A BOUNDARY LAYER PROBLEM FOR AN ASYMPTOTIC PRESERVING SCHEME IN THE QUASI-NEUTRAL LIMIT FOR THE EULER-POISSON SYSTEM

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Abstract. We consider the two-fluid Euler-Poisson system modeling the expansion of a quasi-neutral plasma in the gap between two electrodes. The plasma is injected from the cathode using boundary conditions which are not at the quasi-neutral equilibrium. This generates a boundary layer at the cathode. We numerically show that classical schemes as well as the asymptotic preserving scheme developed in [9] are unstable for general Roe type solvers when the mesh does not resolve the small scale of the Debye length. We formally derive a model describing the boundary layer. Analysing this problem, we determine well-adapted boundary conditions. These well-adapted boundary conditions stabilize general solvers without resolving the Debye length.

Key words. Boundary layer, quasi-neutral limit, Euler-Poisson system, asymptotic preserving scheme, plasma.

1. Introduction. In this paper, we are interested in a boundary layer problem arising in plasma fluid models when boundary conditions are not well-adapted in quasi-neutral regions. This problem occurs in the modeling of a quasi-neutral plasma bubble expansion in the gap between two electrodes. We study such a device in relation with two physical applications. The first one concerns high current diodes in which the plasma is used to increase the extracted current, see [31]. The second application is related to electric arc phenomena on satellite solar panels, see [5], [13].

The plasma, constituted of electrons and of one ion species, is injected from the cathode and undergoes a thermal expansion towards the anode. Attracted by the positive potential of the anode, some electrons are emitted in the gap between the plasma-vacuum interface and the anode. They form a beam of electrons in the vacuum.

Our starting point is a one dimensional Euler system for each species (ions and electrons) coupled with the Poisson equation. We call it the two-fluid Euler-Poisson system. The classical discretizations of this system are subject to severe numerical constraints in quasi-neutral zones. Indeed, in plasmas, charge imbalances take place at the scale of the Debye length, see [4], [17]. It is given by

$$\lambda_D = \left(\frac{\epsilon_0 k_B T_0}{e^2 n_0} \right)^{1/2}, \tag{1.1}$$

where ϵ_0 is the vacuum permitivity, k_B is the Boltzmann constant, T_0 is the plasma temperature scale, e is the positive elementary charge and n_0 is the plasma density scale. Due to these charge imbalances, there are electric restoring forces and the particles oscillate around their equilibrium positions. The electron plasma period, also called the plasma period, is given by

$$\tau_p = \left(\frac{\varepsilon_0 m_e}{n_0 e^2} \right)^{1/2}, \tag{1.2}$$

where m_e is the electron mass. In [14], it is proved that classical discretizations of the two-fluid Euler-Poisson model must resolve the scale of the plasma period otherwise

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a numerical instability is generated. In quasi-neutral zones, where the plasma period is very small, this constraint is extremely penalizing and simulations require huge computational resources.

There are two possible ways to overcome this limitation. The first one consists in using a quasi-neutral model in quasi-neutral zones. But, this solution is not well-adapted to situations such that both quasi-neutral and non quasi-neutral zones are present in the domain. This is the case in our application since the plasma bubble is quasi-neutral and the beam which contains only electrons, is non quasi-neutral. Indeed, the quasi-neutral model is not valid in non quasi-neutral zones and, we have to use different models for the different regions. Thus, we have to reconnect the models and follow the moving interface between quasi-neutral and non quasi-neutral zones. And, the connection of the models and the interface description are difficult problems especially in two or three dimensions, see [11], [16], [18].

The second way consists in finding an asymptotic preserving discretization for the two-fluid Euler-Poisson model, i.e. a scheme which does not require the resolution of the plasma period. Such a scheme has been developed in [9]. In this article, it is numerically observed that the asymptotic preserving scheme remains stable while the classical scheme develops instabilities for time steps greater than the plasma period. Note that both classical and asymptotic preserving schemes do not need to resolve the Debye length. In [10], the asymptotic stability of the scheme proposed in [9] is established on the linearized one-fluid Euler-Poisson system using Fourier analysis.

