_1, x_2) - m_2|^2 + |e(x_1, x_2) - m_1|^2 \right)$$

Integrating over I and thanks to the inequalities (17) and (18), one has :

$$l(I)\ |m_2 - m_1|^2 \leq 2\, C \left(\sum_{a \in \mathcal{A}_{int,T_1}} \left| e_{K_a^+} - e_{K_a^-} \right|^2 + \sum_{a \in \mathcal{A}_{int,T_2}} \left| e_{K_a^+} - e_{K_a^-} \right|^2 \right)$$

Then :

$$\sum_{j=1}^{L} S(K_j \cap T_1)\ |e_j - m_T|^2 \leq C \left(\sum_{a \in \mathcal{A}_{int,T_1}} \left| e_{K_a^+} - e_{K_a^-} \right|^2 + \sum_{a \in \mathcal{A}_{int,T_2}} \left| e_{K_a^+} - e_{K_a^-} \right|^2 \right)$$

With the same arguments, one can prove the following result :

$$\sum_{j=1}^{L} S(K_j \cap T_2)\ |e_j - m_T|^2 \leq C \left(\sum_{a \in \mathcal{A}_{int,T_1}} \left| e_{K_a^+} - e_{K_a^-} \right|^2 + \sum_{a \in \mathcal{A}_{int,T_2}} \left| e_{K_a^+} - e_{K_a^-} \right|^2 \right)$$

and then :

$$\sum_{j=1}^{L} S(K_j \cap T)\ |e_j - m_T|^2 \leq C \sum_{a \in \mathcal{A}_{int}} \left| e_{K_a^+} - e_{K_a^-} \right|^2$$

As we have supposed that Ω is a finite union of triangles, one has :

$$\sum_{j=1}^{L} S(K_j)\ |e_j - m|^2 \leq C \sum_{a \in \mathcal{A}_{int}} \left| e_{K_a^+} - e_{K_a^-} \right|^2$$

where C does not depend of the triangulation $\mathcal{T}$.

One concludes remarking that $\dfrac{\alpha_1}{2\,\alpha_2} \leq \dfrac{l(a)}{d_a}$ for all $a \in \mathcal{A}_{int}$, so :

$$\sum_{j=1}^{L} S(K_j)\ |e_j - m|^2 \leq \frac{\alpha_2}{2\,\alpha_1}\ C \sum_{a\in\mathcal{A}_{int}} \frac{\left|e_{K_a^+} - e_{K_a^-}\right|^2}{d_a}\ l(a)$$

Remark 1 *If the boundary condition is a Dirichlet condition, then this proof can be generalized, it is closed to those given by R. Herbin in [8].*
In this case one considers a direction $\mathcal{D}$ which is parallel to none edges of the mesh.
Let $j \in \{1,\ldots,L\}$ and $x \in K_j$, then one denotes by $\mathcal{A}_{jx}$, the set of the edges such that the intersection between these sides and the line which contains x and parallel to $\mathcal{D}$ is not empty set.
One can write :

$$|e_j|^2 \leq C\, d_\Omega \sum_{a\in\mathcal{A}_{jx}} \frac{|e_{K_a^+} - e_{K_a^-}|^2}{d_a\ \cos\theta_a}$$

where θ_a is the angle between a and $\mathcal{D}$.

Integrating over K_j, summing over j and swapping summations and integrals, one gets :

$$\sum_{j=1}^{L} S(K_j)\,|e_j|^2 \leq C\, d_\Omega^2 \left(\sum_{a\in\mathcal{A}_{int}} \frac{|e_{K_a^+} - e_{K_a^-}|^2}{d_a}\, l(a) + \sum_{a\in\mathcal{A}_{ext}} \frac{|e_{K_a}|^2}{d_a}\, l(a) \right)$$

and then one concludes with the error estimate in discrete H_0^1 norm (see [8]).

4 Convergence of the numerical scheme for the hyperbolic equation

In this section one will prove the convergence of the solution of (10) toward a weak solution of problem (2), (4), (5), proving the following Theorem :

Theorem 2 *Let $(\mathcal{T}_q, \delta_q)_{q\in I\!N}$ be a sequence of triangulations of Ω and time steps which satisfy the properties (6) and the stability condition (9).*
One notes $\mathcal{T}_q = (K_j)^{(q)}_{1\leq j\leq L^{(q)}}$ and $t^n_{(q)} = n\,\delta_q$.
Let the sequence $(u_{\mathcal{T}_q,\delta_q})_{q\in I\!N}$ with $u_{\mathcal{T}_q,\delta_q}(x,t) = {u^{(q)}}^n_j$ if $x \in K_j^{(q)}$ and $t \in [t^n_{(q)}, t^{n+1}_{(q)}[$, where $\Big\{{u^{(q)}}^n_j$, $j \in \{1,\ldots,L^{(q)}\}$, $n \in I\!N\Big\}$, is solution of the discretized equation (10) associated to the triangulation $\mathcal{T}_q$ and the time step δ_q.
Then :

1. *there exists a subsequence still denoted $(u_{\mathcal{T}_q,\delta_q})_{q\in I\!N}$, which converges toward u when $q \to \infty$, i.e. when h goes to 0, in $L^\infty(\Omega\times I\!R_+)$ for the weak $\star$ topology, i.e. one has :*

$$\lim_{q\to\infty} \iint_{\Omega\times I\!R_+} u_{\mathcal{T}_q,\delta_q}(x,t)\,\varphi(x,t)\,dx\,dt = \iint_{\Omega\times I\!R_+} u(x,t)\,\varphi(x,t)\,dx\,dt$$

for all $\varphi \in L^1(\Omega\times I\!R_+)$.

