

STUDY OF A NEW ASYMPTOTIC PRESERVING SCHEME FOR THE EULER SYSTEM IN THE LOW MACH NUMBER LIMIT

GIACOMO DIMARCO*, RAPHAËL LOUBÈRE† AND MARIE-HÉLÈNE VIGNAL‡

Abstract. This article deals with the discretization of the compressible Euler system for all Mach numbers regimes. For highly subsonic flows, since acoustic waves are very fast compared to the velocity of the fluid, the gas can be considered as incompressible. From the numerical point of view, when the Mach number tends to zero, the classical Godunov type schemes present two main drawbacks: they lose consistency and they suffer of severe numerical constraints for stability to be guaranteed since the time step must follow the acoustic waves speed. In this work, we propose and analyze a new uniformly stable and consistent scheme for all Mach number flows, from compressible to incompressible regimes, stability being only related to the flow speed. We propose two space discretizations valid for all regimes. We present several one and two dimensional simulations which show the AP behavior of our schemes and we observe their L^∞ and/or L^2 stability in all regimes.

Keywords: Low Mach number limit, Asymptotic preserving schemes, Euler system, stability analysis.

1. Introduction. Almost all fluids can be said to be compressible. However, there are many situations in which the changes in density are so small to be considered negligible. We refer to these situations saying that the fluid is in an incompressible regime. From the mathematical point of view, the difference between compressible and incompressible situations is that, in the second case, the equation for the conservation of mass is replaced by the constraint that the divergence of the velocity should be zero. This is due to the fact that when the Mach number tends to zero, the pressure waves can be considered to travel at infinite speed. From the theoretical point of view, researchers try to fill the gap between those two different descriptions by determining in which sense compressible equations tend to incompressible ones [2, 22, 23, 24, 35]. In this article we are interested in the numerical solution of the Euler system when used to describe fluid flows where the Mach number strongly varies. This causes the gas to pass from compressible to almost incompressible situations and consequently it causes most of the numerical methods build for solving compressible Euler equations to fail. In fact, when the Mach number tends to zero, it is well known that classical Godunov type schemes do not work anymore. Indeed, they lose consistency in the incompressible limit. This means that when close to the limit, the accuracy of these schemes is not sufficient to describe the flow. Many efforts have been done in the recent past in order to correct this main drawback of Godunov schemes, for instance by using preconditioning methods [36] or by splitting and correcting the pressure on the collocated meshes [6], [10, 11], [14], [15, 16, 32], [17], [25, 26], [28, 29, 30], or instead by using staggered grids like in the famous MAC scheme, see for instance [3], [18], [19], [21], [20], [33]. Unfortunately, even if these approaches permit to bypass the consistency problem of Godunov methods, they all need to resolve the scale of the acoustic waves in the fluid in order to remain stable. This means that they suffer from a restrictive C.F.L. (Courant-Friedrichs-Levy) condition which is inversely proportional to the Mach number value. In this work, we derive a method which is able

*Department of Mathematics and Computer Science, University of Ferrara, Ferrara, Italy (giacomo.dimarco@unife.it).

†CNRS and Institut de Mathématiques de Toulouse (IMT) Université Paul-Sabatier, Toulouse, France (raphael.loubere@math.univ-toulouse.fr).

‡CNRS and Institut de Mathématiques de Toulouse (IMT) Université Paul-Sabatier, Toulouse, France (mhvignal@math.univ-toulouse.fr).

to overcome both drawbacks of standard schemes: the lack of consistency in the limit and the strong stability requirements for small Mach numbers. For sake of clarity, we start considering the isentropic/isothermal gas dynamics case and we extend the results to the case of the full Euler system. The method derived and studied in this work belongs to the class of schemes called asymptotic preserving (AP). Such type of schemes have been developed in [4, 5] and in [7, 9] for the specific problem related to the compressible-incompressible passage. However, in [4, 5], the method is based on a Lagrange-Remap strategy, which naturally decouples the acoustic and transport waves and on a proper discretization of the space derivatives in the asymptotic limit by controlling the numerical diffusion. While in [9, 7], the authors split the pressure through the introduction of a numerical parameter which must be tuned depending on the problem in order for the scheme to work. Here, we use an alternative approach. Following pioneer works of [8] and of [7, 9], we propose a discretization based on an explicit/implicit methodology of the compressible Euler equations and a Godunov type scheme through the use of a simple Rusanov solver (see [12], [34]). We prove rigorous estimates on the L^∞ or L^2 norms of the approximate solution for the linearized isentropic system depending on the choice of the viscosity. These estimates do not prove the stability of the scheme but give compactness when the Mach number tends to zero. More importantly, from these results, we determine how the numerical viscosity must be chosen for general Mach numbers, leading to a method which is numerically consistent with the incompressible limit and, we observe L^∞ or L^2 numerical stability for the chosen semi-discretization in time. Even if the method proposed in this work is only first order accurate in time and space, the proved result paves the way to high order accurate schemes both in time and space through the use of similar explicit/implicit time discretizations and of high order polynomial reconstructions for the space derivatives which employ the same numerical diffusion needed to assure the stability of the method proposed here. We show with several numerical tests in one and two dimensions that our scheme behaves as expected for different regimes ranging from very low to high Mach number flows. The last part of the paper is sacred to the extension of the scheme proposed for the isentropic/isothermal case to the full Euler equations. Numerical evidences show that the approach is fruitful and the scheme performs well on all speed flows.

The article is organized as follows. In Section 2, we recall the isentropic/ isothermal system of Euler equations and its low Mach number limit. Then, in Section 3, we introduce a classical explicit semi-discretization in time and our new asymptotic preserving scheme. We perform in Section 4, an analysis on the L^∞ and L^2 norms of the approximate solution for the linearized system around a constant solution. In particular, we prove that the L^∞ and/or L^2 norm of the approximate density remains bounded when the Mach number tends to zero. It guides our choice of the viscosity coefficients of the scheme in the general case. Several test problems which prove the accuracy and the strong efficiency of our new method are discussed in this part. Finally, in Section 7, we extend our method to the case of the full Euler system and we test this new method on different problems. This extension is not straightforward due to non linear coupling between equations. A concluding Section ends the paper.

2. The continuous isentropic/isothermal model and its low Mach number limit. We start our discussion by considering an isentropic or isothermal flow in a bounded polygonal (or polyhedral) domain $\Omega \in \mathbb{R}^d$, $d = 1, 2$ or 3 . The extension to the full Euler system will be presented in Section 7. We denote by $x \in \Omega$ the space variable and by $t \geq 0$ the time. Following an usual rescaling and change of vari-

ables [23, 27], the isentropic/isothermal Euler system can be recast in a quasilinear hyperbolic system depending on the squared Mach number ε as

$$\partial_t \rho + \nabla \cdot (\rho U) = 0, \quad (2.1a)$$

$$\partial_t(\rho U) + \nabla \cdot (\rho U \otimes U) + \frac{1}{\varepsilon} \nabla p(\rho) = 0, \quad (2.1b)$$

where $\rho = \rho(t, x) \in [0, +\infty[$ is the density of the fluid, $U = U(t, x) \in \mathbb{R}^d$ its velocity and $p(\rho) = \rho^\gamma$ the pressure with $\gamma = 1$ for isothermal fluids and $\gamma > 1$ for isentropic ones. The above system is supplemented by initial and boundary conditions as for instance

$$U(t, x) \cdot \nu(x) = 0, \quad \text{for all } x \in \partial\Omega \text{ and } t > 0, \quad (2.2a)$$

$$(\rho, U)(0, x) = (\rho^0(x), U^0(x)), \quad \text{for all } x \in \Omega, \quad (2.2b)$$

where ν denotes the unit normal to the boundary $\partial\Omega$, outward to Ω . We suppose that the flow to be subsonic. This means that only one boundary condition is needed at each point of the boundary. As observed in [23] well prepared or equivalently consistent initial data are necessary for system (2.1). Thus, we assume that

$$\rho^0(x) = \rho_0 + \varepsilon \tilde{\rho}_0(x) \geq 0, \quad U^0(x) = U_0(x) + \sqrt{\varepsilon} \tilde{U}_0(x), \quad \nabla \cdot U_0(x) = 0, \quad (2.3)$$

where we stress that $\rho_0 > 0$ is a constant. Solutions to the isentropic/isothermal Euler system are known to exist for time intervals independent of the small parameter ε . For this problem, the low-Mach number limit is also proved rigorously to exist [1, 23, 24, 27, 35]. In the next Section, we recall the formal low-Mach number limit $\varepsilon \rightarrow 0$ which will be used in the sequel.

2.1. Low-Mach number limit for the isentropic/isothermal Euler system. When $\varepsilon \rightarrow 0$, the momentum equation (2.1b), yields $\nabla \rho(t, x) = 0$ and so $\rho(t, x) = \rho(t)$, for all $t \geq 0$ and $x \in \Omega$. Inserting, this result in the mass equation (2.1a), integrating on Ω and using the Green formula, gives

$$|\Omega| \rho'(t) + \rho(t) \int_{\partial\Omega} U \cdot \nu = 0,$$

where $|\Omega|$ denotes the measure of Ω . Then, using the initial and boundary conditions (2.2), (2.3) and assuming that

$$\pi_1 = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (p(\rho) - p(\rho_0)) < +\infty,$$

we obtain, the incompressible isentropic Euler system

$$\rho = \rho_0, \quad (2.4a)$$

$$\nabla \cdot U = 0, \quad (2.4b)$$

$$\rho_0 \partial_t U + \rho_0 \nabla \cdot (U \otimes U) + \nabla \pi_1 = 0, \quad (2.4c)$$

with the well-prepared initial condition $U(x, 0) = U_0(x)$ such that $\nabla \cdot U_0(x) = 0$. Note that in system (2.4) the order one correction of the pressure, π_1 , is determined thanks to the incompressibility constraint $\nabla \cdot U = 0$. It is possible to derive an explicit equation for this pressure subtracting the time derivative of the incompressibility

constraint to the divergence of the momentum equation. Performing these operations, we obtain

$$-\Delta\pi_1 = \rho_0 \nabla^2 : (U \otimes U), \quad (2.5)$$

where ∇^2 and $:$ are respectively the tensor of second order derivatives and the contracted product of two tensors. Observe that equations (2.4a), (2.5), (2.4c) are equivalent to the incompressible isentropic Euler system (2.4), since (2.5) and (2.4c) yields $\partial_t \nabla \cdot U = 0$ and the well prepared initial condition gives the incompressibility constraint $\nabla \cdot U = 0$.

We now derive the pressure wave equation which is responsible of the fast waves appearing in the Euler system. This is obtained by performing the same manipulations on system (2.1) as those done on the limit system for determining the explicit equation for the pressure π_1 . Then, we take the divergence of the momentum equation and we subtract it to the time derivative of the mass equation, this leads to

$$\partial_{tt}^2 \rho - \frac{1}{\varepsilon} \Delta p(\rho) = \nabla^2 : (\rho U \otimes U). \quad (2.6)$$

It is important to note that an explicit discretization of this pressure wave equation will be constrained by a stability condition of the form $\Delta t = O(\sqrt{\varepsilon}) \Delta x$. This means that for being uniformly stable with respect to the Mach number a scheme should yield an implicit discretization of the previous pressure wave equation.

3. Classical and AP semi-discretizations for the isentropic system. In the previous Section, we have seen that the low Mach number limit yields numerical difficulties for the time discretization since the pressure waves are very fast compared to the scale of the fluid motion. This may lead to think that, in order to construct a scheme which is consistent with the incompressible limit and which avoids too severe time step restrictions to remain stable, one needs to focus on the semi-discretization in time of (2.1). Thanks to a numerical analysis of the scheme, we will see in the next section that, indeed, this is not sufficient. This is a well known result for hyperbolic systems: the way in which the space derivatives are discretized is crucial in order to get stability and consistency with the limit equations. In particular, the numerical viscosity must be properly chosen in order to guarantee stability.

3.1. Classical explicit Euler scheme. A standard way to discretize in time the Euler system relies on the use of explicit time integrator schemes. Among all the different possibilities, the first and simpler first order accurate scheme reads

$$\frac{\rho^{n+1} - \rho^n}{\Delta t^n} + \nabla \cdot (\rho U)^n = 0, \quad (3.1a)$$

$$\frac{(\rho U)^{n+1} - (\rho U)^n}{\Delta t^n} + \nabla \cdot (\rho U \otimes U)^n + \frac{1}{\varepsilon} \nabla p(\rho^n) = 0. \quad (3.1b)$$

The stability of such discretization, which is the less restrictive among all the possible explicit time discretizations and which will be frequently used next, is ensured under a C.F.L. condition of type

$$\Delta t^n \leq \frac{\Delta x}{|u_i \pm \sqrt{p'(\rho)/\varepsilon}|} = O(\sqrt{\varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (3.2)$$

where $U = (u_i)_{1 \leq i \leq d}$ and where Δx measures the size of the space mesh. This leads to obvious stability issues because the time step Δt^n becomes tremendously small in the

low-Mach number limit. Moreover, in this limit, the consistency of the discretization is not ensured because system (3.1) is reduced to $\nabla p(\rho^n) = 0$, which is nothing but a constraint on the initial condition which may or may not be assured. Instead one would like to relate the consistency of the scheme to the model and not on the initial datum. Let us now recover the discretized pressure wave equation we get from the explicit first order time discretization. Taking the difference between equation (3.1a) at step $n + 2$ and $n + 1$, we get

$$\frac{\rho^{n+2} - \rho^{n+1}}{\Delta t^{n+1}} - \frac{\rho^{n+1} - \rho^n}{\Delta t^n} + \nabla \cdot (\rho U)^{n+1} - \nabla \cdot (\rho U)^n = 0.$$

Now, subtracting the divergence of equation (3.1b), yields

$$\frac{\rho^{n+2} - \rho^{n+1}}{\Delta t^{n+1}} - \frac{\rho^{n+1} - \rho^n}{\Delta t^n} - \frac{\Delta t^n}{\varepsilon} \Delta p(\rho^n) = \Delta t^n \nabla^2 : (\rho U \otimes U)^n,$$

which is indeed an explicit discretization of the pressure equation (2.6), constrained by a time step of order $\sqrt{\varepsilon}$ for stability to be assured.

3.2. A new asymptotic preserving scheme for the isentropic/isothermal Euler equations. The new proposed scheme is based on an explicit/implicit time discretization strategy like in [8]. It reads

$$\frac{\rho^{n+1} - \rho^n}{\Delta t^n} + \nabla \cdot (\rho U)^{n+1} = 0, \quad (3.3a)$$

$$\frac{(\rho U)^{n+1} - (\rho U)^n}{\Delta t^n} + \nabla \cdot (\rho U \otimes U)^n + \frac{1}{\varepsilon} \nabla p(\rho^{n+1}) = 0, \quad (3.3b)$$

where the difference with respect to the full explicit scheme is that now the pressure term and the density flux are taken as implicit. Let us note that in [9] a similar algorithm is proposed. However, in this previous work, the term $1/\varepsilon \nabla p(\rho^{n+1})$ is substituted by $\alpha \nabla p(\rho^n) + (1/\varepsilon - \alpha) \nabla p(\rho^{n+1})$ where $\alpha > 0$ is a numerical parameter which must be correctly fixed to ensure stability and which depends on the considered problem. In this article, the authors explain that ‘‘Numerically, we find some nonphysical oscillations when $\alpha = 0$. These oscillations can even be more severe when the time-step Δt is reduced and are produced by too small a numerical diffusion.’’ In most cases, the value $\alpha = 1$ seems to be the preferred one.