In [9], two test cases in one dimension are considered. The first test case is a periodic perturbation of a quasi-neutral uniform stationary plasma with a non-zero current. The second test case concerns the one dimensional plasma expansion in the gap between two electrodes previously mentioned. Simulations are performed using the modified Lax-Friedrichs solver which is well known to be very diffusive but, also to be a robust solver. When, we perform simulations with more general Roe type solvers, the results are identical in the first test case: the periodic perturbation of a quasi-neutral uniform stationary plasma with a non-zero current. But, in the second test case (the plasma expansion), the classical and the asymptotic preserving schemes develop instabilities when the mesh does not resolve the Debye length. This constraint is also penalizing in quasi-neutral regions since the Debye is very small.

Here, we numerically show that these instabilities are related to the presence of a boundary layer (or a sheath) generated by boundary conditions not well-adapted to the quasi-neutral regime. The physical problem of ion sheath problems have received much interest, we refer the reader to [15], [24], [25], [26], [27], [28], [30], [2], [16], [22] and references therein. The question of boundary conditions at sheath edges has been numerically investigated in [20]. Here, introducing a formal expansion of the solution in the boundary layer, we obtain a differential system which models the boundary layer. The analysis of this differential system allows to construct well-adapted boundary conditions and to stabilize general solvers without resolving the Debye length.

The paper is organized as follows. In section 2, we present the two-fluid Euler-Poisson model and its quasi-neutral limit. Then, we recall the classical discretization and the asymptotic preserving scheme proposed in [9]. In section 3, we present numerical results for the plasma expansion test case and the numerical difficulties related to the presence of the boundary layer. We numerically show that general Roe type solvers, for the classical and the asymptotic preserving schemes, are unstable when the mesh does not resolve the Debye length. In section 4, we formally establish the boundary layer problem and we analyze it. Finally, in section 5, we determine well-adapted boundary conditions and present numerical results. These numerical results show that the well-adapted boundary conditions stabilize general solvers for both classical and asymptotic preserving schemes.

2. The classical and asymptotic preserving schemes for the two-fluid Euler-Poisson system.

2.1. The two-fluid Euler Poisson system and its quasi-neutral limit. We consider the two-fluid Euler-Poisson system written in scaled variables, we refer to [9] for the rescaling step. We denote by $n_i(x,t)$ and $n_e(x,t)$ the ion and electron densities, by $q_i(x,t)$ and $q_e(x,t)$ the ion and electron momenta and by $\phi(x,t)$ the electric potential, where $x \in \mathbb{R}$ is the space variable and $t > 0$ is the time. The scaled two-fluid Euler-Poisson system is written

$$\begin{cases} \partial_t n_{i,e}^\lambda + \partial_x q_{i,e}^\lambda = 0, \\ \partial_t q_i^\lambda + \partial_x f_i(n_i^\lambda, q_i^\lambda) = -n_i^\lambda\, \partial_x \phi^\lambda, \\ \varepsilon\, \partial_t q_e^\lambda + \varepsilon\, \partial_x f_e(n_e^\lambda, q_e^\lambda) = n_e^\lambda\, \partial_x \phi^\lambda, \end{cases} \tag{2.1}$$

$$-\lambda^2\, \partial_{xx}^2 \phi^\lambda = n_i^\lambda - n_e^\lambda, \tag{2.2}$$

for $x \in]0,1[$ and $t > 0$, where the momentum fluxes are defined by $f_i(n,q) = q^2/n + p_i(n)$ and $f_e(n,q) = q^2/n + p_e(n)/\varepsilon$. The isentropic pressure laws are given by $p_{i,e}(n) = C_{i,e} n^{\gamma_{i,e}}$, with $C_{i,e} > 0$ and $\gamma_{i,e} > 1$.

Initially, we suppose the domain devoid of plasma then, $n_i(x,0) = n_e(x,0) = 0$. The hyperbolic systems are assumed supersonic at the point $x = 1$ then, we do not need boundary conditions for fluid quantities at the point $x = 1$. The cathode and the anode are respectively located at $x = 0$ and $x = 1$ and a quasi-neutral plasma is present before the cathode. Then, we set

$$(n_i^\lambda, q_i^\lambda)(0,t) = (n_{i0}^\lambda, q_{i0}^\lambda)(t), \qquad (n_e^\lambda, q_e^\lambda)(0,t) = (n_{e0}^\lambda, q_{e0}^\lambda)(t), \tag{2.3}$$

$$\phi^\lambda(0,t) = 0, \qquad \phi^\lambda(1,t) = \phi_A(t) > 0, \tag{2.4}$$

for all $t > 0$ and where $(n_{i0}^\lambda, q_{i0}^\lambda)$ and $(n_{e0}^\lambda, q_{e0}^\lambda)$ are the respective solutions at the point $x = 0$ of the following Riemann problems