2. *u is a weak solution of problem (2), (4), (5), i.e. one has :*

$$\int_\Omega\int_{I\!R_+} u(x,t)\frac{\partial\varphi}{\partial t}(x,t)\,dx\,dt - \int_\Omega\int_{I\!R_+} u(x,t)\,\nabla P(x).\nabla\varphi(x,t)\,dx\,dt$$
$$+ \int_\Omega u_0(x)\,\varphi(x,0)\,dx + \int_\Gamma\int_{I\!R_+} \overline{u}(\gamma,t)\,\varphi(\gamma,t)\,g^+(\gamma)\,d\gamma\,dt = 0$$

for all $\varphi \in C_c^\infty(\Omega^+\times I\!R_+)$ with $\Omega^+ = \Omega\cup\Gamma^+$.

In order to prove the first part of this Theorem, one will prove in the subsection 4.1 an $L^\infty(\Omega \times I\!R_+)$ estimate on the approximate solution.
Then to prove the second part, one will first show, in subsection 4.2, a weak estimate on the variation of the approximate solution, and then, using this estimate, one will prove that u is a weak solution of problem (2), (4), (5).

4.1 $L^\infty(\Omega \times I\!R_+)$ estimate on the approximate solution

Here one proves that the family $(u_{\mathcal{T}_q,\delta_q})_{q\in I\!N}$ is bounded in $L^\infty(\Omega \times I\!R_+)$, then, thanks to the sequential weak $\star$ relative compactness of the bounded sets of $L^\infty(\Omega \times I\!R_+)$, it shows the existence of a subsequence, still denoted $(u_{\mathcal{T}_q,\delta_q})_{q\in I\!N}$, which converges in $L^\infty(\Omega \times I\!R_+)$ for the weak $\star$ topology when h goes to 0.

Let $\mathcal{T}$ be a triangulation of Ω and $\delta \in I\!R_+^*$ which satisfy the property (6) and the stability condition (9). Another expression of the equation (10) is :

$$\begin{aligned} u_j^{n+1} &= u_j^n \left(1 + \frac{\delta}{S(K_j)}\left(\sum_{k\in\tau_j,\, P_j>P_k} l(c_{jk})\,\frac{(P_k-P_j)}{d_{jk}} - \sum_{i\in\tau_{ext}(j)} g_{ij}^-\right)\right) \\ &\quad + \frac{\delta}{S(K_j)}\left(\sum_{k\in\tau_j,\, P_k>P_j} u_k^n\; l(c_{jk})\,\frac{(P_k-P_j)}{d_{jk}} + \sum_{i\in\tau_{ext}(j)} \overline{u}_{ji}^n\; g_{ij}^+\right) \end{aligned}$$

for all $j \in \{1,\ldots,L\}$.
So u_j^{n+1} is a linear combination of the u_k^n, $1 \le k \le L$ and $\overline{u}_{ji}^n$ $i = 1, 2$ or 3.

Then like in [5], thanks to (8) and to the stability condition (9), one has the following properties :

1. the sum of the coefficients of the combination is equal to 1
2. the coefficients of the combination are all positive

and one can write :

$$\begin{aligned} |u_j^{n+1}| &\le \max\left(\sup_{K_j\in\mathcal{T}} |u_j^n|, \sup_{j\in\mathcal{T}_{ext},\ i=1,2,3} |\overline{u}_{ji}^n|\right) \le \ldots \le \max\left(\sup_{K_j\in\mathcal{T}} |u_j^0|, ||\overline{u}||_{L^\infty(\Gamma^+\times I\!R_+)}\right) \\ &\le \max\left(||u_0||_{L^\infty(\Omega)}, ||\overline{u}||_{L^\infty(\Gamma^+\times I\!R_+)}\right) \end{aligned}$$

Then $(u_{\mathcal{T}_q,\delta_q})_{q\in I\!N}$ is bounded in $L^\infty(\Omega \times I\!R_+)$.

4.2 Convergence of the numerical scheme

The $L^\infty(\Omega \times I\!R_+)$ stability gives the existence of a subsequence which converges to u in $L^\infty(\Omega \times I\!R_+)$ for the weak $\star$ topology.
One will show that u is a solution of (2), (4), (5) in a weak sense.

Let $\mathcal{T}$ be a triangulation of Ω and $\delta \in I\!R_+^*$ which satisfy the property (6) and the stability condition (9). One considers $\varphi \in C_c^\infty(\Omega^+ \times I\!R_+)$ with $\Omega^+ = \Omega \cup \Gamma^+$, and one defines T such that, $\forall\ x \in \Omega^+$, $supp\big(\varphi(x,.)\big)\subset [0, T-1]$ and such that there exists $N \in I\!N$ such that $N\,\delta \le T < (N+1)\,\delta$.
Multiplying (10) by $\dfrac{1}{S(K_j)}\,\varphi(x,t^n)$ and integrating over K_j, then summing over j and n, one gets :

$$E_{1h} + E_{2h} = 0$$

with :

$$
\begin{aligned}
E_{1h} &= \sum_{n=0}^{N}\sum_{j=1}^{L}\int_{K_j} S(K_j)\,(u_j^{n+1}-u_j^n)\,\frac{1}{S(K_j)}\,\varphi(x,t^n)\,dx \\
E_{2h} &= -\sum_{n=0}^{N}\sum_{j=1}^{L}\delta\left(\sum_{k\in\tau_j} u_{jk}^n\,(P_k-P_j)\frac{l(c_{jk})}{d_{jk}}+\sum_{i\in\tau_{ext}(j)}(g_{ij}^+\,\overline{u}_{ji}^n-g_{ij}^-\,u_j^n)\right) \\
&\quad \times\int_{K_j}\frac{1}{S(K_j)}\,\varphi(x,t^n)\,dx
\end{aligned}
$$