In this work, contrarily, we numerically observe that provided the numerical diffusion is well chosen (thanks to a stability analysis of the scheme), the choice $\alpha = 0$ ensures stability and preservation of the asymptotic state in the low Mach limit. Note that this strategy is close to that used in [5], but the authors use a Lagrange remap strategy, instead of an implicit treatment, to separate the acoustic and transport waves. We will see that $\alpha = 0$ is enough to ensure the stability and preservation of the asymptotic state provided the numerical viscosity of the scheme to be properly chosen. We immediately see that this scheme is consistent for all $\varepsilon \geq 0$ with systems (2.1) and (2.4). Indeed, the limit $\varepsilon \rightarrow 0$ formally gives

$$\nabla p(\rho^{n+1}) = 0, \quad \Rightarrow \quad \nabla \rho^{n+1} = 0,$$

which means that the scheme projects the solution over the asymptotic incompressible limit even if the initial datum is not consistent with this limit. In fact, if we operate as in the continuous case, integrating the mass equation on the domain and using the

boundary condition $U^{n+1} \cdot \nu = 0$ on $\partial\Omega$, we obtain $\rho^{n+1} = \rho^n$ and using the initial condition we get $\rho^{n+1} = \rho_0$, for all $n \geq 0$. Inserting this result in the mass equation, we recover the incompressibility constraint $\nabla \cdot U^{n+1} = 0$ and then, a semi-discretization in time of the limit system (2.4)

$$\begin{aligned}\rho^{n+1} &= \rho_0, \\ \nabla \cdot U^{n+1} &= 0, \\ \rho_0 \frac{U^{n+1} - U^n}{\Delta t^n} + \rho_0 \nabla \cdot (U \otimes U)^n + \nabla \pi_1^{n+1} &= 0,\end{aligned}$$

where $\pi_1^{n+1} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (p(\rho^{n+1}) - p(\rho_0))$. Moreover, taking the discrete time derivative of the mass equation (i.e. the difference between equations (3.3a) at step $n+1$ and n) and subtracting the divergence of the momentum equation, we obtain the following discretization for the pressure wave equation

$$\frac{\rho^{n+1} - \rho^n}{\Delta t^n} - \frac{\rho^n - \rho^{n-1}}{\Delta t^{n-1}} - \frac{\Delta t^n}{\varepsilon} \Delta p(\rho^{n+1}) = \Delta t^n \nabla^2 : (\rho U \otimes U)^n,$$

which is an unconstrained implicit discretization of the pressure wave equation (2.6). A key observation is that the scheme even if it looks fully implicit, it can be solved partly explicitly by decoupling the mass and the momentum equations. To this aim, taking the divergence of the momentum equation (3.3b), we get

$$\nabla \cdot (\rho U)^{n+1} = \nabla \cdot (\rho U)^n - \Delta t^n \nabla^2 : (\rho U \otimes U)^n - \frac{\Delta t^n}{\varepsilon} \Delta p(\rho^{n+1}),$$

which further substituted into (3.3a) gives

$$\frac{\rho^{n+1} - \rho^n}{\Delta t^n} + \nabla \cdot (\rho U)^n - \Delta t^n \nabla^2 : (\rho U \otimes U)^n - \frac{\Delta t^n}{\varepsilon} \Delta p(\rho^{n+1}) = 0, \quad (3.4a)$$

$$\frac{(\rho U)^{n+1} - (\rho U)^n}{\Delta t^n} + \nabla \cdot (\rho U \otimes U)^n + \frac{1}{\varepsilon} \nabla p(\rho^{n+1}) = 0. \quad (3.4b)$$

Thus, one can start by solving the mass equation (3.4a) which is a non linear elliptic equation determining the density at step t^{n+1} . Then, the momentum equation (3.4b) provides the momentum at time t^{n+1} explicitly. This is of great importance when dealing with multidimensional equations since the computational cost is strongly reduced compared to a standard fully implicit discretization.

4. Analysis of the full AP scheme for the linearized system.

In the previous section, we have presented our AP semi-discretization in time of (2.1). In this part we focus on the space-time discretization. Indeed, it is well known the numerical viscosity of the fluxes must be properly chosen in order to retain the stability in the low Mach limit ensured by the time discretization. This is related to the fact that the C.F.L. (Courant-Friedrichs-Levy) condition depends on the numerical diffusion of the scheme.

Here, in order to choose this numerical diffusion, we propose to analyze the full AP scheme in a simpler context: the one dimensional linearized isentropic system. So, let us start by rewriting system (2.1) on $\Omega = (0, 1)$, denoting $q = \rho U \in \mathbb{R}$ and linearizing the system around the constant state $(\rho_0, q_0 = \rho_0 U_0) \in (0, +\infty) \times \mathbb{R}^*$. This leads to

$$\partial_t W + A^\varepsilon \partial_x W = 0,$$

where $W = (\rho, q)$, and $A^\varepsilon = A_e + A_i^\varepsilon$, with A_e and A_i^ε being the explicit and implicit sub-matrices related to the flux function. Denoting by $c_0 = \sqrt{p'(\rho_0)}$ the sound speed (for $\varepsilon = 1$), these matrices are given by

$$A_e = \begin{pmatrix} 0 & 0 \\ -U_0^2 & 2U_0 \end{pmatrix}, \quad \text{and} \quad A_i^\varepsilon = \begin{pmatrix} 0 & 1 \\ c_0^2/\varepsilon & 0 \end{pmatrix}.$$

The above system is supplemented by periodic boundary conditions and initial conditions which read

$$\begin{cases} \rho(t = 0, x) = \rho_0 + \varepsilon \tilde{\rho}_0(x), \\ q(t = 0, x) = q_0 + \sqrt{\varepsilon} \tilde{q}_0(x). \end{cases}$$

In the following, space and time meshes are chosen uniform for clarity but a more general choice will not be restrictive for our analysis. Thus, given Δt and Δx , the finite volume discretization of the linearized system is written as

$$\begin{aligned} \frac{W_j^{n+1} - W_j^n}{\Delta t} + \frac{\mathcal{F}_e(W_j^n, W_{j+1}^n) - \mathcal{F}_e(W_{j-1}^n, W_j^n)}{\Delta x} \\ + \frac{\mathcal{F}_i(W_j^{n+1}, W_{j+1}^{n+1}) - \mathcal{F}_i(W_{j-1}^{n+1}, W_j^{n+1})}{\Delta x} = 0, \end{aligned} \quad (4.1)$$

for all $j = 1, \dots, L = 1/\Delta x$ and all $n = 0, \dots, N-1 = T/\Delta t - 1$, where T is the final time. While the initial conditions are $W_j^0 = \begin{pmatrix} \rho_0 + \varepsilon \tilde{\rho}_{0,j} \\ q_0 + \sqrt{\varepsilon} \tilde{q}_{0,j} \end{pmatrix}$. In the following the fluxes are both discretized, the explicit and implicit part separately, with the modified Lax-Friedrichs flux. A similar analysis with the Godunov solver will give analogous results. The fluxes are then

$$\mathcal{F}_e(W_j^n, W_{j+1}^n) = A_e \frac{W_j^n + W_{j+1}^n}{2} - D_e (W_{j+1}^n - W_j^n), \quad (4.2a)$$

$$\mathcal{F}_i(W_j^{n+1}, W_{j+1}^{n+1}) = A_i^\varepsilon \frac{W_j^{n+1} + W_{j+1}^{n+1}}{2} - D_i^\varepsilon (W_{j+1}^{n+1} - W_j^{n+1}), \quad (4.2b)$$

where $D_e \geq 0$ and $D_i^\varepsilon \geq 0$ are positive constants which measure the numerical viscosity and which values will be specified later. Assuming periodic boundary conditions, we also set

$$W_0^n = W_L^n, \quad W_{L+1}^n = W_1^n, \quad \text{for all } n \geq 0. \quad (4.3)$$

Now, we prove the following result

PROPOSITION 4.1. *Let $\varepsilon > 0$ and $(\rho_0, U_0) \in]0, +\infty[\times \mathbb{R}^*$. Let Δt and Δx be the time and space steps satisfying*

$$\Delta t \leq \frac{\Delta x}{2|U_0|}. \quad (4.4)$$

Let $D_e = |U_0|$, and (W_j^n) be the approximate solution given by the scheme (4.1)-(4.3). Then

1. If the viscosity of the implicit part is $D_i^\varepsilon = 0$, the following estimate holds

$$\sum_{j=1}^L \Delta x \left(|\rho_j^{n+1}|^2 + \frac{\varepsilon}{c_0^2} |q_j^{n+1}|^2 \right) \leq \left(4 + \frac{3U_0^2 \varepsilon}{2c_0^2} \right) \sum_{j=1}^L \Delta x \left(|\rho_j^n|^2 + \frac{\varepsilon}{c_0^2} |q_j^n|^2 \right) \quad (4.5)$$

for all $n \geq 0$.

2. If the viscosity of the implicit part is $D_i^\varepsilon = \frac{c_0}{2\sqrt{\varepsilon}}$, the following estimate holds

$$\max_{j=1}^L |\rho_j^{n+1}| + \frac{\sqrt{\varepsilon}}{c_0} \max_{j=1}^L |q_j^{n+1}| \leq \left(2 + \frac{2|U_0|\sqrt{\varepsilon}}{c_0}\right) \left(\max_{j=1}^L |\rho_j^n| + \frac{\sqrt{\varepsilon}}{c_0} \max_{j=1}^L |q_j^n|\right). \quad (4.6)$$

REMARK 4.2.

1. Let us remark that these estimates do not ensure the L^∞ or L^2 stability of the scheme for $\varepsilon > 0$. But, we believe that under the uniform C.F.L. (4.4):

- when the viscosity coefficient $D_i^\varepsilon = 0$, which corresponds to a centered implicit discretization, the scheme is only L^2 stable but not L^∞ stable.
- when the viscosity coefficient $D_i^\varepsilon = \frac{c_0}{2\sqrt{\varepsilon}}$, which corresponds to an upwind implicit discretization, the scheme is L^∞ stable and L^2 stable.

On the toy equation $\partial_t u + c_1 \partial_x u + \frac{c_2}{\varepsilon} \partial_x u = 0$, this result is also true and it can be easily proved. Indeed, we consider an explicit upwind discretization for the term $c_1 \partial_x u$ and a C.F.L $\Delta t \leq \Delta x / c_1$. If we use an implicit upwind discretization for the term $\frac{c_2}{\varepsilon} \partial_x u$, then using the same arguments as in the previous Proposition, we prove that the scheme is L^∞ stable. Contrarily, if we use an implicit centered discretization for this term, the scheme is only L^2 stable. And, we can construct a counter-example for the L^∞ stability.

In section 5, we calibrate the numerical diffusion of our full scheme for the non linear system using these conjectures (based on the estimates proved in Proposition 4.1). Moreover, these conjectures are confirmed by the numerical results presented in section 6, see the discussion related to Fig. 6.1.

2. The estimates of Proposition 4.1 are not sufficient to obtain compactness when Δx (and so also Δt) tends to 0: the approximate solution is not uniformly bounded with respect to the mesh since the constants $\left(4 + \frac{3U_0^2 \varepsilon}{2c_0^2}\right)$ and $\left(2 + \frac{|U_0|\sqrt{\varepsilon}}{c_0}\right)$ are bigger than 1. Nonetheless, these estimates give compactness in the limit $\varepsilon \rightarrow 0$.

Indeed, if we denote by $(\varepsilon_k)_{k \in \mathbb{N}} \subset]0, 1]$ a sequence such that $\varepsilon_k \rightarrow 0$ when $k \rightarrow \infty$, and by $\vec{\rho}^{\varepsilon_k} = (\rho_1^0, \dots, \rho_L^0, \rho_1^1, \dots, \rho_L^1, \dots, \rho_1^N, \dots, \rho_L^N) \in \mathbb{R}^{L \times (N+1)}$, and similarly for \vec{q}^{ε_k} where $(\rho_j^n, q_j^n)_{\substack{1 \leq j \leq L \\ 0 \leq n \leq N}}$ is the approximate solution given by the scheme (4.1)-(4.3) with $\varepsilon = \varepsilon_k$ the sequences $(\vec{\rho}^{\varepsilon_k})_{k \in \mathbb{N}}$ and $(\sqrt{\varepsilon_k} \vec{q}^{\varepsilon_k})_{k \in \mathbb{N}}$ are bounded independently of ε_k . More precisely, they satisfy

$$\begin{aligned} \|\vec{\rho}^{\varepsilon_k}\|_\infty + \frac{\sqrt{\varepsilon_k}}{c_0} \|\vec{q}^{\varepsilon_k}\|_\infty &= \max_{j=1}^L \max_{n=0}^N |\rho_j^n| + \frac{\sqrt{\varepsilon_k}}{c_0} \max_{j=1}^L \max_{n=0}^N |q_j^n| \\ &\leq 2 \max_{n=0}^N \left(\max_{j=1}^L |\rho_j^n| + \frac{\sqrt{\varepsilon_k}}{c_0} \max_{j=1}^L |q_j^n| \right) \leq 2 \max_{n=0}^N \left(\sum_{j=1}^L |\rho_j^n|^2 + \frac{\varepsilon_k}{c_0^2} \sum_{j=1}^L |q_j^n|^2 \right)^{1/2}. \end{aligned}$$

By induction, using (4.6) and (4.5), we have for any $n = 0, \dots, N$

$$\begin{aligned} \max_{j=1}^L |\rho_j^n| + \frac{\sqrt{\varepsilon_k}}{c_0} \max_{j=1}^L |q_j^n| &\leq \left(2 + \frac{2|U_0|}{c_0}\right)^N \left(\max_{j=1}^L |\rho_j^0| + \frac{\sqrt{\varepsilon_k}}{c_0} \max_{j=1}^L |q_j^0| \right), \\ \sum_{j=1}^L |\rho_j^n|^2 + \frac{\varepsilon_k}{c_0^2} \sum_{j=1}^L |q_j^n|^2 &\leq \left(4 + \frac{3U_0^2}{2c_0^2}\right)^N \left(\sum_{j=1}^L |\rho_j^0|^2 + \frac{\varepsilon_k}{c_0^2} \sum_{j=1}^L |q_j^0|^2 \right), \end{aligned}$$

depending if $D_i^{\varepsilon_k} = 0$ or $D_i^{\varepsilon_k} = \frac{c_0}{2\sqrt{\varepsilon_k}}$.

Then, using the initial condition

$$\|\bar{\rho}^{\varepsilon_k}\|_\infty + \frac{\sqrt{\varepsilon_k}}{c_0} \|\bar{q}^{\varepsilon_k}\|_\infty \leq 2K_0^N \left((\rho_0 + \|\tilde{\rho}_0\|_\infty) + \frac{1}{c_0^2} (|q_0| + \|\tilde{q}_0\|_\infty) \right),$$

where $K_0 = \left(4 + \frac{3U_0^2}{2c_0^2}\right)^{1/2}$ if $D_i^{\varepsilon_k} = 0$ and $K_0 = \left(2 + \frac{2|U_0|}{c_0}\right)$ $D_i^{\varepsilon_k} = \frac{c_0}{2\sqrt{\varepsilon_k}}$.