$$\begin{cases} \partial_t n_i + \partial_x q_i = 0, \\ \partial_t q_i + \partial_x f_i(n_i, q_i) = 0, \\ \begin{pmatrix} n_i \\ q_i \end{pmatrix}(x,0) = \begin{cases} \begin{pmatrix} n_0 \\ q_0 \end{pmatrix}(t), & x < 0, \\ \begin{pmatrix} n_i^\lambda \\ q_i^\lambda \end{pmatrix}(0^+,t), & x > 0, \end{cases} \end{cases} \qquad \begin{cases} \partial_t n_e + \partial_x q_e = 0, \\ \varepsilon\, \partial_t q_e + \varepsilon\, \partial_x f_e(n_e, q_e) = 0, \\ \begin{pmatrix} n_e \\ q_e \end{pmatrix}(x,0) = \begin{cases} \begin{pmatrix} n_0 \\ q_0 \end{pmatrix}(t), & x < 0, \\ \begin{pmatrix} n_e^\lambda \\ q_e^\lambda \end{pmatrix}(0^+,t), & x > 0, \end{cases} \end{cases} \tag{2.5}$$

where $(n_{i,e}^\lambda, q_{i,e}^\lambda)(0^+,t) = \lim_{x\to 0^+} (n_{i,e}^\lambda, q_{i,e}^\lambda)(x,t)$ and where (n_0, q_0) is a given subsonic state, i.e. such that $q_0/n_0 + \sqrt{p_i'(n_0)} > 0$, $q_0/n_0 + \sqrt{p_e'(n_0)/\varepsilon} > 0$, $q_0/n_0 - \sqrt{p_i'(n_0)} < 0$ and $q_0/n_0 - \sqrt{p_e'(n_0)/\varepsilon} < 0$.

The mathematical theory of the Euler-Poisson system has been studied in [7] and [23] for the one dimensional isothermal case, in [19] for the one dimensional isentropic case and in [1] for the multi-dimensional case and for the two species model.

In (2.1), (2.2), the dimensionless parameter ε is the ratio between the ion and electron masses and the dimensionless parameter λ is the scaled Debye length, i.e.

the ratio between the Debye length λ_D, given by (1.1), and the macroscopic length scale. In the physical applications related to this work, high current diodes or arc phenomena on satellite, the scaled Debye length, λ, is a very small parameter in the plasma bubble and an order one parameter in the beam. The dimensionless parameter ε is also small but, we do not neglect it. We note that the rescaled plasma period, i.e. the ratio between the plasma period given by (1.2) and the macroscopic time scale, is given by $\tau = \sqrt{\varepsilon}\,\lambda$.

We refer to [29] for the study of the quasi-neutral limit of the one dimensional steady Euler-Poisson system for well-adapted boundary conditions and to [21] for general boundary data. The quasi-neutral limit of the transient one species Euler-Poisson system has been studied in [6] for the isothermal case and in [22], [34] for the isentropic case. The quasi-neutral limit of the two species model has been formally studied in [8], [9] and [11], we recall the results included in these works. The formal limit $\lambda \to 0$ of (2.1), (2.2) yields:

$$\begin{cases} \partial_t \bar{n}_{i,e} + \partial_x \bar{q}_{i,e} = 0, \\ \partial_t \bar{q}_i + \partial_x f_i(\bar{n}_i, \bar{q}_i) = -\bar{n}_i\, \partial_x \bar{\phi}, \\ \varepsilon\, \partial_t \bar{q}_e + \varepsilon\, \partial_x f_e(\bar{n}_e, \bar{q}_e) = \bar{n}_e\, \partial_x \bar{\phi}, \\ \bar{n}_i - \bar{n}_e = 0. \end{cases} \tag{2.6}$$

Then, in the passage from the Euler-Poisson system (2.1), (2.2) to the quasi-neutral system (2.6), the equation for the potential changes dramatically, from the elliptic Poisson equation (2.2) into the algebraic quasi-neutrality constraint $\bar{n}_i = \bar{n}_e$. In the quasi-neutral system, $\bar{\phi}$ is the Lagrange multiplier of the constraint $\bar{n}_i = \bar{n}_e$. An elliptic equation for the potential can be obtained taking the space derivative of the difference between the momentum conservation laws and remarking that $\partial_x j = \partial_x(q_i - q_e) = 0$ thanks to the quasi-neutrality constraint and to the mass equations, see [9] for more details. This elliptic equation is given by

$$-\partial_x \left((\varepsilon\, \bar{n}_i + \bar{n}_e)\, \partial_x \bar{\phi} \right) = \varepsilon\, \partial^2_{xx} (f_i(\bar{n}_i, \bar{q}_i) - f_e(\bar{n}_e, \bar{q}_e)). \tag{2.7}$$

It is important to note that numerical schemes can not be consistent with the quasi-neutral limit if, in the limit $\lambda \to 0$, the discrete potential does not give an approximate solution of (2.7).