In a classical way one could show that :

$$
\lim_{h\to 0} E_{1h} = -\int_\Omega\int_{I\!R_+} u(x,t)\frac{\partial\varphi}{\partial t}(x,t)\,dx\,dt-\int_\Omega u_0(x)\,\varphi(x,0)\,dx
$$

The proof of the following result will be given :

$$
\lim_{h\to 0} E_{2h} = \int_\Omega\int_{I\!R_+} u(x,t)\,\nabla P(x).\nabla\varphi(x,t)\,dx\,dt-\int_\Gamma\int_{I\!R_+}\overline{u}(\gamma,t)\,\varphi(\gamma,t)\,g^+(\gamma)\,d\gamma\,dt
$$

One defines E_{3h} and E_{4h} by :

$$
\begin{aligned}
E_{3h} = \sum_{n=0}^{N}\delta\Bigg(&\sum_{(j,k)\in\mathcal{S}}(u_j^n-u_k^n)\,\frac{(P_k-P_j)}{d_{jk}}\int_{c_{jk}}\varphi(\gamma,t^n)\,d\gamma\,+ \\
&+\sum_{j=1}^{L}\sum_{i\in\tau_{ext}(j)}(u_j^n-\overline{u}_{ji}^n)\int_{c_i(j)}\varphi(\gamma,t^n)\,g(\gamma)\,d\gamma\Bigg)
\end{aligned}
$$

$$
E_{4h} = \sum_{n=0}^{N}\delta\left(\int_\Omega u_{\mathcal{T},\delta}(x,t^n)\,\nabla P(x).\nabla\varphi(x,t^n)\,dx-\int_\Gamma\overline{u}_{\mathcal{T},\delta}(\gamma,t^n)\varphi(\gamma,t^n)\,g(\gamma)\,d\gamma\right)
$$

The result will be proved in three steps.
The first step is the proof of the existence of a constant $C\geq 0$, independent of $\mathcal{T}$ and δ, such that :

$$
\Big|E_{3h}-E_{2h}\Big|\leq C\,h^{1/2}
$$

According to (8) one has :

$$
\begin{aligned}
E_{2h} = \sum_{n=0}^{N}\sum_{j=1}^{L}\delta&\left(\sum_{k\in\tau_j}(u_{jk}^n-u_j^n)\,(P_j-P_k)\frac{l(c_{jk})}{d_{jk}}+\sum_{i\in\tau_{ext}(j)}(u_j^n-\overline{u}_{ji}^n)\,g_{ij}^+\right) \\
&\times\int_{K_j}\frac{1}{S(K_j)}\,\varphi(x,t^n)\,dx
\end{aligned}
$$

$$
\begin{aligned}
E_{2h} = \sum_{n=0}^{N}\delta\Bigg(&\sum_{(j,k)\in\mathcal{S}}\frac{l(c_{jk})}{S(K_k)}\,(u_j^n-u_k^n)\,\frac{(P_j-P_k)}{d_{jk}}\int_{K_k}\varphi(x,t^n)\,dx \\
&+\sum_{j=1}^{L}\sum_{i\in\tau_{ext}(j)}\frac{1}{S(K_j)}\,(u_j^n-\overline{u}_{ji}^n)\,g_{ij}^+\int_{K_j}\varphi(x,t^n)\,dx\Bigg)
\end{aligned}
$$

thus :

$$\left|E_{2h}-E_{3h}\right| \leq \sum_{n=0}^{N} \delta \Bigg(\sum_{(j,k)\in\mathcal{S}} |u_j^n - u_k^n| \; \frac{(P_j-P_k)}{d_{jk}} \left| \int_{c_{jk}} \varphi(\gamma,t^n)\,d\gamma - \frac{l(c_{jk})}{S(K_k)} \int_{K_k} \varphi(x,t^n)\,dx \right|$$

$$+ \sum_{j=1}^{L} \sum_{i\in\tau_{ext}(j)} |u_j^n - \overline{u}_{ji}^n| \; \left| \frac{g_{ij}^+}{S(K_j)} \int_{K_j} \varphi(x,t^n)\,dx - \int_{c_i(j)} \varphi(\gamma,t^n)\,g(\gamma)\,d\gamma \right| \Bigg)$$

One defines :

$$D_1 = \left| \int_{c_{jk}} \varphi(\gamma,t^n)\,d\gamma - \frac{l(c_{jk})}{S(K_k)} \int_{K_k} \varphi(x,t^n)\,dx \right|$$

$$D_2 = \left| \frac{g_{ij}^+}{S(K_j)} \int_{K_j} \varphi(x,t^n)\,dx - \int_{c_i(j)} \varphi(\gamma,t^n)\,g(\gamma)\,d\gamma \right|$$

Remarking that if $\varphi \equiv C$ is constant then $D_1 = 0$, one gets :

$$\begin{aligned} D_1 &= \left| \int_{c_{jk}} (\varphi(\gamma,t^n) - \varphi(x_k,t^n))\,d\gamma - \frac{l(c_{jk})}{S(K_k)} \int_{K_k} (\varphi(x,t^n) - \varphi(x_k,t^n))\,dx \right| \\ &\leq C\,h_j\,l(c_{jk}) \end{aligned}$$