This proves that there exists a subsequence, still denoted by $(\bar{\rho}^{\varepsilon_k}, \sqrt{\varepsilon_k} \bar{q}^{\varepsilon_k})_{k \in \mathbb{N}}$, converging toward a vector $(\bar{\rho}^{lim}, \bar{Q}^{lim})$ when $k \rightarrow +\infty$. Passing to the limit $k \rightarrow +\infty$ in the scheme, we can prove in both cases ($D_i^{\varepsilon_k} = 0$ and $D_i^{\varepsilon_k} = \frac{c_0}{2\sqrt{\varepsilon_k}}$) that $(\bar{\rho}^{lim})_j^n = \rho_0$ and $(\bar{Q}^{lim})_j^n = 0$ for all $j = 1, \dots, L$ and all $n = 0, \dots, N$.

We emphasize that we have proved compactness only for ρ and $\sqrt{\varepsilon} q$. Obviously, the interesting result would be for ρ and q which seems much more difficult and beyond the scope of this paper. Taking inspiration from the continuous framework study (see [22]), this result could be possibly obtained proving energy estimates. Note that there exist in the literature, some stability results for finite volumes schemes solving compressible Euler or Navier-Stokes. But, either they are not dedicated to the uniform behavior in the low Mach limit and they concern staggered schemes (see for instance [20, 13]), or they concern semi-discretization in space schemes (see [10]).

Proof of Proposition 4.1: We first need to introduce the following quantities: $\lambda_\pm^\varepsilon = \pm c_0/\sqrt{\varepsilon}$ the eigenvalues of A_i^ε , $r_\pm^\varepsilon = (1, \pm c_0/\sqrt{\varepsilon})$, $l_\pm^\varepsilon = (\pm c_0/\sqrt{\varepsilon}, 1)$ some associated right and left eigenvectors, $\alpha_{-,j}^{n+1} = \rho_j^{n+1}/2 - \sqrt{\varepsilon} q_j^{n+1}/(2c_0)$ and $\alpha_{+,j}^{n+1} = \rho_j^{n+1}/2 + \sqrt{\varepsilon} q_j^{n+1}/(2c_0)$ the coordinates in this right eigenvectors basis. Similarly we introduce, $\lambda_0 = 0$ and $\lambda_2 = 2U_0$ the eigenvalues of A_e , $r_0 = (2, U_0)$ and $r_2 = (0, U_0)$ some associated right eigenvectors and $\alpha_{0,j}^n = \rho_j^n/2$ and $\alpha_{2,j}^n = q_j^n/U_0 - \rho_j^n/2$ the coordinates in the right eigenvector basis (r_0, r_2) .

The scheme (4.1), (4.2) can be rewritten separating the explicit and the implicit terms as $Z_j^{n+1} = B_j^n$, where

$$Z_j^{n+1} = W_j^{n+1} + \Delta t A_i^\varepsilon \frac{W_{j+1}^{n+1} - W_{j-1}^{n+1}}{2\Delta x} - \frac{D_i^\varepsilon \Delta t}{\Delta x} \left(W_{j+1}^{n+1} - 2W_j^{n+1} + W_{j-1}^{n+1} \right),$$

$$B_j^n = W_j^n - \Delta t A_e \frac{W_{j+1}^n - W_{j-1}^n}{2\Delta x} + \frac{D_e \Delta t}{\Delta x} \left(W_{j+1}^n - 2W_j^n + W_{j-1}^n \right).$$

Now, passing to the right eigenvectors basis $(r_-^\varepsilon, r_+^\varepsilon)$ for the implicit part leads to

$$\begin{aligned} Z_j^{n+1} = & \left[\alpha_{-,j}^{n+1} \left(1 + \frac{2D_i^\varepsilon \Delta t}{\Delta x} \right) - \frac{\Delta t}{\Delta x} \alpha_{-,j+1}^{n+1} \left(\frac{c_0}{2\sqrt{\varepsilon}} + D_i^\varepsilon \right) + \frac{\Delta t}{\Delta x} \alpha_{-,j-1}^{n+1} \left(\frac{c_0}{2\sqrt{\varepsilon}} - D_i^\varepsilon \right) \right] r_-^\varepsilon \\ & + \left[\alpha_{+,j}^{n+1} \left(1 + \frac{2D_i^\varepsilon \Delta t}{\Delta x} \right) + \frac{\Delta t}{\Delta x} \alpha_{+,j+1}^{n+1} \left(\frac{c_0}{2\sqrt{\varepsilon}} - D_i^\varepsilon \right) - \frac{\Delta t}{\Delta x} \alpha_{+,j-1}^{n+1} \left(\frac{c_0}{2\sqrt{\varepsilon}} + D_i^\varepsilon \right) \right] r_+^\varepsilon, \end{aligned}$$

while choosing $D_i^\varepsilon = 0$ and summing over the cells, leads to

$$\begin{aligned} & \sum_{j=1}^L \Delta x \frac{Z_j^{n+1} \cdot l_-^\varepsilon}{r_-^\varepsilon \cdot l_-^\varepsilon} \alpha_{-,j}^{n+1} \\ & = \sum_{j=1}^L \Delta x |\alpha_{-,j}^{n+1}|^2 + \frac{c_0 \Delta t}{2\sqrt{\varepsilon}} \left(-\alpha_{-,L+1}^{n+1} \alpha_{-,L}^{n+1} + \alpha_{-,0}^{n+1} \alpha_{-,1}^{n+1} \right) = \sum_{j=1}^L \Delta x |\alpha_{-,j}^{n+1}|^2, \end{aligned}$$

since the boundary conditions are periodic and $\alpha_{-,L+1}^{n+1} \alpha_{-,L}^{n+1} = \alpha_{-,1}^{n+1} \alpha_{-,0}^{n+1}$. A similar inequality holds also true for $|\alpha_{+,j}^{n+1}|$. Now, performing the same passage to the right eigenvalues basis for B_j^n gives

$$B_j^n = \left[\alpha_{0,j}^n \left(1 - \frac{2D_e \Delta t}{\Delta x} \right) + \frac{D_e \Delta t}{\Delta x} \alpha_{0,j+1}^n + \frac{D_e \Delta t}{\Delta x} \alpha_{0,j-1}^n \right] r_{0+} + \left[\alpha_{2,j}^n \left(1 - \frac{2D_e \Delta t}{\Delta x} \right) + \frac{\Delta t}{2\Delta x} \alpha_{2,j+1}^n (2D_e - 2U_0) + \frac{\Delta t}{2\Delta x} \alpha_{2,j-1}^n (2D_e + 2U_0) \right] r_{2+}.$$

Observe that for each term to be a convex combination it is enough to assure that $1 - \frac{2D_e \Delta t}{\Delta x} \geq 0$ and $2D_e \geq 2|U_0|$. Choosing finally $D_e = |U_0|$, leads to condition (4.4). Then, using $Z_j^{n+1} = B_j^n$, Cauchy-Schwarz's and Young's inequalities, we obtain

$$\begin{aligned} \sum_{j=1}^L \Delta x |\alpha_{\pm,j}^{n+1}|^2 &\leq \frac{1}{2} \sum_{j=1}^L \Delta x |\alpha_{\pm,j}^{n+1}| + \left| \frac{r_0 \cdot l_{\pm}^{\varepsilon}}{r_{\pm}^{\varepsilon} \cdot l_{\pm}^{\varepsilon}} \right| \left[\sum_{j=1}^L \Delta x |\alpha_{0,j}^n|^2 \left(1 - \frac{2|U_0| \Delta t}{\Delta x} \right) \right. \\ &\quad \left. + \sum_{j=2}^{L+1} \Delta x |\alpha_{0,j}^n|^2 \frac{|U_0| \Delta t}{\Delta x} + \sum_{j=0}^{L-1} \Delta x |\alpha_{0,j}^n|^2 \frac{|U_0| \Delta t}{\Delta x} \right] \\ &\quad + \left| \frac{r_2 \cdot l_{\pm}^{\varepsilon}}{r_{\pm}^{\varepsilon} \cdot l_{\pm}^{\varepsilon}} \right| \left[\sum_{j=1}^L \Delta x |\alpha_{2,j}^n|^2 \left(1 - \frac{2|U_0| \Delta t}{\Delta x} \right) + \right. \\ &\quad \left. \sum_{j=2}^{L+1} \Delta x |\alpha_{2,j}^n|^2 (|U_0| - U_0) + \sum_{j=0}^{L-1} \Delta x |\alpha_{2,j}^n|^2 (|U_0| + U_0) \right]. \end{aligned}$$

And, thanks to the periodic boundary conditions

$$\sum_{j=1}^L \Delta x |\alpha_{\pm,j}^{n+1}|^2 \leq \frac{1}{2} \sum_{j=1}^L \Delta x |\alpha_{\pm,j}^{n+1}|^2 + \left| \frac{r_0 \cdot l_{\pm}^{\varepsilon}}{r_{\pm}^{\varepsilon} \cdot l_{\pm}^{\varepsilon}} \right|^2 \sum_{j=1}^L \Delta x |\alpha_{0,j}^n|^2 + \left| \frac{r_2 \cdot l_{\pm}^{\varepsilon}}{r_{\pm}^{\varepsilon} \cdot l_{\pm}^{\varepsilon}} \right|^2 \sum_{j=1}^L \Delta x |\alpha_{2,j}^n|^2.$$

Finally, we remark that we have

$$\left| \frac{r_0 \cdot l_{\pm}^{\varepsilon}}{r_{\pm}^{\varepsilon} \cdot l_{\pm}^{\varepsilon}} \right| = \left| 1 \pm \frac{U_0}{2c_0/\sqrt{\varepsilon}} \right| \leq 1 + \frac{|U_0| \sqrt{\varepsilon}}{2c_0}, \quad \left| \frac{r_2 \cdot l_{\pm}^{\varepsilon}}{r_{\pm}^{\varepsilon} \cdot l_{\pm}^{\varepsilon}} \right| = \frac{|U_0| \sqrt{\varepsilon}}{2c_0}, \quad (4.7)$$

then

$$\sum_{j=1}^L \Delta x |\alpha_{\pm,j}^{n+1}|^2 \leq 2 \left(1 \pm \frac{U_0 \sqrt{\varepsilon}}{2c_0} \right)^2 \sum_{j=1}^L \Delta x |\alpha_{0,j}^n|^2 + \frac{|U_0|^2 \varepsilon}{2c_0^2} \sum_{j=1}^L \Delta x |\alpha_{2,j}^n|^2. \quad (4.8)$$

But,

$$\begin{aligned}
\sum_{j=1}^L \Delta x \left(|\rho_j^{n+1}|^2 + \frac{\varepsilon}{c_0^2} |q_j^{n+1}|^2 \right) &= \sum_{j=1}^L \Delta x \left((\alpha_{+,j}^{n+1} + \alpha_{-,j}^{n+1})^2 + (\alpha_{+,j}^{n+1} - \alpha_{-,j}^{n+1})^2 \right), \\
&= 2 \sum_{j=1}^L \Delta x \left(|\alpha_{+,j}^{n+1}|^2 + |\alpha_{-,j}^{n+1}|^2 \right), \\
&\leq 4 \left[\left(1 - \frac{U_0 \sqrt{\varepsilon}}{2c_0} \right)^2 + \left(1 + \frac{U_0 \sqrt{\varepsilon}}{2c_0} \right)^2 \right] \sum_{j=1}^L \Delta x |\alpha_{0,j}^n|^2 + \frac{2|U_0|^2 \varepsilon}{c_0^2} \sum_{j=1}^L \Delta x |\alpha_{2,j}^n|^2, \\
&\leq \left(2 + \frac{U_0^2 \varepsilon}{2c_0^2} \right) \sum_{j=1}^L \Delta x |\rho_j^n|^2 + \frac{4\varepsilon}{c_0^2} \sum_{j=1}^L \Delta x |q_j^n|^2 + \frac{|U_0|^2 \varepsilon}{c_0^2} \sum_{j=1}^L \Delta x |\rho_j^n|^2, \\
&\leq \left(4 + \frac{3U_0^2 \varepsilon}{2c_0^2} \right) \sum_{j=1}^L \Delta x \left(|\rho_j^n|^2 + \frac{\varepsilon}{c_0^2} |q_j^n|^2 \right).
\end{aligned}$$

Now, let us turn to the second part of the proposition, setting $D_i^\varepsilon = c_0/(2\sqrt{\varepsilon})$ gives

$$\begin{aligned}
Z_j^{n+1} &= \left[\alpha_{-,j}^{n+1} + \frac{c_0 \Delta t}{\sqrt{\varepsilon} \Delta x} (\alpha_{-,j}^{n+1} - \alpha_{-,j+1}^{n+1}) \right] r_-^\varepsilon \\
&\quad + \left[\alpha_{+,j}^{n+1} + \frac{c_0 \Delta t}{\sqrt{\varepsilon} \Delta x} (\alpha_{+,j}^{n+1} - \alpha_{+,j-1}^{n+1}) \right] r_+^\varepsilon.
\end{aligned}$$

Denoting by $j_0 \in \{1, \dots, L\}$ the integer such that $|\alpha_{-,j_0}^{n+1}| = \max_{j=1}^L |\alpha_{-,j}^{n+1}| = \max_{j=0}^{L+1} |\alpha_{-,j}^{n+1}|$, since α_{-,j_0}^{n+1} and $\alpha_{-,j_0}^{n+1} - \alpha_{-,j_0+1}^{n+1}$ have the same sign, we get for all $j \in \{1, \dots, L\}$

$$|\alpha_{-,j}^{n+1}| \leq |\alpha_{-,j_0}^{n+1}| \leq \left| \alpha_{-,j_0}^{n+1} + \frac{c_0 \Delta t}{\sqrt{\varepsilon} \Delta x} (\alpha_{-,j_0}^{n+1} - \alpha_{-,j_0+1}^{n+1}) \right| = \frac{|Z_{j_0}^{n+1} \cdot l_-|}{|r_- \cdot l_-|},$$

where, as before, a similar inequality holds for $|\alpha_{+,j}^{n+1}|$. Using the expression for B_j^n , $Z_j^{n+1} = B_j^n$ and (4.7), the previous inequalities and remarking that

$$\frac{|B_j^n \cdot l_0|}{|l_0 \cdot r_0|} \leq \max_{k=0}^{L+1} |\alpha_{0,k}^n| = \max_{k=1}^L |\alpha_{0,k}^n|, \quad \frac{|B_j^n \cdot l_2|}{|l_2 \cdot r_2|} \leq \max_{k=1}^L |\alpha_{2,k}^n|,$$

we get

$$\max_{j=1}^L |\alpha_{\pm,j}^{n+1}| \leq \left| 1 \pm \frac{U_0 \sqrt{\varepsilon}}{2c_0} \right| \max_{j=1}^L |\alpha_{0,j}^n| + \frac{|U_0| \sqrt{\varepsilon}}{2c_0} \max_{j=1}^L |\alpha_{2,j}^n|, \quad \text{for all } n \geq 0.$$