2.2. The classical and asymptotic preserving schemes. First, we present the classical scheme for the two-fluid Euler-Poisson system (2.1), (2.2). It is an explicit finite volume discretization with an implicit treatment of source terms. This implicit treatment is a necessary condition for the stability property, see [14]. We discretize the domain $(0, 1)$ with a uniform grid of step $\Delta x = 1/N$ where N is the number of cells. We set $x_{k+1/2} = k\,\Delta x$ for $k = 0, \cdots, N$. We consider a sequence of positive real numbers $(t^m)_{m \geq 0}$ with $t^0 = 0$ and we denote by $\Delta t^m = t^{m+1} - t^m$, the time steps, for all $m \geq 0$. For $l = i$ or e, $k = 1, \cdots N$ and $m \geq 0$, let $U^m_{l,k} = (n^m_{l,k}, q^m_{l,k})$, ϕ^m_k be approximations of $(n^\lambda_l, q^\lambda_l)$, ϕ^λ on $(x_{k-1/2}, x_{k+1/2}) \times (t^m, t^{m+1})$. The classical scheme is given by

$$\frac{U_{i,k}^{m+1}-U_{i,k}^{m}}{\Delta t^m}+\frac{G_i^m(U_{i,k+1}^m,U_{i,k}^m)-G_i^m(U_{i,k}^m,U_{i,k-1}^m)}{\Delta x}=\begin{pmatrix}0\\ n_{i,k}^{m+1}\,\frac{E_{k+1/2}^{m+1}+E_{k-1/2}^{m+1}}{2}\end{pmatrix}, \quad (2.8)$$

$$\frac{U_{e,k}^{m+1}-U_{e,k}^{m}}{\Delta t^m}+\frac{G_e^m(U_{e,k+1}^m,U_{e,k}^m)-G_e^m(U_{e,k}^m,U_{e,k-1}^m)}{\Delta x}=\begin{pmatrix}0\\ -n_{e,k}^{m+1}\,\frac{E_{k+1/2}^{m+1}+E_{k-1/2}^{m+1}}{2}\end{pmatrix}, \quad (2.9)$$

$$-\Delta_{app}(\phi_k^{m+1})=\frac{E_{k+1/2}^{m+1}-E_{k-1/2}^{m+1}}{\Delta x}=n_{i,k}^{m+1}-n_{e,k}^{m+1}, \quad (2.10)$$

where the approximate electric field is given by

$$E_{k+1/2}^{m+1}=-\frac{\phi_{k+1}^{m+1}-\phi_k^{m+1}}{\Delta x}, \quad E_{1/2}^{m+1}=-\frac{\phi_1^{m+1}}{\Delta x/2}, \quad E_{N+1/2}^{m+1}=-\frac{\phi_A(t^{m+1})-\phi_N^{m+1}}{\Delta x/2},$$

for $k=1,\cdots,N-1$. The numerical fluxes, G_i^m, G_e^m, depend on the considered solver. In sections 3 and 5, we give numerical results for three different solvers: the Riemann solver, see [32], and the degree 2 and 0 polynomial schemes developed in [12] which are Roe type solvers. Let $l=i$ or e, we denote by $G_l(n_l,q_l)$ the continuous flux $(q_l,f_l(n_l,q_l))$. The polynomial fluxes $G_l^m(U_{l,k+1}^m,U_{l,k}^m)=(Q_l^m(U_{l,k+1}^m,U_{l,k}^m),F_l^m(U_{l,k+1}^m,U_{l,k}^m))$, are given by

$$G_l^m(U_{l,k+1}^m,U_{l,k}^m)=\frac{G_l(U_{l,k+1}^m)+G_l(U_{l,k}^m)}{2}+\frac{1}{2}\,P_{j,k+1/2}^{l,m}\,(U_{l,k}^m-U_{l,k+1}^m), \quad (2.11)$$

where the matrix $P_{j,k+1/2}^{l,m}$ is a degree j polynomial approximation of $|DG_l((U_{l,k+1}^m+U_{l,k}^m)/2)|$.