Similarly one could show that :

$$D_2 \leq C\,h_j\,g_{ij}^+$$

and then :

$$\left|E_{2h}-E_{3h}\right| \leq C\,\delta \sum_{n=0}^{N} \sum_{(j,k)\in\mathcal{S}} h_j\,l(c_{jk}) \; \frac{(P_j-P_k)}{d_{jk}} \; |u_j^n - u_k^n|$$

$$+\delta \sum_{n=0}^{N} \sum_{j\in\mathcal{T}_{ext}} \sum_{i\in\tau_{ext}(j)} h_j\,g_{ij}^+ \; |u_j^n - \overline{u}_{ji}^n|$$

To conclude, the following Lemma is proved :

Lemma 2 *Let $\mathcal{T}$ be a triangulation of Ω and $\delta \in I\!R_+^*$ which satisfy the property (6) and the stability condition (9). Let $T>0$, one defines N by $N\,\delta \leq T < (N+1)\,\delta$ and one supposes $N \geq 1$, then one notes*

$$EF_h(T) = \delta \sum_{n=0}^{N} \Bigg(\sum_{(j,k)\in\mathcal{S}} h_j\,l(c_{jk}) \; \frac{(P_j-P_k)}{d_{jk}} \; |u_j^n - u_k^n| + \sum_{j\in\mathcal{T}_{ext}} \sum_{i\in\tau_{ext}(j)} h_j\,g_{ij}^+ \; |u_j^n - \overline{u}_{ji}^n| \Bigg)$$

Then there exists a constant $C \geq 0$ independent of $\mathcal{T}$ and δ such that :

$$EF_h(T) \leq C\; h^{1/2}$$

Proof of Lemma 2

Let $n \in I\!N$, $K_j \in \mathcal{T}$, one multiplies (10) by u_j^n and one sums over n and j, then using the following property :

$$u_j^{n+1}\; u_j^n - (u_j^n)^2 = -\frac{1}{2}\;(u_j^{n+1} - u_j^n)^2 - \frac{1}{2}\;(u_j^n)^2 + \frac{1}{2}\;(u_j^{n+1})^2$$

one gets :

$$-\frac{1}{2}\sum_{n=0}^{N}\sum_{j=1}^{L}S(K_j)\ (u_j^{n+1}-u_j^n)^2-\delta\sum_{n=0}^{N}\sum_{j=1}^{L}\Bigg(\sum_{k\in\tau_j}u_j^n\ u_{jk}^n\ l(c_{jk})\ \frac{(P_k-P_j)}{d_{jk}}$$

$$+\sum_{i\in\tau_{ext}(j)}\Big(u_j^n\ \overline{u}_{ji}^n\ g_{ij}^+-(u_j^n)^2\ g_{ij}^-\Big)\Bigg)\leq\frac{1}{2}\sum_{j=1}^{L}S(K_j)\,(u_j^0)^2 \tag{19}$$

Let B_1 and B_2 be defined by :

$$\begin{aligned}B_1 &=\sum_{n=0}^{N}\sum_{j=1}^{L}S(K_j)\ (u_j^{n+1}-u_j^n)^2\\ B_2 &=\delta\sum_{n=0}^{N}\sum_{j=1}^{L}\Bigg(\sum_{k\in\tau_j}u_j^n\ u_{jk}^n\ l(c_{jk})\ \frac{(P_k-P_j)}{d_{jk}}+\sum_{i\in\tau_{ext}(j)}\Big(u_j^n\ \overline{u}_{ji}^n\ g_{ij}^+-(u_j^n)^2\ g_{ij}^-\Big)\Bigg)\end{aligned}$$

Then one has :

$$B_2=\delta\sum_{n=0}^{N}\Bigg(\sum_{(j,k)\in\mathcal{S}}\Big((u_j^n)^2-u_j^n\ u_k^n\Big)\ l(c_{jk})\ \frac{(P_k-P_j)}{d_{jk}}+\sum_{j=1}^{L}\sum_{i\in\tau_{ext}(j)}\Big(u_j^n\ \overline{u}_{ji}^n\ g_{ij}^+-(u_j^n)^2\ g_{ij}^-\Big)\Bigg)$$

Remarking that :

$$(u_j^n)^2-u_j^n\,u_k^n=\frac{1}{2}(u_j^n-u_k^n)^2+\frac{1}{2}(u_j^n)^2-\frac{1}{2}(u_k^n)^2\qquad\forall\ (j,k)\in\mathcal{S}$$

$$\sum_{(j,k)\in\mathcal{S}}\Big((u_j^n)^2-(u_k^n)^2\Big)\ l(c_{jk})\ \frac{(P_k-P_j)}{d_{jk}}=\sum_{j=1}^{L}\sum_{k\in\tau_j}(u_j^n)^2\ l(c_{jk})\ \frac{(P_k-P_j)}{d_{jk}}$$

and thanks to (8) :

$$(u_j^n)^2\Bigg(\sum_{k\in\tau_j}l(c_{jk})\ \frac{(P_k-P_j)}{d_{jk}}+\sum_{i\in\tau_{ext}(j)}(g_{ij}^+-g_{ij}^-)\Bigg)=0\qquad\forall\ j\in\{1,\ldots,L\}$$

one can write :

$$\begin{aligned}B_2=\ \frac{\delta}{2}\sum_{n=0}^{N}\Bigg(&\sum_{k\in\tau_j}(u_j^n-u_k^n)^2\ l(c_{jk})\ \frac{(P_k-P_j)}{d_{jk}}\quad-\sum_{j=1}^{L}\sum_{i\in\tau_{ext}(j)}g_{ij}^+\Big(u_j^n-\overline{u}_{ji}^n\Big)^2\\ &+\sum_{j=1}^{L}\sum_{i\in\tau_{ext}(j)}\Big((\overline{u}_{ji}^n)^2\ g_{ij}^+-(u_j^n)^2\ g_{ij}^-\Big)\Bigg)\end{aligned}$$