We conclude thanks to the following inequalities

$$\begin{aligned}
\max_{j=1}^L |\rho_j^{n+1}| &= \max_{j=1}^L |\alpha_{+,j}^{n+1} + \alpha_{-,j}^{n+1}| \leq 2 \left(1 + \frac{|U_0| \sqrt{\varepsilon}}{2c_0} \right) \max_{j=1}^L |\alpha_{0,j}^n| + \frac{|U_0| \sqrt{\varepsilon}}{c_0} \max_{j=1}^L |\alpha_{2,j}^n|, \\
&= \left(1 + \frac{|U_0| \sqrt{\varepsilon}}{2c_0} \right) \max_{j=1}^L |\rho_j^n| + \frac{|U_0| \sqrt{\varepsilon}}{c_0} \max_{j=1}^L |q_j^n / U_0 - \rho_j^n / 2|, \\
&\leq \left(1 + \frac{|U_0| \sqrt{\varepsilon}}{c_0} \right) \left(\max_{j=1}^L |\rho_j^n| + \frac{\sqrt{\varepsilon}}{c_0} \max_{j=1}^L |q_j^n| \right),
\end{aligned}$$

and

$$\frac{\sqrt{\varepsilon}}{c_0} \max_{j=1}^L |q_j^{n+1}| = \max_{j=1}^L |\alpha_{+,j}^{n+1} - \alpha_{-,j}^{n+1}| \leq \left(1 + \frac{|U_0| \sqrt{\varepsilon}}{c_0} \right) \left(\max_{j=1}^L |\rho_j^n| + \frac{\sqrt{\varepsilon}}{c_0} \max_{j=1}^L |q_j^n| \right).$$

5. The full Cartesian discretization for the non linear two-dimensional isentropic Euler system. In this Section, we focus on the space discretization of the two-dimensional system (2.1) discretized in time by scheme (3.3). This scheme can be recast in the uncoupled system (3.4) which has clearly advantages in terms of computational cost. This leads to consider two alternatives for the full space-time discretization both permitting to maintain the decoupled structure of the semi-discrete scheme proposed. The first possibility is to discretize the space derivative in equations (3.3) and successively reformulate the system as done in the semi-discrete case. Alternatively, we can directly discretize equations (3.4) which naturally gives an uncoupled fully discretized scheme. Even if the first choice seems more natural, it typically yields to space discretizations with larger stencils being the full scheme obtained by two consecutive applications of the discrete Lax-Friedrichs flux functions. Instead, the second choice permits to get less diffusive schemes since the order two terms ∇^2 : and Δ can be directly discretized by using compact stencils. In this work, we pursue the second possibility. Note that both possibilities yield a conservative scheme. To this aim, we rewrite system (3.4) as

$$\frac{W^{n+1} - W^n}{\Delta t^n} + \nabla \cdot F_e(W^n) + \nabla \cdot F_i(W^{n+1/2}) - \Delta t^n \begin{pmatrix} \frac{\Delta p(\rho)^{n+1}}{\varepsilon} + \nabla^2 : (\rho U \otimes U)^n \\ 0 \\ 0 \end{pmatrix} = 0,$$

where $W = (\rho, \rho u, \rho v)$, $F_e(W) = (0, \rho U \otimes U)$, $F_i(W^{n+1/2}) = ((\rho U)^n, p^{n+1}/\varepsilon Id)$ and $U = (u, v)$. We consider now a rectangular domain Ω and we define by $(x, y) \in \Omega$ the components of the space variables. We discretize on a non uniform Cartesian mesh with L_x and L_y the number of cells in the two spatial directions. For all $t \geq 0$ the unknown $W(x, t)$ is approximated by the piecewise constant approximation $W_{k,j}^n = (\rho_{k,j}^n, (\rho u)_{k,j}^n, (\rho v)_{k,j}^n)$ with mesh steps respectively $\Delta x_k = x_{k+1/2} - x_{k-1/2}$ and $\Delta y_j = y_{j+1/2} - y_{j-1/2}$ with $k = 1, \dots, L_x$ and $j = 1, \dots, L_y$. In this setting, the full discretized scheme reads

$$\begin{aligned} \Delta x_k \Delta y_j \frac{W_{k,j}^{n+1} - W_{k,j}^n}{\Delta t^n} + (\mathcal{F}_e)_{k+1/2,j}^n - (\mathcal{F}_e)_{k-1/2,j}^n + (\mathcal{F}_e)_{k,j+1/2}^n - (\mathcal{F}_e)_{k,j-1/2}^n \\ + (\mathcal{F}_i)_{k+1/2,j}^{n+1/2} - (\mathcal{F}_i)_{k-1/2,j}^{n+1/2} + (\mathcal{F}_i)_{k,j+1/2}^{n+1/2} - (\mathcal{F}_i)_{k,j-1/2}^{n+1/2} \\ - \Delta t^n \Delta x_k \Delta y_j \begin{pmatrix} \frac{(\Delta p(\rho))_{k,j}^{n+1}}{\varepsilon} + (\nabla^2 : (\rho U \otimes U))_{k,j}^n \\ 0 \\ 0 \end{pmatrix} = 0, \end{aligned}$$

for all $k = 1, \dots, L_x$ and all $j = 1, \dots, L_y$. In the above equation, the explicit fluxes considered are the Rusanov or modified Lax-Friedrichs solvers. They read

$$(\mathcal{F}_e)_{k+1/2,j}^n = \Delta y_j \left(\frac{F_e(W_{k,j}^n) + F_e(W_{k+1,j}^n)}{2} - (D_e)_{k+1/2,j}^n (W_{k+1,j}^n - W_{k,j}^n) \right),$$

where, following the analysis done in the linearized context and presented in the previous section (see remark 4.2), $(D_e)_{k+1/2,j}^n$ is the viscosity coefficient defined thanks to the eigenvalues of the Jacobian matrix associated to the explicit flux \mathcal{F}_e

$$(D_e)_{k+1/2,j}^n = \begin{cases} \max(|u_{k+1,j}^n|, |u_{k,j}^n|), & \text{for the Rusanov solver,} \\ \max_{\substack{k=1, \dots, L_x \\ j=1, \dots, L_y}} |u_{k,j}^n|, & \text{for the modified Lax-Friedrichs solver.} \end{cases}$$

Similarly in the y direction we have

$$(\mathcal{F}_e)_{k,j+1/2}^n = \Delta x_k \left(\frac{F_e(W_{k,j}^n) + F_e(W_{k,j+1}^n)}{2} - (D_e)_{k,j+1/2}^n (W_{k,j+1}^n - W_{k,j}^n) \right),$$

with

$$(D_e)_{k,j+1/2}^n = \begin{cases} \max(|v_{k,j+1}^n|, |v_{k,j}^n|), & \text{for the Rusanov solver,} \\ \max_{\substack{k=1, \dots, L_x \\ j=1, \dots, L_y}} |v_{k,j}^n|, & \text{for the modified Lax-Friedrichs solver.} \end{cases}$$

The implicit fluxes are discretized by the same type of numerical fluxes used for the explicit part where, however, the choice of the numerical diffusion is done accordingly to the results of the previous section. They read, in the x direction

$$(\mathcal{F}_i)_{k+1/2,j}^{n+1/2} = \Delta y_j \left(\frac{F_i(W_{k,j}^{n+1/2}) + F_i(W_{k+1,j}^{n+1/2})}{2} - (D_i)_{k+1/2,j}^n (W_{k+1,j}^{n+1} - W_{k,j}^{n+1}) \right),$$

where, following the analysis done in the linearized context and presented in the previous section (see remark 4.2), the viscosity coefficient can be either zero $(D_i)_{k+1/2,j}^n = 0$ (we expect in this case an L^2 stable scheme) or defined through the use of the eigenvalues of the Jacobian matrix associated to the implicit flux \mathcal{F}_i (we expect in this case a L^∞ stable scheme)

$$(D_i)_{k+1/2,j}^n = \frac{1}{2} \begin{cases} \max(\sqrt{p'(\rho_{k+1,j}^n)/\varepsilon}, \sqrt{p'(\rho_{k,j}^n)/\varepsilon}), & \text{for the Rusanov solver,} \\ \max_{\substack{k=1, \dots, L_x \\ j=1, \dots, L_y}} \sqrt{p'(\rho_{k,j}^n)/\varepsilon}, & \text{for the modified L-F solver.} \end{cases}$$

Similarly, in the y direction

$$(\mathcal{F}_i)_{k,j+1/2}^{n+1/2} = \Delta x_k \left(\frac{F_i(W_{k,j}^{n+1/2}) + F_i(W_{k,j+1}^{n+1/2})}{2} - (D_i)_{k,j+1/2}^n (W_{k,j+1}^{n+1} - W_{k,j}^{n+1}) \right),$$

with $(D_i)_{k,j+1/2}^n = 0$ or

$$(D_i)_{k,j+1/2}^n = \frac{1}{2} \begin{cases} \max(\sqrt{p'(\rho_{k,j+1}^n)/\varepsilon}, \sqrt{p'(\rho_{k,j}^n)/\varepsilon}), & \text{for the Rusanov solver,} \\ \max_{\substack{k=1, \dots, L_x \\ j=1, \dots, L_y}} \sqrt{p'(\rho_{k,j}^n)/\varepsilon}, & \text{for the modified L-F solver.} \end{cases}$$

Finally, concerning the discretization of the elliptic operator for the pressure, we set

$$\begin{aligned} (\Delta p(\rho))_{k,j}^{n+1} &= \frac{1}{\Delta x_k} \left(\frac{p(\rho_{k+1,j}^{n+1}) - p(\rho_{k,j}^{n+1})}{\Delta x_{k+1/2}} - \frac{p(\rho_{k,j}^{n+1}) - p(\rho_{k-1,j}^{n+1})}{\Delta x_{k-1/2}} \right) \\ &\quad + \frac{1}{\Delta y_j} \left(\frac{p(\rho_{k,j+1}^{n+1}) - p(\rho_{k,j}^{n+1})}{\Delta y_{j+1/2}} - \frac{p(\rho_{k,j}^{n+1}) - p(\rho_{k,j-1}^{n+1})}{\Delta y_{j-1/2}} \right), \end{aligned}$$

while the last operator is discretized by two consecutive applications of a central difference scheme

$$\begin{aligned}
(\nabla^2 : (\rho U \otimes U))_{k,j}^n &= \frac{1}{\Delta x_k} \left(\frac{(\rho u^2)_{k+1,j}^n - (\rho u^2)_{k,j}^n}{\Delta x_{k+1/2}} - \frac{(\rho u^2)_{k,j}^n - (\rho u^2)_{k-1,j}^n}{\Delta x_{k-1/2}} \right) \\
&+ \frac{1}{\Delta x_k \Delta y_j} \left(\frac{(\rho u v)_{k+1,j+1}^{n+1} - (\rho u v)_{k+1,j-1}^{n+1}}{2} - \frac{(\rho u v)_{k-1,j+1}^{n+1} - (\rho u v)_{k-1,j-1}^{n+1}}{2} \right) \\
&+ \frac{1}{\Delta y_j} \left(\frac{(\rho v^2)_{k,j+1}^n - (\rho v^2)_{k,j}^n}{\Delta y_{j+1/2}} - \frac{(\rho v^2)_{k,j}^n - (\rho v^2)_{k,j-1}^n}{\Delta y_{j-1/2}} \right),
\end{aligned}$$

where $\Delta x_{k+1/2} = (\Delta x_{k+1} + \Delta x_k)/2$ for all $k = 1, \dots, L_x - 1$, $\Delta x_{1/2} = \Delta x_1/2$ and $\Delta x_{L_x+1/2} = \Delta x_{L_x}/2$. Similarly, $\Delta y_{j+1/2} = (\Delta y_{j+1} + \Delta y_j)/2$ for all $j = 1, \dots, L_y - 1$, $\Delta y_{1/2} = \Delta y_1/2$ and $\Delta y_{L_y+1/2} = \Delta y_{L_y}/2$.

6. Numerical results. In this part, we present several numerical test cases to illustrate the main features of the new proposed method, the differences and the improvements with respect to classical explicit discretization schemes as the one discussed in (3.1). The explicit scheme employed for comparisons uses the same Rusanov numerical flux of our AP method where, however, standard numerical viscosity depending on the eigenvalues of the compressible Euler equation is used. We refer to this method to as the classical scheme (CL). We test our new method against the CL method on two one dimensional problems and on a two dimensional one. For every test studied well prepared initial and boundary conditions are employed [9, 23]. The analysis of numerical schemes for non well prepared initial data are postponed to a future work. The CL and the AP, introduced in (3.3), schemes are run for different choices of the Mach number ranging from compressible to incompressible flows. Simulations in compressible regimes permits to observe the behavior of our scheme in situations in which explicit schemes can be employed with acceptable time step sizes. Simulations in incompressible regimes permits to observe the capability of our method to capture the limit solution without the burden of solving all micro-structures due to the presence of fast waves in the fluid. For this last situation, we also study the possibility for the AP scheme to retrieve the fast scale behavior when the time step is artificially reduced. For all problems considered, we compare the numerical results, the number of time steps needed to get the final solution and the evolution of the time step in time for the CL and the AP schemes. This permits to extrapolate the computational costs related to the two methods.

6.1. Riemann problem from Degond-Tang [9]. We consider the same test case studied in [9]. The initial data are

$$\rho^0(x) = \begin{cases} 1 & \text{if } x \in [0, 0.2] \\ 1 + \varepsilon & \text{if } x \in]0.2, 0.3] \\ 1 & \text{if } x \in]0.3, 0.7] \\ 1 - \varepsilon & \text{if } x \in]0.7, 0.8[\\ 1 & \text{if } x \in [0.8, 1] \end{cases} \quad (\rho u)^0(x) = \begin{cases} 1 - \varepsilon/2 & \text{if } x \in [0, 0.2] \\ 1 & \text{if } x \in]0.2, 0.3] \\ 1 + \varepsilon/2 & \text{if } x \in]0.3, 0.7] \\ 1 & \text{if } x \in]0.7, 0.8[\\ 1 - \varepsilon/2 & \text{if } x \in [0.8, 1] \end{cases}$$

We choose an equation of state with $\gamma = 2$, which gives $p(\rho) = \rho^2$ for the pressure. The final time is fixed to $T_f = 0.051$ and 300 space cells are employed. In Figure 6.1, top panels, we report the solutions for the density on the left and the momentum on the right for the CL (red line) and the AP (blue line) schemes when the numerical

viscosity is fixed equal to zero for the implicit terms in AP, i.e. $D_i = 0$. A reference solution is also present which has been computed by the CL scheme with 3000 spatial cells. We observe that our scheme is able to reproduce the same results of the CL scheme with a slightly larger numerical diffusion due to its semi-implicit character. In addition, we clearly see some small overshoots after the shock on the left part of the domain (around $x \simeq 0.14$). These overshoots not only remain bounded but their intensity diminishes in time. They are linked to the choice of $D_i = 0$. This result is consistent with our conjecture (see Remark 4.2, point 1): we believe that we can only expect L^2 stability with $D_i = 0$ and not L^∞ stability. Figure 6.1 bottom panels presents the results of the same simulation where now the numerical diffusion of the implicit terms is not zero anymore, instead it has been fixed proportional to the Mach number as demanded in Proposition 4.1 to guarantee L^∞ estimates. As before, we report the results for the density on the left and the momentum on the right. As expected, since the implicit discretization is now upwind for the AP scheme, the spurious overshoots have completely disappeared, the AP scheme seems to be L^∞ stable. Remark that some additional dissipation is observed, this is due to the fact that the implicit upwind discretization has a larger diffusion than the centered one. Note also that, the price to pay, in order to have a scheme which assures a stronger stability results, is that the momentum equation requires the resolution of a linear system since the viscosity terms are implicit which may become costly in higher spatial dimensions.