It is proved in [14], that the classical scheme is conditionally stable. It must resolve the scaled plasma period, i.e. $\Delta t^m\le\sqrt{\varepsilon}\,\lambda$ for all $m\ge 0$. This condition is particularly expansive in quasi-neutral zones where λ is a small parameter.

Let us recall the principle of the asymptotic preserving scheme proposed in [9]. First, in [9] it is shown that equation (2.2) can be reformulated into an equation in which the transition from equation (2.2) to equation (2.7) is explicit when $\lambda\to 0$. This formulation, called the reformulated Poisson equation, is given by

$$\tau^2\,\partial_{tt}^2(-\partial_{xx}^2\phi^\lambda)-\partial_x\left(\left(\varepsilon n_i^\lambda+n_e^\lambda\right)\partial_x\phi^\lambda\right)=\varepsilon\,\partial_{xx}^2(f_i(n_i^\lambda,q_i^\lambda)-f_e(n_e^\lambda,q_e^\lambda)), \quad (2.12)$$

where $\tau=\sqrt{\varepsilon}\,\lambda$ is the scaled plasma period. This equation is equivalent to the Poisson equation provided $n_{i,e}^\lambda,q_{i,e}^\lambda$ satisfy the Euler systems (2.1) and that the Poisson equation and its time derivative are satisfied at $t=0$, see [9] for more details. Furthermore, remark that the formal limit $\lambda\to 0$ gives the quasi-neutral elliptic equation (2.7).

The asymptotic preserving scheme is based on this reformulated equation for consistency reasons with the quasi-neutral limit. This scheme consists in changing in the classical scheme (2.8)-(2.10), the fluid equations by the following discretizations

$$
\begin{aligned}
&\frac{n_{i,k}^{m+1}-n_{i,k}^{m}}{\Delta t^m}+\frac{Q_i^m(U_{i,k+1}^{m+1/2},U_{i,k}^{m+1/2})-Q_i^m(U_{i,k}^{m+1/2},U_{i,k-1}^{m+1/2})}{\Delta x}=0,\\
&\frac{n_{e,k}^{m+1}-n_{e,k}^{m}}{\Delta t^m}+\frac{Q_e^m(U_{e,k+1}^{m+1/2},U_{e,k}^{m+1/2})-Q_e^m(U_{e,k}^{m+1/2},U_{e,k-1}^{m+1/2})}{\Delta x}=0,\\
&\frac{q_{i,k}^{m+1}-q_{i,k}^{m}}{\Delta t^m}+\frac{F_i^m(U_{i,k+1}^{m},U_{i,k}^{m})-F_i^m(U_{i,k}^{m},U_{i,k-1}^{m})}{\Delta x}=n_{i,k}^m\,\frac{E_{k+1/2}^{m+1}+E_{k-1/2}^{m+1}}{2},\\
&\frac{q_{e,k}^{m+1}-q_{e,k}^{m}}{\Delta t^m}+\frac{F_e^m(U_{e,k+1}^{m},U_{e,k}^{m})-F_e^m(U_{e,k}^{m},U_{e,k-1}^{m})}{\Delta x}=-n_{e,k}^m\,\frac{E_{k+1/2}^{m+1}+E_{k-1/2}^{m+1}}{2},
\end{aligned}
\tag{2.13}
$$

where $U_{l,k+1}^{m+1/2}=(n_{l,k}^m,q_{l,k}^{m+1})$ for $l=i$ or e, for all $m\geq 0$ and all $k=1,\cdots,N$. In [9] it is shown that the discrete Poisson equation (2.10) is equivalent up to terms of order Δt^2 or Δx^2 to the following discretization of the reformulated Poisson equation (2.12)