Furthermore according to (10) one has :

$$\sum_{n=0}^{N}\sum_{j=1}^{L}S(K_j)\ (u_j^{n+1}-u_j^n)^2=\sum_{n=0}^{N}\sum_{j=1}^{L}\frac{\delta^2}{S(K_j)}\Bigg(\sum_{k\in\tau_j}(u_{jk}^n-u_j^n)\ l(c_{jk})\ \frac{(P_k-P_j)}{d_{jk}}$$

$$+\sum_{i\in\tau_{ext}(j)}g_{ij}^+\Big(\overline{u}_{ji}^n-u_j^n\Big)\Bigg)^2$$

The number of the neighbours of a triangle K_j is lower than 3, and the number of its sides on the boundary is lower than 2, then thanks to the Cauchy-Schwarz inequality :

$$\begin{aligned} B_1 \quad &\le \sum_{n=0}^{N} \delta^2 \Bigg(\sum_{(j,k)\in\mathcal{S}} \frac{5}{S(K_k)} \quad (l(c_{jk}))^2 \; (u_j^n - u_k^n)^2 \; \left(\frac{(P_j - P_k)}{d_{jk}}\right)^2 \\ &\qquad\qquad + \sum_{j=1}^{L} \frac{5}{S(K_j)} \sum_{i\in\tau_{ext}(j)} (g_{ij}^+)^2 \; \left(\overline{u}_{ji}^n - u_j^n\right)^2 \Bigg) \end{aligned}$$

Thus using this inequality and the last expression of B_2 in (19) one gets :

$$\begin{aligned} &\delta \sum_{n=0}^{N} \Bigg(\sum_{(j,k)\in\mathcal{S}} \theta_{jk} \; l(c_{jk}) \; \frac{(P_j - P_k)}{d_{jk}} \; (u_j^n - u_k^n)^2 + \sum_{j=1}^{L} \sum_{i\in\tau_{ext}(j)} \theta_j \; g_{ij}^+ \left(\overline{u}_{ji}^n - u_j^n\right)^2 \\ &\qquad + \sum_{j=1}^{L} \sum_{i\in\tau_{ext}(j)} (u_j^n)^2 \; g_{ij}^- \Bigg) \le \delta \sum_{n=0}^{N} \sum_{j=1}^{L} \sum_{i\in\tau_{ext}(j)} (\overline{u}_{ji}^n)^2 \; g_{ij}^+ + \sum_{j=1}^{L} S(K_j)\,(u_j^0)^2 \end{aligned}$$

with :

$$\theta_{jk} = 1 - \delta \; \frac{5}{S(K_k)} \; l(c_{jk}) \; \frac{(P_j - P_k)}{d_{jk}} > \alpha$$

$$\theta_j = 1 - \delta \; \frac{5}{S(K_j)} \; g_{ij}^+ > \alpha$$

So one can write the following inequality :

$$\begin{aligned} &\delta \sum_{n=0}^{N} \sum_{(j,k)\in\mathcal{S}} l(c_{jk}) \; \frac{(P_j - P_k)}{d_{jk}} \; (u_j^n - u_k^n)^2 + \delta \sum_{n=0}^{N} \sum_{j=1}^{L} \sum_{i\in\tau_{ext}(j)} g_{ij}^+ \left(\overline{u}_{ji}^n - u_j^n\right)^2 \\ &\qquad \le \frac{1}{\alpha} \; S(\Omega) \; ||u_0||^2_{L^\infty(\Omega)} + \; \frac{1}{\alpha} \; ||\overline{u}||^2_{L^\infty(\Gamma^+\times I\!R_+)} \; T \; \int_{\Gamma^+} g^+(\gamma) \; d\gamma = \frac{K}{\alpha} \end{aligned}$$

Using the Cauchy Schwarz inequality and the previous property one gets :

$$EF_h(T) \le h^{1/2} \left(\frac{K}{\alpha}\right)^{1/2} \times \left(\delta \sum_{n=0}^{N} \sum_{(j,k)\in\mathcal{S}} h_j \, l(c_{jk}) \; \frac{(P_j - P_k)}{d_{jk}} + \delta \sum_{n=0}^{N} \sum_{j=1}^{L} \sum_{i\in\tau_{ext}(j)} h_j \; g_{ij}^+ \right)^{1/2}$$

But $\delta \sum_{n=0}^{N} \sum_{j=1}^{L} \sum_{i\in\tau_{ext}(j)} h_j \; g_{ij}^+ \le T\,h \int_{\Gamma} g^+(\gamma) \; d\gamma = T\,h\,G^+$

with T and G^+ independent of $\mathcal{T}$ and δ.

Still using the Cauchy Schwarz inequality, one has :

$$\sum_{(j,k)\in\mathcal{S}} h_j \, l(c_{jk}) \; \frac{(P_j - P_k)}{d_{jk}} \le \left(\sum_{(j,k)\in\mathcal{S}} h_j^2 \, \frac{l(c_{jk})}{d_{jk}} \right)^{1/2} \left(\sum_{(j,k)\in\mathcal{S}} \frac{l(c_{jk})}{d_{jk}} \; (P_j - P_k)^2 \right)^{1/2}$$

But $\dfrac{\alpha_1}{2\,\alpha_2} \le \dfrac{l(c_{jk})}{d_{jk}} \le C \quad \forall \; (j,k) \in \mathcal{S}$, where C only depends on α_1, α_2 and on η, so :

$$\left(\sum_{(j,k)\in\mathcal{S}} h_j^2 \, \frac{l(c_{jk})}{d_{jk}} \right)^{1/2} \le \left(C \sum_{(j,k)\in\mathcal{S}} h_j^2 \right)^{1/2} \le \left(C \, \frac{S(\Omega)}{\alpha_1} \right)^{1/2}$$