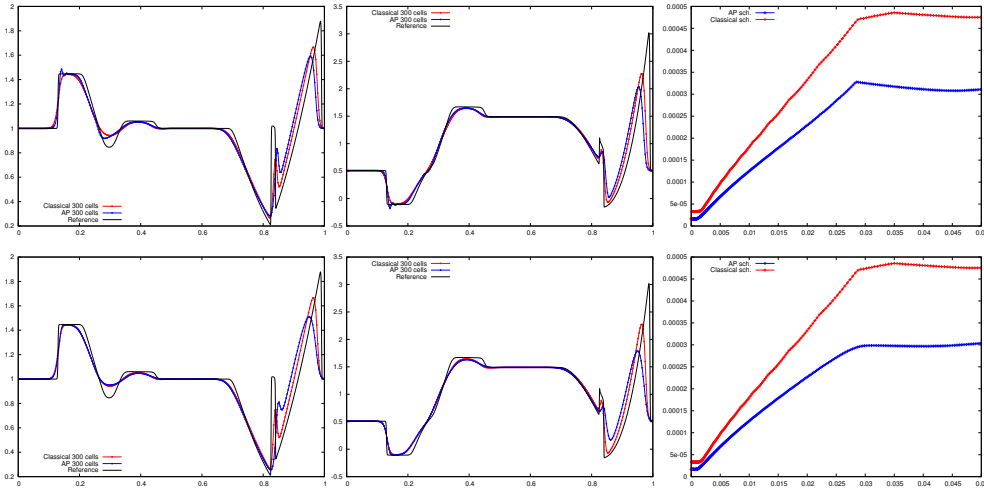


FIG. 6.1. *Solution of the Riemann problem at $t = 0.051$ for 300 cells and $\varepsilon = 0.99$. Left panel density ρ , middle panel momentum (ρu), right panel time steps Δt as a function of time. Reference solution in black line, classical explicit scheme CL in red and AP scheme in blue. Top panels: AP scheme without extra implicit dissipation ($D_i = 0$). Bottom panels: AP scheme with implicit dissipation ($D_i \neq 0$).*

In order to emphasize the L^2 and L^∞ stability behavior we show in Figure 6.2 the numerical density when $\varepsilon = 0.99$ for the scheme with $D_i = 0$ (red crosses) or $D_i \neq 0$ (blue squares). Two meshes made of 300 and 1000 cells respectively are considered. The panels are zoomed on domain $[0.125 : 0.27]$ for early times $t = 0.0125$ (left panels) and $t = 0.0171875$ (right panels).

Furthermore, in Figure 6.1 are also reported the values of the time steps Δt^n for

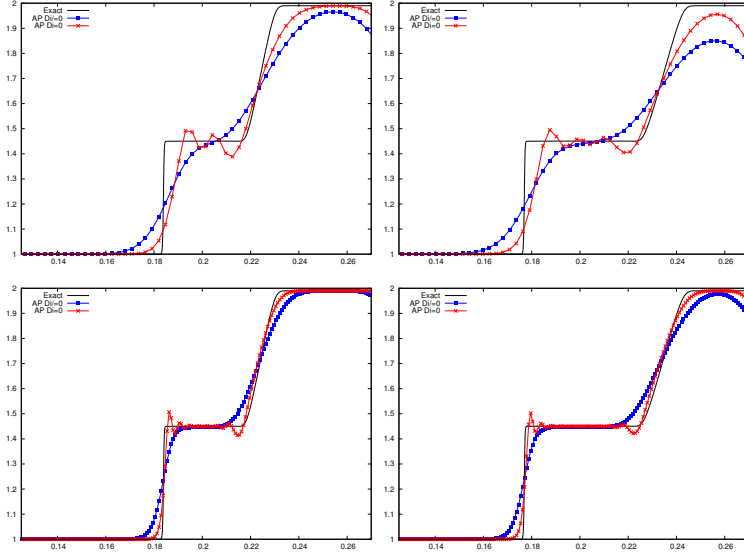


FIG. 6.2. Solution of the Riemann problem for 300 (top panels) and 1000 (bottom panels) cells and $\varepsilon = 0.99$. Intermediate time $t = 0.0125$ (left) and $t = 0.0171875$ (right) — Zooms on domain $[0.125 : 0.27]$ — Reference solution in black line, AP scheme with $D_i = 0$ in red (symbol \times) and with $D_i \neq 0$ in blue (symbol \blacksquare).

the CL and the AP schemes (with and without extra implicit dissipation). We observe that for this relatively large Mach numbers, our AP scheme demands same order but a little bit smaller time steps compared to explicit discretizations. On the other hand, the time steps restrictions for the L^2 ($D_i = 0$) and the L^∞ ($D_i \neq 0$) schemes are almost the same. In Figure 6.3, we report the behaviors of our scheme when the mesh is refined. In particular, we show simulation results for 300, 900 and 1500 cells. We observe that the numerical solution does converge towards the reference solution as expected.

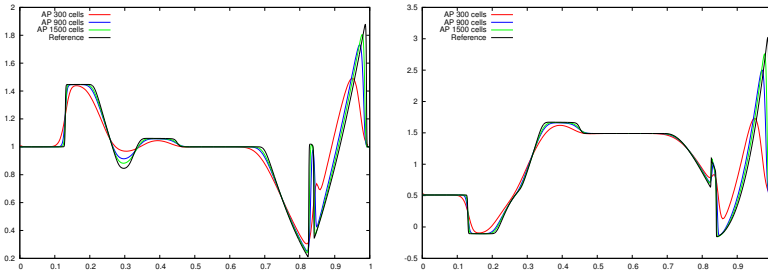


FIG. 6.3. Solution of the Riemann problem at $t = 0.051$ for $\varepsilon = 0.99$. The number of cells employed is 300 (red), 900 (blue) and 1500 (green). Left panel density ρ , right panel momentum (p_u). Convergence for the L^∞ stable AP scheme. Reference solution is in black line.

We focus now on a low Mach regime. In Figure 6.4 are presented the results for the two schemes, CL and AP, when a 300 cell mesh is employed. Blue line for the AP, red line for CL, reference solution black line. The reference solution has been computed by using the CL scheme with 3000 cells. On the left top panel the density

is reported while on the top right panel the momentum. In this case, the Mach number is $M = 0.01$ which gives $\varepsilon = M^2 = 10^{-4}$. The asymptotic stable scheme projects the solution on to the limit incompressible solution faster than the explicit scheme. This is exactly the scope of the method, we want to avoid to capture the micro-structures of the solution, instead the goal is to capture the limit solution at reasonable costs. To this aim, we report the size of time steps in the bottom panel of Figure 6.4. The time steps of the AP scheme are around $70 \approx 1/\sqrt{\varepsilon} = 100$ time smaller than the ones required by an explicit method. This shows that the AP scheme can employ time steps which are independent with respect the small parameter ε while time steps of explicit schemes remain close to the Mach number values. In order to prove that the

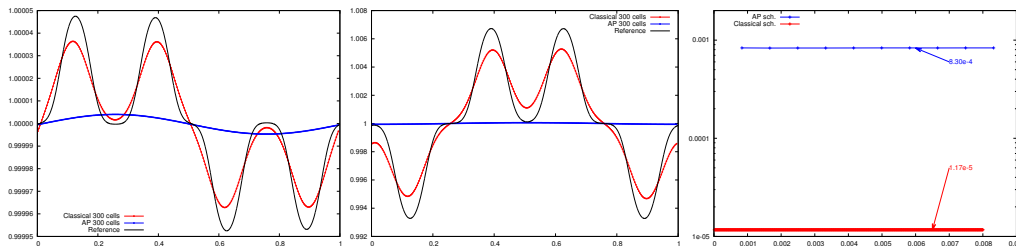


FIG. 6.4. Solution of the Riemann problem at $t = 0.008$ for $\varepsilon = 10^{-4}$ and 300 cells for the AP and CL schemes. Left panel density ρ , middle panel momentum (ρu), right panel time steps Δt as a function of time. Reference solution in black line, CL scheme red line and AP scheme blue line.

AP scheme converges to the correct solution we report in Figure 6.5 the same results as in Figure 6.4 but with a small time step for the AP scheme. The AP scheme employs 604 time steps to capture this new solution shown in the Figure while the CL scheme 684 time steps. From the results we can conclude that the AP scheme is able to compute the correct rate of convergence to the limit solution if the time step is reduced and made closer to the time step of the explicit method.

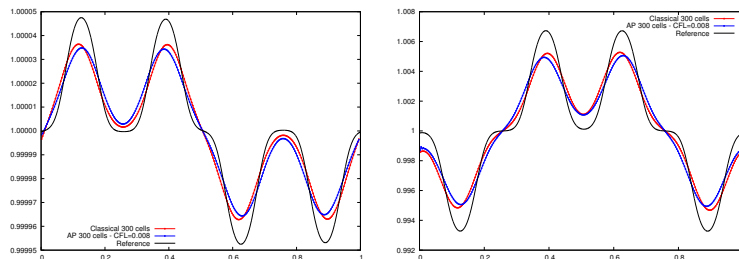


FIG. 6.5. Solution of the Riemann problem at $t = 0.008$ for $\varepsilon = 10^{-4}$ and 300 cells for the AP and the CL schemes. Left panel density ρ , right panel momentum (ρu). Reference solution in black line, CL red line and AP scheme blue line. The time step of the AP scheme is reduced to the order of the time step of the explicit scheme.

We finally measure in table 6.1 the number of time steps and the CPU times needed to reach the final time for different values of the Mach number, from compressible to incompressible situations.

As expected, we observe that in the case of large ε the cost of our AP scheme is more important than the classical one, even if the costs are still comparable. This is due to the time needed to invert the linear system which characterizes the AP

	Mach #	# time steps		Ratio	CPU time		Ratio
		CL	AP	CL/AP	CL	AP	CL/AP
300 cells	$\varepsilon = 1$	150	245	0.6	0.076s	0.12 s	0.63
	$\varepsilon = 0.9$	51	94	0.5	0.066 s	0.0083 s	0.8
	$\varepsilon = 0.75$	24	39	0.6	0.035 s	0.0060 s	0.58
	$\varepsilon = 0.5$	18	20	0.9	0.031 s	0.0033 s	0.94
	$\varepsilon = 0.25$	21	16	1.3	0.031 s	0.0025 s	1.2
	$\varepsilon = 10^{-1}$	28	14	2.0	0.039 s	0.0033 s	1.2
	$\varepsilon = 10^{-2}$	74	11	6.7	0.078 s	0.0057 s	1.4
	$\varepsilon = 10^{-3}$	220	10	22	0.085 s	0.0055 s	1.6
	$\varepsilon = 10^{-4}$	684	11	62	0.092 s	0.0054 s	1.7
	$\varepsilon = 10^{-5}$	2152	11	196	0.29 s	0.0051 s	5.7
	$\varepsilon = 10^{-6}$	6797	11	618	0.83 s	0.0027 s	30.7
	$\varepsilon = 10^{-7}$	21472	11	1952	2.58 s	0.0024 s	108
	$\varepsilon = 10^{-8}$	67688	10	6768	8.10 s	0.0026 s	311.5

TABLE 6.1

Number of time steps and CPU time needed for the CL and the AP schemes to reach $t = 0.008$ for the Riemann problem for different values of ε .

scheme. However, when ε becomes small enough then the AP scheme only needs 11 cycles to get to the final time whereas the number of iterations for the CL scheme increases indefinitely. The CPU time needed to update one AP cell in one time step is about constant of the order 4.265×10^{-6} s whereas for the CL scheme is obviously less expensive and of the order of 2.5×10^{-7} with our implementation.

Recall that, in the case of the scheme of [9], the authors choose the following discretization for the pressure term: $\alpha \nabla p(\rho^n) + (1/\varepsilon - \alpha) \nabla p(\rho^{n+1})$ where $\alpha > 0$ is a numerical parameter which must be correctly fixed to ensure stability and which depends on the considered problem. The value $\alpha = 0$ leading to non physical oscillations, in general, the authors choose the value $\alpha = 1$ and a time step Δt fixed to a constant value during all the simulations. They also propose an explicit formula for the time step which depends on α , however it does not seem sufficient to erase the non physical oscillations.

6.2. Time varying Mach number test. In this part we consider a Riemann problem where the Mach number changes with time. In these cases, if one is interested only to the main features described by the limit incompressible solution and not to the fast scale dynamics which ends in the limit problem, the use of an Asymptotic Preserving scheme is indicated since it permits to bypass the solution of the fast waves without losing the asymptotic behaviors. In Figure 6.6, on the left panel, is shown the value of $\varepsilon(t)$ as a function of time in logarithmic scale. The schemes, CL and AP, use both 600 spatial cells to furnish a solution. On the right panel is shown the time steps needed by the two methods as a function of time in logarithmic scale. As expected while the explicit scheme has a time step which follows the variation of $\varepsilon(t)$, the AP scheme is able to maintain a relative constant time step throughout the simulation. The total number of time steps needed by our scheme is 163 while an explicit method needs 24897 steps in order to remain stable. In Figure 6.7 we report the density profiles for eight intermediate times for both schemes with embedded a Figure showing the squared Mach number $\varepsilon(t)$ with a vertical line showing the current time of the plot. Up to time $t = 0.05$ the AP scheme still captures the main structures of the solution but with additional dissipation (first two rows). However, we see that a larger time step can be used compared to CL. When ε jumps to small values (third row) then the AP scheme bypasses the solution of the small scale dynamics and starts to project the solution to the limit incompressible solution but still it remains able to capture the main phenomena. Finally, when ε decreases to 0, our scheme captures

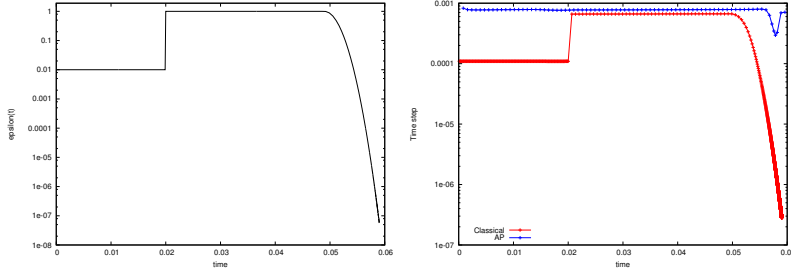


FIG. 6.6. Riemann problem with time varying Mach number at $t = 0.06$ for 600 cells. Left panel shows the value of $\epsilon(t)$ as a function of time. Right panel shows time steps as a function of time. The scales are logarithmic. The number of time steps is 163 for the AP scheme and 24897 for the classical explicit scheme.

the limit solution which is 1, while the CL scheme still tries to solve the small scale paying the price of extremely small time steps.