$$
\begin{aligned}
-\varepsilon\,\lambda^2\,\Delta x&\left(\frac{\Delta_{app}(\phi_k^{m+1})-\Delta_{app}(\phi_k^m)}{\Delta t^{m+1}}-\frac{\Delta_{app}(\phi_k^m)-\Delta_{app}(\phi_k^{m-1})}{\Delta t^m}\right)\\
&-\Delta x\,\Delta t^{m+1}\left((\varepsilon\,n_i+n_e)_{k+1/2}^m\,E_{k+1/2}^{m+1}-(\varepsilon\,n_i+n_e)_{k-1/2}^m\,E_{k-1/2}^{m+1}\right)\\
&\qquad=\varepsilon\,\Delta t^{m+1}\left(\frac{f_{i,k+1}^m-2\,f_{i,k}^m+f_{i,k-1}^m}{\Delta x}+\frac{f_{e,k+1}^m-2\,f_{e,k}^m+f_{e,k-1}^m}{\Delta x}\right),
\end{aligned}
\tag{2.14}
$$

where

$$
(\varepsilon\,n_i+n_e)_{k+1/2}^m=\frac{(\varepsilon\,n_{i,k+1}^m+n_{e,k+1}^m)+(\varepsilon\,n_{i,k}^m+n_{e,k}^m)}{2}\quad\text{and}\quad f_{l,k}^m=f_l(n_{l,k}^m,q_{l,k}^m),
$$

for $l=i$ or e. This formulation gives an uncoupled formulation of the asymptotic preserving scheme. Indeed, using (2.14) we compute the potential at time t^{m+1} as a function of the variables at time t^m. Then, with (2.13) we update the momenta and the densities. Furthermore, note that the discretization (2.14) of (2.12) is implicit. This is the key point for the asymptotic stability property of the scheme. Indeed, in classical discretizations, see [9], the resulting discretization of (2.12) is explicit. But, equation (2.12) is an harmonic oscillator equation on the total charge $-\partial_{xx}^2\phi^\lambda=n_i^\lambda-n_e^\lambda$. And, it is well known that an explicit time discretization of this equation gives a conditionally stable scheme while an implicit time discretization is unconditionally stable with respect to the scaled plasma period τ. This implicit discretization is a consequence of the implicit discretization, in term of momenta, of the mass fluxes. It is rigorously proved in [10], that the scheme is stable without resolving the small scale of the plasma period $\tau=\sqrt{\varepsilon}\,\lambda$. It is also consistent with the quasi-neutral limit since the limit $\lambda\to 0$ of the reformulated Poisson equation (2.12) gives exactly the elliptic equation (2.7) of the quasi-neutral model. Last, it is important to remark that the cost of the asymptotic preserving scheme is the same as the one of the classical scheme.

3. Numerical difficulties related to the boundary layer. In [9], two test cases in one dimension of space, are presented in order to compare the classical and asymptotic preserving schemes. The first test case is a periodic perturbation of a quasi-neutral uniform stationary plasma with a non-zero current. For this test-case, an exact solution of the linearized Euler-Poisson system about the considered steady state is analytically known. For small perturbations, the solutions of the linearized and non-linear problems are believed to be close. The classical and asymptotic preserving schemes are compared. It is numerically observed that the asymptotic preserving scheme remains stable while the classical scheme develops instabilities for time steps greater than the plasma period. For the space discretization, the modified Lax-Friedrichs solver has been used. This solver is known to be very diffusive but, it is simple and very robust for the validation phase. The extension to more accurate order 1 solvers like polynomial solvers (degree 0 and 2), HLLE, HLLC or to the order 2 Lax-Wendroff solver, give the same results. When the time steps are greater than the plasma period, the asymptotic preserving scheme remains stable while the classical scheme is unstable. Note that in this case the boundary conditions for the fluid quantities are periodic.

The second test case is the one dimensional quasi-neutral plasma expansion in the vacuum separating two electrodes. This test case is particularly well-adapted to the asymptotic preserving scheme, since a transition between a quasi-neutral region (the plasma) to a non quasi-neutral one (the beam) occurs. As in the previous test case, the numerical simulations presented in [9], are performed using the modified Lax-Friedrichs solver. They show that the asymptotic preserving remains stable while the classical scheme is unstable when the time steps are greater than the plasma period. Here, we present results obtained with more general order 1 solvers: the degree 0 and 2 polynomial solvers. In this case, we see on numerical results, that a boundary layer problem appears at the injection point $x = 0$. This boundary layer destabilizes the classical and asymptotic preserving schemes, for the degree 2 polynomial solver, when they do not resolve the small scale of the Debye length λ.