Thus to complete the proof of Lemma 2, it just remains to show that there exists a constant $C \geq 0$, independent of $\mathcal{T}$ and δ, such that :

$$\sum_{(j,k)\in\mathcal{S}} \frac{(P_j - P_k)^2}{d_{jk}}\, l(c_{jk}) \leq C \tag{20}$$

This result will be proved in four steps (another proof is possible, using the error estimate (11), see Remark 2) :

Step 1 :
One shows that there exists a constant $C_1 \geq 0$ such that :

$$\sum_{j=1}^{L}\sum_{k\in\tau_j} \frac{(P_j - P_k)^2}{d_{jk}}\, l(c_{jk}) \leq C_1 \left(\sum_{j\in\mathcal{T}_{ext}} h_j\ (P_j)^2 \right)^{1/2} \tag{21}$$

Multiplying (8) by P_j and summing over j, one gets :

$$\left| \sum_{j=1}^{L}\sum_{k\in\tau_j} l(c_{jk})\ \frac{(P_k\ P_j - (P_j)^2)}{d_{jk}} \right| = \left| \sum_{j=1}^{L}\sum_{i\in\tau_{ext}(j)} g_{ij}\ P_j \right|$$

Then thanks to Cauchy-Schwarz :

$$\begin{aligned}
\left| \sum_{j=1}^{L}\sum_{i\in\tau_{ext}(j)} g_{ij}\ P_j \right| &\leq \alpha_2^{1/2}\ ||g||_{L^\infty(\Gamma)} \left(\sum_{j\in\mathcal{T}_{ext}} h_j\, |P_j|^2 \right)^{1/2} \left(\sum_{j\in\mathcal{T}_{ext}} l(c_i(j)) \right)^{1/2} \\
&\leq \alpha_2^{1/2}\ meas(\Gamma)\, ||g||_{L^\infty(\Gamma)} \left(\sum_{j\in\mathcal{T}_{ext}} h_j\, |P_j|^2 \right)^{1/2}
\end{aligned}$$

where $meas(\Gamma)$ is the 1D Lebesgue measure of Γ.

Furthermore one has :

$$\left| \sum_{j=1}^{L}\sum_{k\in\tau_j} \frac{l(c_{jk})}{d_{jk}} (P_k\ P_j - (P_j)^2) \right| = \frac{1}{2}\sum_{j=1}^{L}\sum_{k\in\tau_j} \frac{l(c_{jk})}{d_{jk}} (P_j - P_k)^2$$

and then :

$$\sum_{j=1}^{L}\sum_{k\in\tau_j} \frac{l(c_{jk})}{d_{jk}} (P_j - P_k)^2 \leq 2\ \alpha_2^{1/2}\ meas(\Gamma)\ ||g||_{L^\infty(\Gamma)} \left(\sum_{j\in\mathcal{T}_{ext}} h_j\ (P_j)^2 \right)^{1/2}$$

Step 2 :
One shows that there exists C_2 and C_3 positive such that :

$$\sum_{j\in\mathcal{T}_{ext}} h_j\ (P_j)^2 \leq C_2 \sum_{j=1}^{L}\sum_{k\in\tau_j} \frac{l(c_{jk})}{d_{jk}} (P_j - P_k)^2 + C_3 \sum_{j=1}^{L} S(K_j)\ (P_j)^2 \tag{22}$$

Ω is a bounded polygonal open set, first Ω will be supposed convex. If Ω is convex, then one defines two directions $\mathcal{D}_1$ and $\mathcal{D}_2$ and one breaks up the boundary of Ω in four parts, not necessary disjoint as on the figure 3.

(252,39)(426,39) (390,172)(390,20) 4.000(294,42)(314,23)(378,32)(385,42) (34,175)(34,20) (0,37)(173,37) (137,169)(1

Figure 3:

One notes $\mathcal{T}_{\Gamma_1}$ (respectively $\mathcal{T}_{\Gamma_2}$, $\mathcal{T}_{\Gamma_3}$, $\mathcal{T}_{\Gamma_4}$) the set of the suffix of the meshes which have sides on Γ_1 (respectively on Γ_2, Γ_3, Γ_4).

Let $j \in \mathcal{T}_{\Gamma_1}$ and $\tilde{x} \in \partial K_j$, then one defines :

$$\mathcal{D}^{(1)}_{j\tilde{x}} = \{(x_1, x_2) \in \Omega \ ; \ (x_1, x_2) \text{ is in the line which contains } \tilde{x} \text{ and parallel to } \mathcal{D}_1\}$$

$$\mathcal{A}^{(1)}_{j\tilde{x}} = \{a \in \mathcal{A}_{int} \ ; \ a \cap \mathcal{D}^{(1)}_{j\tilde{x}} \neq \emptyset\}$$

Let $\varphi \in C^\infty(\overline{\Omega})$ such that $\forall\ x \in \Omega,\ 0 \leq \varphi(x) \leq 1$, $\varphi(x) = 1\ \forall\ x \in \Gamma_1$, and $\varphi(x) = 0\ \forall\ x \in \Gamma_3$.