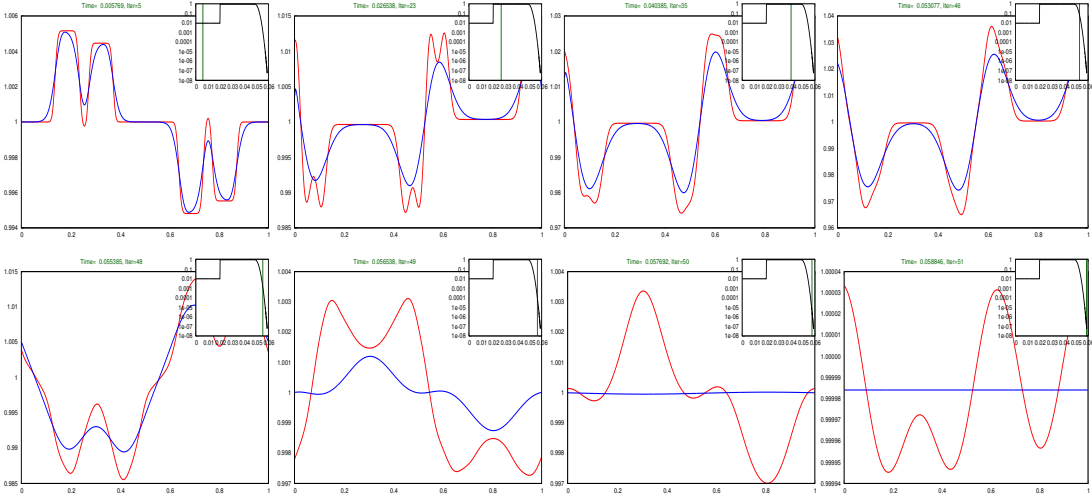


FIG. 6.7. Time varying Mach number test. CL (red line) and AP (blue line), 600 cells. Snapshots of the density at different times. From left to right: eight different intermediate times corresponding to different $\epsilon(t)$ values (straight vertical green line on the embedded panels).

6.3. Double rarefaction test problem. In this test we measure the errors produced by our scheme. To this aim, we have computed an exact solution of the isentropic Euler equations composed by two rarefaction waves emanating from the following initial condition

$$\rho^0(x) = \begin{cases} 1 + \varepsilon & \text{if } x \in [0, 0.5], \\ 1 & \text{if } x \in]0.5, 1.0], \end{cases} \quad (\rho u)^0(x) = \begin{cases} (1 + \varepsilon)(1 - \sqrt{\varepsilon}) & \text{if } x \in [0, 0.5], \\ 1 + \sqrt{\varepsilon} & \text{if } x \in]0.5, 1.0]. \end{cases}$$

We choose an equation of state with $\gamma = 2$, which gives $p(\rho) = \rho^2$ for the pressure. The structure of the exact solution is constituted of left and right moving rarefaction waves separated by a constant state. We choose three different values for the squared Mach number i.e. $\varepsilon = M^2 = 0.99, 0.1, 0.001$. For these three values we compute the

solution up to final times 0.1, 0.05 and 0.007 for the AP scheme with respectively $D_i = 0$ and $D_i \neq 0$. The exact solution is calculated explicitly and projected onto a uniform mesh made of 100000 cells. The results of the error in L^2 and L^∞ norms are provided in Table 6.2 and Figure 6.8. They clearly show the convergence in L^2 norm, of the AP scheme for both space discretizations and for this particular test case. They also show the stability of the AP schemes in both norms. Note that for this particular test case, the AP scheme with a centered implicit discretization (i.e. $D_i = 0$) is L^∞ stable but we recall that this not always true, see Figure 6.2. Note also, that the L^∞ norm of the errors for $\varepsilon = 0.001$ seem saturated. This is due to the fact that in this case, the rarefaction waves drastically steepen as their size tends to 0 when ε tends to 0. Numerically speaking these rarefaction waves have a shock-like shape. Consequently, in these zones the consistency error is large but, this is a localized problem since the L^2 errors are converging. Furthermore, we recover good convergence rates (even for the L^∞ norm), if one considers more regular test cases with ill prepared initial conditions for the low Mach regime.

	N	AP $D_i = 0$				AP $D_i \neq 0$			
		L^2 error	rate	L^∞ error	rate	L^2 error	rate	L^∞ error	rate
$\varepsilon = 0.99$	200	2.04E-03	—	9.81E-02	—	2.45E-03	—	1.04E-01	—
	400	9.30E-04	1.136	7.09E-02	0.468	1.17E-03	1.061	7.91E-02	0.399
	800	4.05E-04	1.200	5.05E-02	0.490	5.36E-04	1.131	5.89E-02	0.425
	1600	1.70E-04	1.249	3.57E-02	0.501	2.36E-04	1.185	4.33E-02	0.444
	3200	6.68E-05	1.350	2.45E-02	0.542	9.78E-05	1.269	3.09E-02	0.484
	6400	2.54E-05	1.393	1.71E-02	0.518	3.94E-05	1.313	2.20E-02	0.493
$\varepsilon = 0.1$	200	1.88E-04	—	4.89E-02	—	2.13E-04	—	4.90E-02	—
	400	1.17E-04	0.687	4.25E-02	0.202	1.34E-04	0.667	4.31E-02	0.184
	800	7.08E-05	0.719	3.74E-02	0.183	8.28E-05	0.698	3.91E-02	0.143
	1600	4.18E-05	0.763	3.15E-02	0.247	4.95E-05	0.741	3.28E-02	0.252
	3200	2.32E-05	0.850	2.61E-02	0.274	2.80E-05	0.823	2.79E-02	0.236
	6400	1.22E-05	0.925	2.09E-02	0.316	1.50E-05	0.898	2.24E-02	0.317
$\varepsilon = 0.001$	200	7.93E-08	—	6.77E-04	—	7.99E-08	—	6.77E-04	—
	400	6.11E-08	0.377	6.62E-04	0.0325	6.16E-08	0.377	6.60E-04	0.0347
	800	4.47E-08	0.451	6.23E-04	0.0873	4.52E-08	0.447	6.22E-04	0.0866
	1600	3.23E-08	0.466	6.11E-04	0.0269	3.28E-08	0.464	6.12E-04	0.0244
	3200	2.35E-08	0.461	6.52E-04	-0.092	2.38E-08	0.461	6.51E-04	-0.090
	6400	1.62E-08	0.532	6.29E-04	0.0500	1.65E-08	0.532	6.29E-04	0.0496

TABLE 6.2

Double rarefaction test problem — L^2 and L^∞ errors and rates of convergence for the AP schemes with a centered implicit discretization $D_i = 0$ or with an upwind implicit discretization $D_i \neq 0$.

6.4. 2D isentropic cylindrical explosion problem. In this section we focus on a genuinely 2D cylindrical problem with $\gamma = 1$. The computational domain is set to $\Omega = [0, 1]^2$ paved with $L_x \times L_y$ uniform cells. The initial density is given by

$$\rho_{k,j} = \begin{cases} 1 + \varepsilon & \text{if } r_{k,j}^2 \leq 1/4 \\ 1 & \text{else} \end{cases},$$

where $r_{k,j}$ is the distance from the cell center. The velocity is set as $u_{k,j} = -\alpha_{k,j}x_i / (r_{k,j}\rho_{k,j})$, $v_{k,j} = -\alpha_{k,j}y_j / (r_{k,j}\rho_{k,j})$, where the coefficients $\alpha_{k,j}$ are given by $\alpha_{k,j} =$

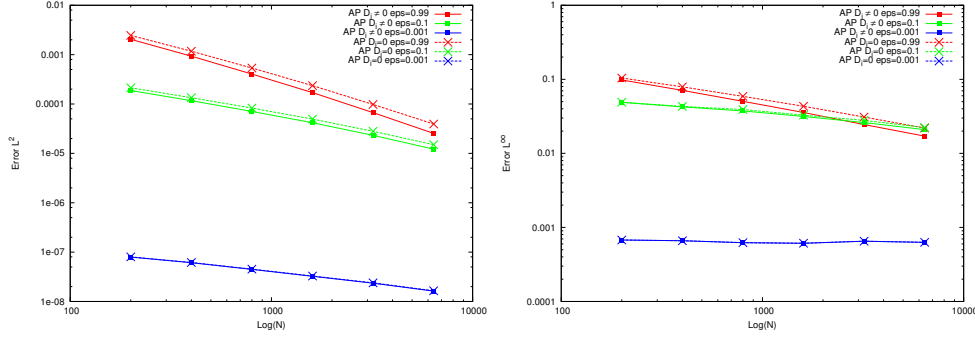


FIG. 6.8. Double rarefaction test case. Logscale of the L^2 and L^∞ norms of the error for the density for the AP schemes with or without implicit diffusion (denoted APi and AP respectively) as a function of the number of cells for $\varepsilon = 0.99, 0.1, 0.001$ (in red, green, blue respectively). Data from table 6.2.

$\max(0, 1 - r_{k,j})(1 - e^{-16r_{k,j}^2})$ and we set $u_{k,j} = v_{k,j} = 0$ if $r_{k,j} \leq 10^{-15}$, see figure 6.9 for a representation of the initial data. Periodic boundary conditions are set everywhere. We take 200 cells in both directions. We compute the solution for different values of the squared Mach number ε by using our AP scheme and the CL scheme for $\varepsilon = 1$ and $\varepsilon = 10^{-4}$. The AP scheme uses $D_i = 0$ which means that we consider the L^2 stable scheme. In Figures 6.10-6.11 are presented the density variable and the velocity

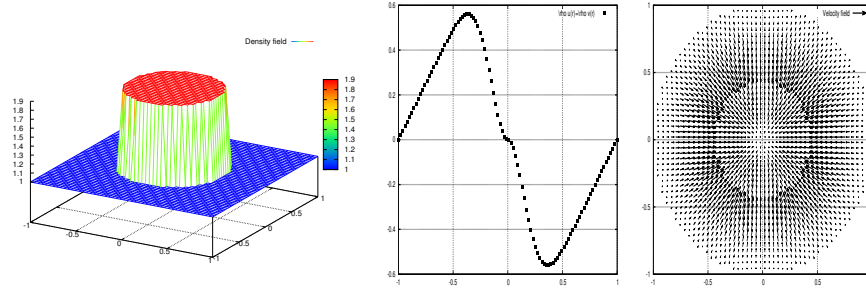


FIG. 6.9. 2D cylindrical problem for $\varepsilon = 1$. Initial density left panel, (ρu) and (ρv) as a function of the distance from the center middle panel, two dimensional velocity field right panel.

field for the CL and the AP schemes (top and bottom panels respectively) for $\varepsilon = 1.0$ at $t = 0.1$ (short time), $t = 0.24$ (focusing of waves) and $t_{\text{final}} = 0.5$ the final time. From these Figures, we can observe that the AP scheme can reproduce the numerical results of a classical explicit scheme in the case $\varepsilon = 1$. Concerning the time steps, the explicit scheme demands 144 iterations while the AP scheme needs only 64 iterations to get to the final solution. However the AP scheme scheme being more expensive due to its implicit character, the ratio of CPU time is still of order 3 in favor of the explicit method. Finally, in Figures 6.12 are presented the density variable and the velocity field for the CL and AP schemes (top and bottom panels respectively) for $\varepsilon = 10^{-4}$, i.e. close to the incompressible limit for a final time $t_{\text{final}} = 0.05$. In this case, the classical explicit scheme demands 1109 time steps while the AP scheme only needs 3 time steps to get to the final solution. From the computational point of view, the AP scheme is 5 time less expensive in our implementation. From the images, we see that while the AP scheme has captured the limit solution the explicit scheme is

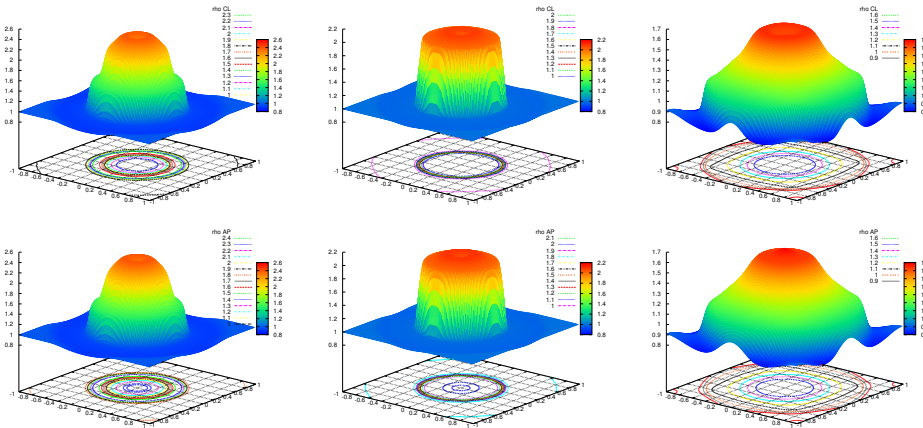


FIG. 6.10. 2D cylindrical problem for $\varepsilon = 1$ at $t = 0.1$ left panel, $t = 0.24$ middle panel and $t_{final} = 0.5$ right panel. Top images show the density profile for the CL scheme. Bottom images show the density profile for the AP scheme.

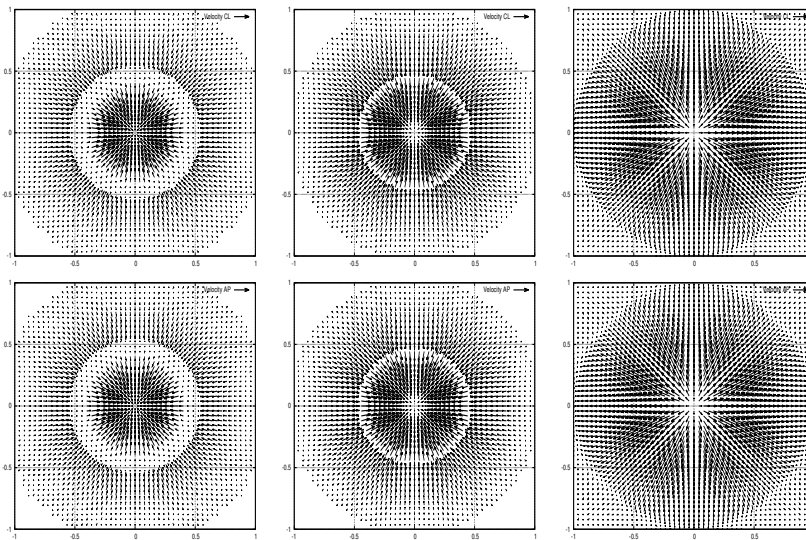


FIG. 6.11. 2D cylindrical problem for $\varepsilon = 1$ at $t = 0.1$ left panel, $t = 0.24$ middle panel and $t_{final} = 0.5$ right panel. Top images show the velocity field for the CL scheme. Bottom images show the velocity field for the AP scheme.

still resolving small waves dynamic. The difference in time steps between the two schemes being of the order of the Mach number $\sqrt{\varepsilon}$.