Then for all $j \in \mathcal{T}_{\Gamma_1}$ one has :

$$(P_j)^2 \leq |(P_j)^2 - (P_j\,\varphi_j)^2| + \sum_{a \in \mathcal{A}^{(1)}_{j\tilde{x}}} |(P_{K_a^+}\,\varphi_{K_a^+})^2 - (P_{K_a^-}\,\varphi_{K_a^-})^2| + (P_{j_{0\tilde{x}}}\,\varphi_{j_{0\tilde{x}}})^2$$

where $j_{0\tilde{x}}$ is the suffix of the element of $\mathcal{T}_{\Gamma_3}$ such that there exists $i \in \{1,2,3\}$ such that $c_i(j_{0\tilde{x}}) \cap \mathcal{D}^{(1)}_{j\tilde{x}} \neq \emptyset$, so :

$$\begin{aligned} {P_j}^2 \leq \sum_{a \in \mathcal{A}^{(1)}_{j\tilde{x}}} &\left(\left| \left(P_{K_a^+}\,\varphi_{K_a^+}\right)^2 - \left(P_{K_a^-}\,\varphi_{K_a^+}\right)^2 \right| + \left| \left(P_{K_a^-}\,\varphi_{K_a^+}\right)^2 - \left(P_{K_a^-}\,\varphi_{K_a^-}\right)^2 \right| \right) \\ &+ \left| {P_j}^2 - \left(P_j\,\varphi_j\right)^2 \right| + \left| \left(P_{j_{0\tilde{x}}}\,\varphi_{j_{0\tilde{x}}}\right)^2 - \left(P_{j_{0\tilde{x}}} \times 0\right)^2 \right| \end{aligned}$$

But φ is in $C^\infty(\overline{\Omega})$ and belongs to $[0,1]$, then one has :

$${P_j}^2 \leq \sum_{a \in \mathcal{A}^{(1)}_{j\tilde{x}}} \left(\left| {P_{K_a^+}}^2 - {P_{K_a^-}}^2 \right| + C_1\ h_{K_a^-}\ {P_{K_a^-}}^2 \right) + C_2 \left(h_{j_{0\tilde{x}}}\, {P_{j_{0\tilde{x}}}}^2 + h_j\, {P_j}^2 \right)$$

where $C_1 = ||\varphi'||_{L^\infty(\overline{\Omega})} \ \dfrac{2\,(\alpha_2)^2}{\alpha_1}$ and $C_2 = ||\varphi'||_{L^\infty(\overline{\Omega})} \ \alpha_1$.

Integrating over Γ_1, one obtains :

$$\sum_{j\in\mathcal{T}_{\Gamma_1}} \alpha_1\, h_j\, {P_j}^2 \leq \sum_{a\in\mathcal{A}_{int}} \left(\left| {P_{K_a^+}}^2 - {P_{K_a^-}}^2 \right| l(a) + C_1\ h_{K_a^-}\ {P_{K_a^-}}^2\, l(a) \right)$$
$$+C_2\ \alpha_2 \left(\sum_{j\in\mathcal{T}_{\Gamma_1}} S(K_j)\, {P_j}^2 + \sum_{j\in\mathcal{T}_{\Gamma_3}} S(K_j)\, {P_j}^2 \right)$$

According to the Young inequality :

$$\sum_{j\in\mathcal{T}_{\Gamma_1}} \alpha_1\, h_j\, {P_j}^2 \leq \sum_{a\in\mathcal{A}_{int}} \left(\frac{1}{2} \left| P_{K_a^+} - P_{K_a^-} \right|^2 + \left| {P_{K_a^+}}^2 + {P_{K_a^-}}^2 \right| l(a)^2 + C_1\ h_{K_a^-}\ {P_{K_a^-}}^2\, l(a) \right)$$
$$+C_2\ \alpha_2 \left(\sum_{j\in\mathcal{T}_{\Gamma_1}} S(K_j)\, {P_j}^2 + \sum_{j\in\mathcal{T}_{\Gamma_3}} S(K_j)\, {P_j}^2 \right)$$

then as $\dfrac{l(a)}{d_a} \geq \dfrac{\alpha_1}{2\,\alpha_2}$ for all $a \in \mathcal{A}_{int}$ one obtains :

$$\sum_{j\in\mathcal{T}_{\Gamma_1}} h_j\, (P_j)^2 \leq \frac{\alpha_2}{(\alpha_1)^2} \sum_{j=1}^{L} \sum_{k\in\tau_j} \frac{l(c_{jk})}{d_{jk}} |P_j - P_k|^2 + \frac{\alpha_2}{\alpha_1} \left(3\,\alpha_2 + C_1 + C_2 \right) \sum_{j=1}^{L} S(K_j)\, (P_j)^2$$

Similarly one could prove the same result for $\mathcal{T}_{\Gamma_2}$, $\mathcal{T}_{\Gamma_3}$ and $\mathcal{T}_{\Gamma_4}$. Then summing these inequalities, one completes the second step with Ω convex.

If Ω is not convex then one breaks up its boundary Γ in p parts $(\Gamma_i)_{1\leq i\leq p}$ disjoint and for all $i = 1, \ldots, p$, one defines $\Gamma_i^{'}$ as on the figure 4.

4.000(123,68)(130,26)(130,26) 4.000(176,63)(183,21)(183,21) (125,63)(178,51) 4.000(177,146)(16

Figure 4:

Then one defines for all $i = 1, \dots, p$, $\varphi^{(i)} \in C^\infty(\overline{\Omega})$ such that $\forall\ x \in \Omega,\ 0 \leq \varphi^{(i)}(x) \leq 1$, $\varphi^{(i)}(\gamma) = 1$ if $\gamma \in \Gamma_i$ and $\varphi^{(i)}(\gamma) = 0$ if $\gamma \in \Gamma_i^{'}$. Then similarly to the convex case one could prove (22).

Step 3 :

One supposes that :

$$\sum_{j=1}^{L} S(K_j)\, P_j = 0$$

it is always possible since solutions of (8) differ from a constant.
Then as one has proved (11), one could show that there exists a constant $C_4 \geq 0$ such that :

$$\sum_{j=1}^{L} S(K_j)\,(P_j)^2 \leq C_4 \sum_{j=1}^{L} \sum_{k \in \tau_j} \frac{l(c_{jk})}{d_{jk}} (P_j - P_k)^2 \tag{23}$$

Step 4 :
One concludes, remarking that :

$$\sum_{(j,k)\in\mathcal{S}} \frac{l(c_{jk})}{d_{jk}}(P_j-P_k)^2 = \frac{1}{2}\sum_{j=1}^{L}\sum_{k\in\tau_j}\frac{l(c_{jk})}{d_{jk}}(P_j-P_k)^2$$

then thanks to (21), (22) and (23) one has :

$$\sum_{j=1}^{L}\sum_{k\in\tau_j}\frac{l(c_{jk})}{d_{jk}}(P_j-P_k)^2 \leq C_1\sqrt{C_2+C_3\;C_4}\left(\sum_{j=1}^{L}\sum_{k\in\tau_j}\frac{l(c_{jk})}{d_{jk}}(P_j-P_k)^2\right)^{1/2}$$

So the proof of the inequality (20) and of the Lemma 2 are completed, and then one has :

$$\left|E_{3h}-E_{2h}\right| \leq C\,h^{1/2}$$

Remark 2 *As P is supposed to be in $C^2(\overline{\Omega})$, the estimate (20) can also be given by the error estimate in discrete H_0^1 norm (11), but our proof of (20) does not use the property $P\in C^2(\overline{\Omega})$ and then it could be extended to more complex cases.*

The second step is the proof of the following result :

$$\left|E_{3h}-E_{4h}\right| \leq C\,h$$

E_{3h} can also be written :

$$E_{3h} = \sum_{n=0}^{N}\delta\sum_{j=1}^{L}\left(\sum_{k\in\tau_j} \quad u_j^n\,\frac{(P_k-P_j)}{d_{jk}}\int_{c_{jk}}\varphi(\gamma,t^n)\,d\gamma + \right.$$
$$\left.\sum_{i\in\tau_{ext}(j)}(u_j^n-\overline{u}_{ji}^n)\int_{c_i(j)}\varphi(\gamma,t^n)\,g(\gamma)\,d\gamma\right)$$

and :

$$E_{4h} \quad = \sum_{n=0}^{N}\delta\sum_{j=1}^{L}\left(\sum_{k\in\tau_j}u_j^n\int_{c_{jk}}\nabla P(\gamma).n_{K_j}(\gamma)\,\varphi(\gamma,t^n)\,d\gamma + \sum_{i\in\tau_{ext}(j)}(u_j^n-\overline{u}_{ji}^n)\right.$$
$$\left.\int_{c_i(j)}\varphi(\gamma,t^n)\,g(\gamma)\,d\gamma\right)$$

then one gets :

$$\left|E_{3h}-E_{4h}\right| \leq \sum_{n=0}^{N}\delta\sum_{j=1}^{L}\sum_{k\in\tau_j}|u_j^n|\int_{c_{jk}}\left|\frac{(P_k-P_j)}{d_{jk}}-\nabla P(\gamma).n_{K_j}(\gamma)\right|\,|\varphi(\gamma,t^n)|\,d\gamma$$

But according to (1) and (8) if $\varphi\equiv\varphi(x_{jk},t^n)$ is constant, $\left|E_{3h}-E_{4h}\right|=0$, thus :

$$\left|E_{3h}-E_{4h}\right| \quad \leq \quad T\,||u_0||_{L^\infty(\Omega)}\,C_\varphi\,h\sum_{j=1}^{L}\sum_{k\in\tau_j}\left(l(c_{jk})\left|\frac{(P_k-P_j)}{d_{jk}}-\nabla P(x_{jk}).n_{K_j}(x_{jk})\right|\right.$$
$$\left.+\int_{c_{jk}}\left|\nabla P(x_{jk}).n_{K_j}(x_{jk})-\nabla P(\gamma).n_{K_j}(\gamma)\right|\,d\gamma\right)$$

where C_φ only depends on α_2 and the first order derivative of φ.
Then one has :

$$\begin{aligned}\left|E_{3h} - E_{4h}\right| &\leq T\,||u_0||_{L^\infty(\Omega)}\,C_\varphi\,h\sum_{j=1}^{L} 3\,(C_1\,h_j^2 + C_2\,h_j^2)\\ &\leq 3\,(C_1 + C_2)\;S(\Omega)\,T\,||u_0||_{L^\infty(\Omega)}\,C_\varphi\,h\end{aligned}$$

One concludes remarking that :

$$\lim_{h\to 0} E_{4h} = \int_\Omega \int_{I\!R_+} u(x,t)\,\nabla P(x).\nabla\varphi(x,t)\,dx\,dt - \int_\Gamma \int_{I\!R_+} \overline{u}(\gamma,t)\,\varphi(\gamma,t)\,g^+(\gamma)\,d\gamma\,dt$$

which completes the proof of the numerical scheme's convergence.

Remark 3 *One can prove the same results for a system a little bit different than (1)-(5), where (3) is changed by a Fourier condition :*

$$\nabla P(\gamma).n(\gamma) + \lambda\,P(\gamma) = g(\gamma) \quad \gamma \in \Gamma$$

where $\lambda > 0$.
Then one must introduce some unknowns at the boundary. As fluxes are not exact on the boundary, the error estimate, in discrete H^1 *norm, includes boundary terms.*
The technic used to prove the error estimate in discrete H_0^1 *norm is closed to this used in this note.*

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