7. Extension to the full Euler system. In this Section we study the extension of our scheme to the case of the full Euler system. This system in rescaled variables reads

$$\partial_t \rho + \nabla \cdot (\rho U) = 0, \quad (7.1a)$$

$$\partial_t (\rho U) + \nabla \cdot (\rho U \otimes U) + \frac{1}{\varepsilon} \nabla p = 0, \quad (7.1b)$$

$$\partial_t E + \nabla \cdot ((E + p) U) = 0, \quad (7.1c)$$

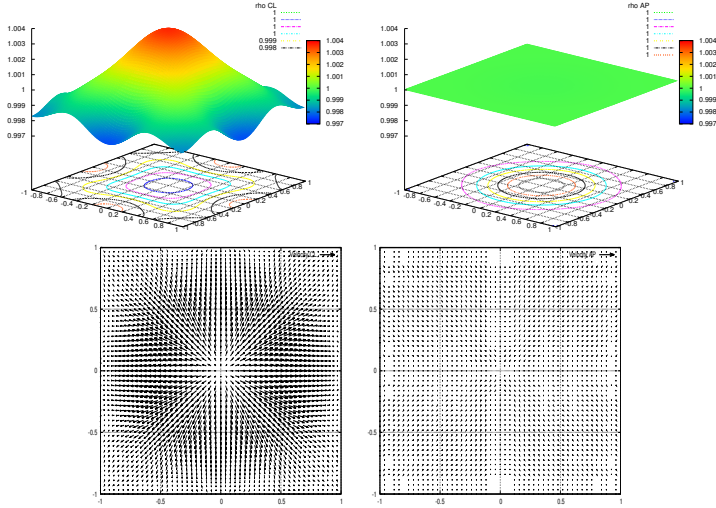


FIG. 6.12. 2D cylindrical problem for $\varepsilon = 10^{-4}$ at $t_{\text{final}} = 0.05$. Top panel density for the CL and the AP schemes. Bottom panel velocity field.

supplemented with the following equation of state

$$p = (\gamma - 1) \left(E - \varepsilon \frac{\rho |U|^2}{2} \right), \quad (7.1d)$$

where $\gamma \geq 1$ is the ratio of specific heats. Like for the isentropic case, appropriate initial and boundary values should be provided

$$U(t, x) \cdot \nu(x) = 0, \text{ for all } x \in \partial\Omega \text{ and } t > 0, \quad (7.1e)$$

$$(\rho, U, p)(0, x) = (\rho^0(x), U^0(x), p^0(x)), \text{ for all } x \in \Omega, \quad (7.1f)$$

where the initial condition is well prepared to the low-Mach number regime

$$p^0(x) = p_0 + \varepsilon \tilde{p}_0(x) \geq 0, \quad \text{and} \quad U^0(x) = U_0(x) + \sqrt{\varepsilon} \tilde{U}_0(x), \quad \nabla \cdot U_0(x) = 0, \quad (7.1g)$$

with $p_0 > 0$ a given constant. In this case, the low Mach number limit gives the incompressible Euler system, see [27], [1]

$$\partial_t \rho + \nabla \cdot (\rho U) = 0, \quad \nabla \cdot U = 0, \quad \partial_t (\rho U) + \nabla \cdot (\rho U \otimes U) + \nabla \pi_1 = 0,$$

with the pressure given by $p(t, x) = (\gamma - 1) E(t, x) = p_0$ and where the constraint $\nabla \cdot U = 0$ implicitly defines the pressure correction $\pi_1 = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (p - p_0) < +\infty$. It is explicitly given by the equation $-\nabla \cdot \left(\frac{1}{\rho} \nabla \pi_1 \right) = \nabla \cdot ((U \cdot \nabla) U)$.

7.1. The semi AP discretization in time. The scheme for the full Euler system follows the isentropic case strategy. The direct extension to this previous

scheme to the full Euler case reads

$$\begin{aligned}
\frac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla \cdot q^n &= 0, \\
\frac{q^{n+1} - q^n}{\Delta t} + \nabla \cdot \left(\frac{q \otimes q}{\rho} \right)^n - \frac{\gamma - 1}{2} \nabla \cdot \left(\frac{|q|^2}{\rho} \right)^n + \frac{\gamma - 1}{\varepsilon} \nabla E^{n+1} &= 0, \\
\frac{E^{n+1} - E^n}{\Delta t} + \nabla \cdot \left(\frac{\gamma E q}{\rho} \right)^{n+1} - \nabla \cdot \left(\frac{(\gamma - 1) \varepsilon |q|^2 q}{\rho^2} \right)^n &= 0, \\
p^{n+1} &= (\gamma - 1) \left(E^{n+1} - \varepsilon \frac{(\rho U^2)^n}{2} \right),
\end{aligned}$$

where we have used $\frac{(E+p)}{\rho} = \frac{\gamma E}{\rho} - \frac{\varepsilon(\gamma-1)}{2} \frac{|q|^2}{\rho}$. We observe that in [7] a similar algorithm is proposed but the term $1/\varepsilon \nabla E^{n+1}$ is changed into $\alpha \nabla E^n + (1/\varepsilon - \alpha) \nabla E^{n+1}$ where $\alpha > 0$, and the energy flux is replaced by $\nabla \cdot \left[\left(\frac{\gamma E}{\rho} - \frac{(\gamma-1)\varepsilon}{2} \frac{|q|^2}{\rho^2} \right)^n q^{n+1} \right]$. We recall that α is a numerical parameter which must be correctly fixed to ensure the uniform stability. Here, like in the isentropic case, we choose $\alpha = 0$. Furthermore, we prefer this implicit-explicit decomposition, because it leads to a ‘‘conservative splitting’’ in the sense that the flux can be expressed into the form $\nabla \cdot F_i(W^{n+1}) + \nabla \cdot F_e(W^n)$. Such conservative splittings allow straightforward extensions to general solvers like Roe, HLLC... Obviously, we must carefully choose the numerical space viscosity of the scheme for stability reasons. However, unfortunately, this approach has the main drawback that the Jacobian matrix associated to the explicit flux F_e may have complex eigenvalues (for $\gamma < 3$). This means that the previous implicit-explicit decomposition is ‘‘conservative’’ but not hyperbolic. In order to avoid this problem, we choose instead the following ‘‘non conservative’’ but hyperbolic splitting:

$$\begin{aligned}
\frac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla \cdot q^{n+1} &= 0, \\
\frac{q^{n+1} - q^n}{\Delta t} + \nabla \cdot \left(\frac{q \otimes q}{\rho} \right)^n - \frac{\gamma - 1}{2} \nabla \cdot \left(\frac{|q|^2}{\rho} \right)^n + \frac{\gamma - 1}{\varepsilon} \nabla E^{n+1} &= 0, \\
\frac{E^{n+1} - E^n}{\Delta t} + \nabla \cdot \left(\left(\frac{\gamma E}{\rho} - \frac{(\gamma - 1) \varepsilon |q|^2}{\rho^2} \right)^n q^{n+1} \right) &= 0, \\
p^{n+1} &= (\gamma - 1) \left(E^{n+1} - \varepsilon \frac{(\rho U^2)^n}{2} \right).
\end{aligned}$$

In one dimension, this semi-discretization can be rewritten in a compact form as

$$\frac{W^{n+1} - W^n}{\Delta t} + A_e(W^n, W^{n+1}) \partial_x W^n + A_i(W^n, W^{n+1}) \partial_x W^{n+1} = 0,$$

where A_e and A_i have real eigenvalues. In doing so, we retrieve the hyperbolicity of the system but the flux can no longer be written as a ‘‘conservative’’ implicit-explicit splitting $\partial_x F_i(W^{n+1}) + \partial_x F_e(W^n)$.

This scheme, as the one proposed for the isentropic system, has an uncoupled formulation. This can be obtained by inserting the value of q^{n+1} given by the momentum equation into the energy equation. This gives the following linear parabolic

equation for the energy at time $n + 1$ which furnishes the energy E^{n+1} :

$$\begin{aligned} & \frac{E^{n+1} - E^n}{\Delta t} + \nabla \cdot \left(\left(\frac{\gamma E}{\rho} - \frac{(\gamma - 1)\varepsilon}{2} \frac{|q|^2}{\rho^2} \right)^n q^n \right) \\ & - \Delta t \nabla \cdot \left(\left(\frac{\gamma E}{\rho} - \frac{(\gamma - 1)\varepsilon}{2} \frac{|q|^2}{\rho^2} \right)^n \left(\nabla \cdot \left(\frac{q \otimes q}{\rho} \right)^n - \frac{\gamma - 1}{2} \nabla \left(\frac{|q|^2}{\rho} \right)^n \right) \right) \\ & - \frac{\Delta t (\gamma - 1)}{\varepsilon} \nabla \cdot \left(\left(\frac{\gamma E}{\rho} - \frac{(\gamma - 1)\varepsilon}{2} \frac{|q|^2}{\rho^2} \right)^n \nabla E^{n+1} \right) = 0. \end{aligned}$$

Then, using the knowledge of the energy, one can update the momentum equation to get q^{n+1} and finally the mass equation gives ρ^{n+1} . This decoupled formulation is of fundamental importance when dealing with multidimensional problems since it avoids the inversion of very large linear systems. In one dimension, on a uniform mesh, denoting by $\mathcal{E}^n = \left(\frac{\gamma E}{\rho} - \frac{\varepsilon(\gamma-1)}{2} \frac{|q|^2}{\rho} \right)^n$ and remarking that $\nabla \cdot \frac{q \otimes q}{\rho} - \frac{\gamma-1}{2} \nabla \frac{|q|^2}{\rho} = \frac{3-\gamma}{2} \partial_x(\rho U)$, the full discretization is the following. First, we solve the following tridiagonal system for the energy:

$$\begin{aligned} & \frac{E_j^{n+1} - E_j^n}{\Delta t} \\ & - \frac{\Delta t (\gamma - 1)}{\varepsilon \Delta x} \left(\frac{\mathcal{E}_{j+1}^n + \mathcal{E}_j^n}{2} \left(\frac{E_{j+1}^{n+1} - E_j^{n+1}}{\Delta x} \right) - \frac{\mathcal{E}_j^n + \mathcal{E}_{j-1}^n}{2} \left(\frac{E_j^{n+1} - E_{j-1}^{n+1}}{\Delta x} \right) \right) \\ & = - \frac{\mathcal{F}_{j+1/2}^{E,n} - \mathcal{F}_{j-1/2}^{E,n}}{\Delta x} + \\ & \frac{\Delta t (3 - \gamma)}{2 \Delta x} \left(\frac{\mathcal{E}_{j+1}^n + \mathcal{E}_j^n}{2} \left(\frac{(\rho U)_{j+1}^{n+1} - (\rho U)_j^{n+1}}{\Delta x} \right) - \frac{\mathcal{E}_j^n + \mathcal{E}_{j-1}^n}{2} \left(\frac{(\rho U)_j^{n+1} - (\rho U)_{j-1}^{n+1}}{\Delta x} \right) \right), \end{aligned}$$

where $\mathcal{F}_{j+1/2}^{E,n}$ is the flux given by

$$\mathcal{F}_{j+1/2}^{E,n} = \frac{(\mathcal{E} q)_{j+1}^n + (\mathcal{E} q)_j^n}{2} - (D_e)_{j+1/2}^n (E_{j+1}^n - E_j^n),$$

with a viscosity coefficient related to the eigen values of the matrix A_e , then $(D_e)_{j+1/2}^n = \max(|3 - \gamma|, \gamma) \max\left(\frac{|q_{j+1}^n|}{\rho_{j+1}^n}, \frac{|q_j^n|}{\rho_j^n}\right)$. Once E^{n+1} is computed, p^{n+1} can be determined and we can explicitly determine q^{n+1} and after ρ^{n+1} , using the following equations

$$q_j^{n+1} = q_j^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}_{j+1/2}^{q,n+1/2} - \mathcal{F}_{j-1/2}^{q,n+1/2} \right), \quad \rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}_{j+1/2}^{\rho,n+1/2} - \mathcal{F}_{j-1/2}^{\rho,n+1/2} \right),$$

where

$$\begin{aligned} \mathcal{F}_{j+1/2}^{q,n+1/2} &= \frac{1}{2} \left(\left(\frac{q^2}{\rho} \right)_{j+1}^n + \frac{p_{j+1}^{n+1}}{\varepsilon} + \left(\frac{q^2}{\rho} \right)_j^n + \frac{p_j^{n+1}}{\varepsilon} \right) - (D_e)_{j+1/2}^n (q_{j+1}^n - q_j^n), \\ \mathcal{F}_{j+1/2}^{\rho,n+1/2} &= \frac{q_{j+1}^{n+1} + q_j^{n+1}}{2} - (D_e)_{j+1/2}^n (\rho_{j+1}^n - \rho_j^n). \end{aligned}$$

The timestep constraint is given by: $\Delta t \leq \frac{\Delta x}{2 \max_j (|U_j^n|)}$.

Note that, even if the splitting is not ‘‘conservative’’, the scheme is by construction conservative in mass, momentum and total energy because it is built in a flux form, likewise the scheme in the isentropic case.

7.2. Numerical tests. In this last part, we present two numerical test cases to illustrate the behavior of this new scheme for the full Euler system in the one dimensional setting. We use the same Rusanov numerical fluxes employed for the isentropic case and a C.F.L. condition related to the eigenvalues of the explicit Jacobian matrix A_e . This gives a time step limitation of type

$$\Delta t^n \leq \frac{\Delta x}{\max(|(3 - \gamma) U^n|, \gamma |U^n|)}.$$

7.2.1. Riemann like problem. In this problem, the computational domain is set to $\Omega = (0, 1)$ with periodic boundary conditions on both sides. Density and pressure are uniform and set to 1 with $\gamma = 7/5$. The velocity is defined by $U(x) = 1 - \varepsilon/2$ if $x \leq 0.2$ or $x \geq 0.8$, $1 + \varepsilon/2$ if $x \leq 0.75$ or $x \geq 0.25$ and 1 elsewhere. The total energy E is deduced from the equation of state when ε is fixed as $p = (\gamma - 1) \left(E - \varepsilon^2 \frac{\rho U^2}{2} \right)$. The computations are stopped at the final time $t_{final} = 0.05$ and the square of the Mach number ε is chosen equal to 0.99 and 10^{-4} . In Figure 7.1 are depicted the pressure and velocity variables for the explicit classical (first order explicit Euler scheme) and AP schemes when 300 cells are employed. A reference solution is also present which has been obtained with the CL scheme and 10000 cells. From these Figures, we observe that both schemes produce rather equivalent numerical solutions although the AP scheme is a little bit more dissipative. The AP scheme demands 40 cycles while the CL one 45. The CPU time is about 0.08 for the AP scheme and 0.02 for the CL. The right panel of Figure 7.1 presents the time steps as a function of time in logarithmic scale for both schemes showing that, as expected, Δt is of the same order in the case $\varepsilon = 0.99$. In Figure 7.2 are reported the same

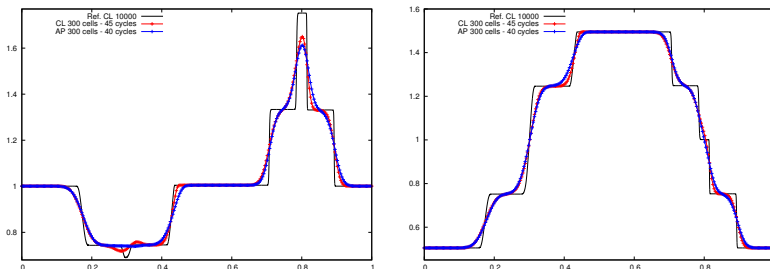


FIG. 7.1. Riemann problem for the one dimensional Euler system for $\varepsilon = 0.99$ at $t_{final} = 0.05$ for 300 cells. Left panel show the pressure profile, middle panel show the velocity profile. CL scheme is depicted in red while the AP scheme is depicted in blue. Reference solution is in black.

result as in Figure 7.1 with a Mach number squared of $\varepsilon = 10^{-4}$. The AP scheme demands only 27 cycles to furnish a solution while the CL 2177 time steps. The CPU time is about 1.02 for the CL scheme and 0.07 for the AP one. Moreover, we can observe in the panels that the AP scheme has captured the limit incompressible solution while the explicit scheme is still resolving the small scale dynamic. The time step employed by the AP scheme is of the order $\Delta t = 1.9 \times 10^{-3}$, while the CL scheme is restricted to $\Delta t = 2.3 \times 10^{-5}$ leading to a ratio of $\sqrt{\varepsilon} = 10^{-2}$ for the two time steps. We finally consider the solution of the Riemann problem in which our scheme is used with a restricted time step, i.e. Δt to 10^{-5} . In this case, we observe that we are able to retrieve the small scale structures of the solution lost in the case of larger time

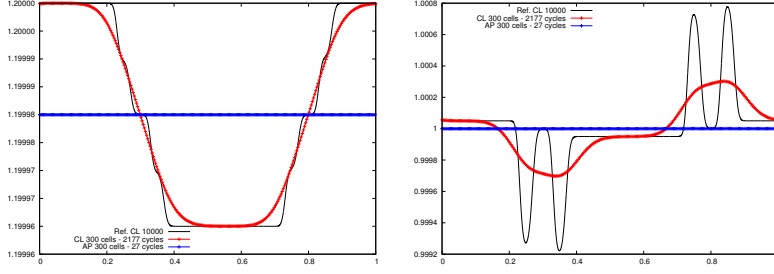


FIG. 7.2. Riemann problem for the one dimensional Euler system for $\varepsilon = 10^{-4}$ at $t_{final} = 0.05$ for 300 cells. Left panel show the pressure profile, middle panel show the velocity profile. CL scheme is depicted in red while the AP scheme is depicted in blue. Reference solution is in black.

steps. The results are reported in Figure 7.3 where a comparison between the explicit results and those of the AP scheme is shown. In this case the AP scheme demands 5000 cycles for a CPU time of the order of 6.2.

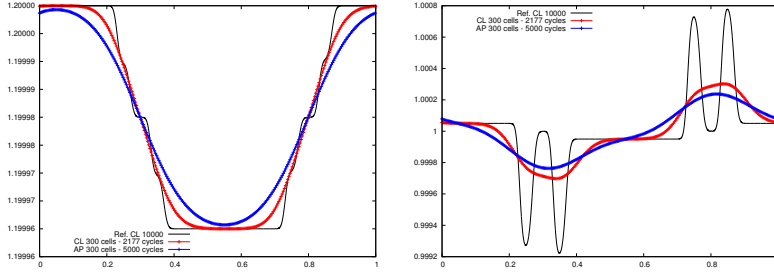


FIG. 7.3. Riemann problem for the one dimensional Euler system for $\varepsilon = 10^{-4}$ at $t_{final} = 0.05$ for 300 cells. Left panel show the pressure profile, right panel show the velocity profile. CL scheme is depicted in red while the AP scheme is depicted in blue. Reference solution is in black. The AP scheme employs a fixed time step $\Delta t = 10^{-5}$ whereas the Classical scheme employs a stability restricted time step like in the previous figure.

7.2.2. Colliding acoustic pulses. The last test we consider is a weakly compressible problem from [25, 31]. The computational domain is $\Omega = (-L, L)$ with $L = \frac{2}{\varepsilon}$. The initial setup consists of two colliding acoustic pulses in a weakly compressible regime

$$\rho(x, 0) = \rho_0 + \frac{\varepsilon}{2} \rho_1 \alpha(x), \quad U(x, 0) = \frac{-U_0}{2} \text{sign}(x) \varepsilon \rho_1 \alpha(x), \quad p(x, 0) = p_0 + \frac{\varepsilon}{2} p_1 \alpha(x),$$

where $\alpha(x) = 1 - \cos\left(\frac{2\pi x}{L}\right)$, $\rho_0 = 0.955$, $\rho_1 = 2$, $U_0 = 2\sqrt{\gamma}$, $p_0 = 1$ and $p_1 = 2\gamma$. In Figure 7.4 we report the results in term of density, velocity and energy at an intermediate time $t = 0.815$ and at the final time $t = 1.63$ along with the initial data for the CL and the AP schemes using 440 cells and $\varepsilon = 1/11$. We can observe that the AP scheme is a little more dissipative but capture the main physical phenomena. The CL scheme needs 1030 cycles to reach the final time while the AP scheme only demands 224 time steps leading to a ratio of about 4.6.

8. Conclusions. The purpose of this article is the development and the analysis of a new Asymptotic Preserving scheme which is able to deal with low Mach number fluid flows bypassing the resolution of the fast waves dynamics. We have proven the

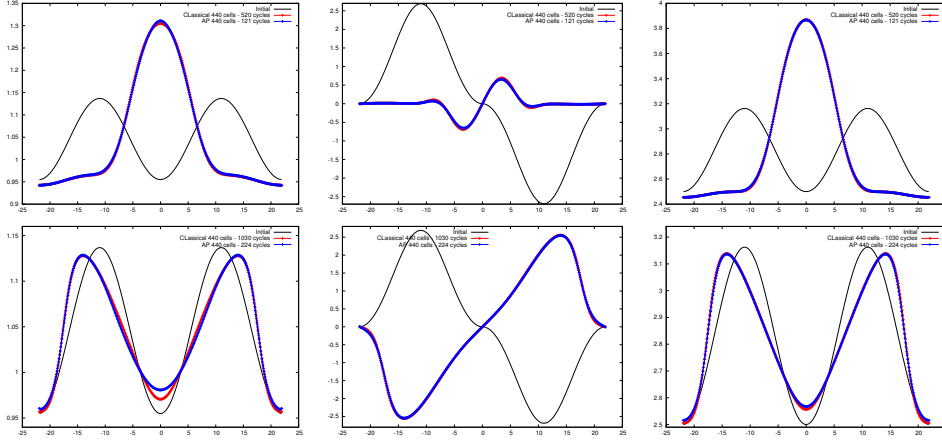


FIG. 7.4. Colliding acoustic pulses problem for the one dimensional Euler system for $\varepsilon = 1/11$ at $t = 0.815$ (top panels) and $t = 1.63$ (bottom panels) for 440 cells. Left panel show the density profile, middle panel shows the velocity profile, right panel show the energy profile. CL scheme is depicted in red while the AP scheme is depicted in blue. Initial condition is in black.

good behaviors of our scheme by performing a stability analysis for the isentropic Euler equations. We have successively extended the proposed scheme to the case of the full Euler system. The numerical results show that the scheme possesses the demanded properties. It is L^2 stable or L^∞ stable depending on the choice of the numerical diffusion and consistent with the low Mach number limit. Close to incompressible regimes this new scheme permits to compute solutions of low Mach number flows at a much lower computational costs. This reduction being much more important when dealing with multidimensional problems. However, results also indicate that the scheme in intermediate regimes is more diffusive than explicit methods. For this reason, we plan to develop in future works high order schemes in time and space which will permit to get more accurate results without losing the nice properties of the scheme here presented.

Acknowledgments. We thank the anonymous reviewers who have contributed to the improvement of the article with their constructive remarks and comments.

G. Dimarco and M.H. Vignal were supported by the French ANR project MOONRISE. The material of this work has been gathered during SHARK-FV conferences (*Sharing Higher-order Advanced Know-how on Finite Volume* www.math.univ-toulouse.fr/SHARK-FV/ held in 2014, 2015, 2016).

REFERENCES

- [1] T. Alazard, Incompressible limit of the non isentropic Euler equations with the solid wall boundary conditions, *Adv. Diff. Eq.* 10 (2005), pp. 19-44.
- [2] K. Asano, On the incompressible limit of the compressible euler equation. *Japan J. Appl. Math.* 4 (1987), pp. 455-488
- [3] M. P. Bonner, Compressible subsonic flow on a staggered grid, Master thesis, The University of British Columbia. (2007).
- [4] C. Chalons, M. Girardin, S. Kokh, Large time step and asymptotic preserving numerical schemes for the gas dynamics equations with source terms, *SIAM J. Sci. Comput.*, 35, (2013), pp. 2874-2902.
- [5] C. Chalons, M. Girardin, S. Kokh, An all-regime Lagrange-Projection like scheme for the gas dynamics equations on unstructured meshes, *Communications in Computational Physics (CICP)*, Vol. 20, No. 1, pp. 188-233 (2016).

- [6] Ph. Colella, K. Pao, A projection method for low speed flows, *J. Comp. Phys.* 149 (1999), pp. 245-269.
- [7] F. Cordier, P. Degond, A. Kumbaro, An Asymptotic-Preserving all-speed scheme for the Euler and Navier Stokes equations, *J. Comp. Phys.*, 231 (2012), pp. 5685-5704.
- [8] P. Degond, F. Deluzet, A. Sangam, M.-H. Vignal, An Asymptotic Preserving Scheme for the Euler equations in a strong magnetic field, *Comp. Phys.*, 228 (2009), pp. 3540-3558.
- [9] P. Degond, M. Tang, All speed scheme for the low mach number limit of the Isentropic Euler equation, *Comm. Comp. Phys.*, 10 (2011), pp. 1-31.
- [10] S. Dellacherie, J. Jung, P. Omnes, P.A. Raviart, Construction of modified Godunov-type schemes accurate at any Mach number for the compressible Euler system, *Math. Models Methods Appl. Sci.*, 26 (2016), pp. 2525-2615
- [11] S. Dellacherie, Analysis of Godunov type schemes applied to the compressible Euler system at low Mach number, *J. Comp. Phys.*, 229 (2010), pp. 978-1016.
- [12] R. Eymard, T. Gallouët, R. Herbin, Finite volume methods, *Handbook of numerical analysis*, Vol. VII, pp. 713–1020, North-Holland, Amsterdam, 2000.
- [13] T. Gallouët, R. Herbin, D. Maltese, A. Novotny, Convergence of the Marker-and-Cell scheme for a semi-stationary compressible Navier-Stokes problem, *Mathematics and Computers in Simulation*, 137 (2017), pp. 325-349.
- [14] N. Grenier, J.-P. Vila, P. Villedieu, An accurate low-Mach scheme for a compressible two-fluid model applied to free-surface flows, *J. Comp. Phys.* 252 (2013), pp. 1-19.
- [15] H. Guillard, A. Murrone, On the behavior of upwind schemes in the low Mach number limit : II. Godunov type schemes, *Comp. & Fluids*, 33 (2004), pp. 655-675.
- [16] H. Guillard, C. Viozat, On the behavior of upwind schemes in the low Mach limit, *Comp. & Fluid*, 28 (1999), pp. 63-86.
- [17] J. Haack, S. Jin and J.G. Liu, An all-speed asymptotic-preserving method for the isentropic Euler and Navier-Stokes equations, *Commun. Comput. Phys.*, 12 (2012), pp. 955-980.
- [18] F. H. Harlow, and A. Amsden, A numerical fluid dynamics calculation method for all flow speeds, *J. Comput. Phys*, 8 (1971), pp. 197-213.
- [19] D. R. van der Heul, C. Vuik and P. Wesseling, A conservative pressure-correction method for flow at all speeds, *Comp. & Fluids*, 32 (2003), pp. 1113-1132.
- [20] R. Herbin, W. Kheriji, J.-C. Latché, Pressure correction staggered schemes for barotropic one-phase and two-phase flows, *Comp. & Fluids*, 88 (2013), pp. 524-542.
- [21] R.I. Issa, A.D.Gosman, A.P. Watkins, The computation of compressible and incompressible flow of fluid with a free surface, *Phys. Fluids*, 8 (1965), pp. 2182-2189.
- [22] P.-L. Lions and N. Masmoudi, Incompressible limit for a viscous compressible fluid. *J. Math. Pures Appl.* 77 (1998), pp. 585-627.
- [23] S. Klainerman, A. Majda Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, *Comm. Pure Appl. Math.*, 34 (1981), pp. 481-524.
- [24] S. Klainerman, A. Majda, Compressible and incompressible fluids, *Comm. Pure Appl. Math.*, 35 (1982), pp. 629-651.
- [25] R. Klein, Semi-implicit extension of a Godunov-type scheme based on low Mach number asymptotics I: One-dimensional flow, *J. Comp. Phys.*, 121 (1995), pp. 213-237.
- [26] R. Klein, N. Botta, T. Schneider, C.D. Munz, S. Roller, A. Meister, L. Hoffmann, T. Sonar, Asymptotic adaptive methods for multi-scale problems in fluid mechanics, *J. Eng. Math.*, 39 (2001), pp. 261-343.
- [27] G. Metivier, S. Schochet, The incompressible limit of the non-isentropic Euler equations, *Arch. Rational Mech. Anal.* 158 (2001), pp. 61-90.
- [28] C. D. Munz, S. Roller, R. Klein, K. J. Geratz, The extension of incompressible flow solvers to the weakly compressible regime, *Comp. Fluid*, 32 (2002), pp. 173-196.
- [29] C. D. Munz, M. Dumbser, S. Roller, Linearized acoustic perturbation equations for low Mach number flow with variable density and temperature, *J. Comput. Phys.*, 224

- (2007), pp. 352-364.
- [30] J. H. Park, C. D. Munz, Multiple pressure variables methods for fluid flow at all Mach numbers, *Int. J. Numer. Meth. Fluid*, 49 (2005), 905-931.
 - [31] K.R. Arun, S. Noelle, M. Lukáčova-Medvidova, C.D. Munz, An asymptotic preserving all Mach number scheme for the Euler equations of gas dynamics, preprint, October 2012.
 - [32] H. Paillère, C. Viozat, A. Kumbaro, I. Toumi, Comparison of low mach number models for natural convection problems. *Heat & Mass Tran.*, 36 (2000), pp. 567-573.
 - [33] S. V. Patankar, *Numerical heat transfer and fluid flow*, New York: McGraw-Hill, 1980.
 - [34] V.V. Rusanov, Calculation of interaction of non-steady shock waves with obstacles. *J. Comput. Math. Phys. USSR* 1 (1961), pp. 267-279.
 - [35] S. Schochet, The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit, *Comm. Math. Phys.* 104 (1986), pp. 49-75.
 - [36] E. Turkel, Preconditioned methods for solving the incompressible and low speed compressible equations, *J. Comp. Phys.*, 72 (1987), pp. 277-298.