The parameters are issued from plasma arc physics (see e.g. [5], [13]) this leads to the following values: $\gamma_i = \gamma_e = 5/3$, $C_i = C_e = 1$, $\varepsilon = 10^{-4}$, $\lambda = 10^{-4}$, $\phi_A = 100$ and we set $(n_0, q_0) = (1, 1)$. Note that this gives a scaled plasma period $\tau = \sqrt{\varepsilon}\,\lambda = 10^{-6}$. We consider results for the classical scheme with the Riemann solver (see [32]) in the resolved case, i.e. when the space step, Δx, is lower than the scaled Debye length, λ, and when the time steps, Δt^m for all $m \geq 0$, are lower than the scaled plasma period, $\tau = \sqrt{\varepsilon}\lambda$. The curves given by this scheme will be the reference curves. We compare this reference solution to the classical and asymptotic preserving schemes in all the different cases, full resolved, partially resolved and not resolved. They respectivelly correspond to the cases ($\Delta x \leq \lambda$ and $\Delta t^m \leq \tau$ for all $m \geq 0$), ($\Delta x > \lambda$ and $\Delta t^m \leq \tau$ for all $m \geq 0$) and ($\Delta x > \lambda$ and $\Delta t^m > \tau$ for all $m \geq 0$). We use two different solvers the degree 0 and 2 polynomial solvers. In these cases, the numerical fluxes are given by (2.11). Note that the degree 0 polynomial solver has diagonal numerical viscosity matrices $(1/2\, P^{l,m}_{0,k+1/2})$. This is also the case for the modified Lax-Friedrichs solver used in [9]. On the contrary, the degree 2 polynomial solver has non diagonal numerical viscosity matrices $(1/2\, P^{l,m}_{2,k+1/2})$. This is also the case for general Roe type solvers like the Roe, HLLE, HLLC,$\cdots$ solvers.

On Figures 3.1, 3.2, 3.3, we present results obtained in the resolved case. In this case, we have $\Delta x \leq \lambda$ and $\Delta t^m \leq \tau$ for all $m \geq 0$ for all schemes and all solvers. We can remark that for a given solver, the results are identical for the classical and asymptotic preserving schemes. Furthermore, the different solvers give same results in the core of the plasma. But, on the boundary we can see several differences. First, at the cathode $x = 0$, on Figure 3.1 for the electron and ion densities and on the left picture of Figure 3.3 for the ion velocity, we can see that there is a boundary layer. Indeed, all solvers do not give the same resolution of the boundary layer but, all curves meet at the end of the boundary layer. Furthermore, at the anode $x = 1$, on Figure 3.2 for the electron velocity, we can see that there are oscillations on the curves given by the asymptotic preserving scheme for both degree 0 and 2 polynomial solvers. We will see that these oscillations disappear when Δx is bigger than the scaled Debye length λ. Here, we focus our attention on the boundary layer at the cathode, $x = 0$, and we defer to a future work the study of the problems on the electron velocity at the anode.

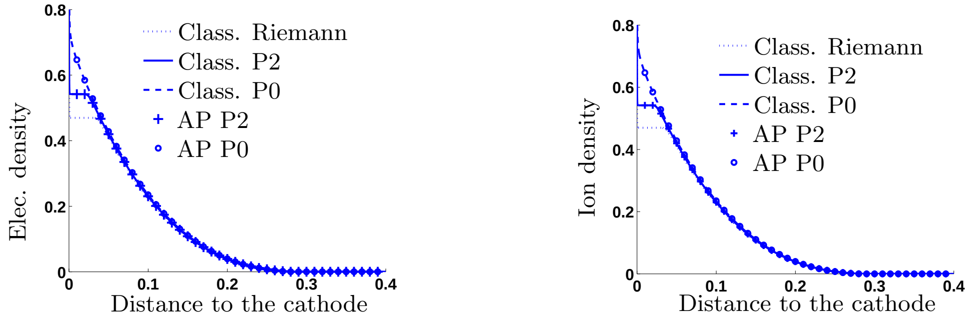

FIG. 3.1. *Resolved case: $\Delta x = 10^{-4} = \lambda$ and $\Delta t \leq \tau = 10^{-6}$. Electron (on the left hand side) and ion (on the right hand side) densities as functions of x, the distance to the cathode, at the rescaled time $t = 0.05$. The results are computed with the classical scheme using the Riemann solver (dotted line), the degree 2 polynomial solver (solid line) and the degree 0 polynomial solver (dashed line) and with the asymptotic preserving scheme using the degree 2 polynomial solver (cross markers) and the degree 0 polynomial solver (circle markers).*

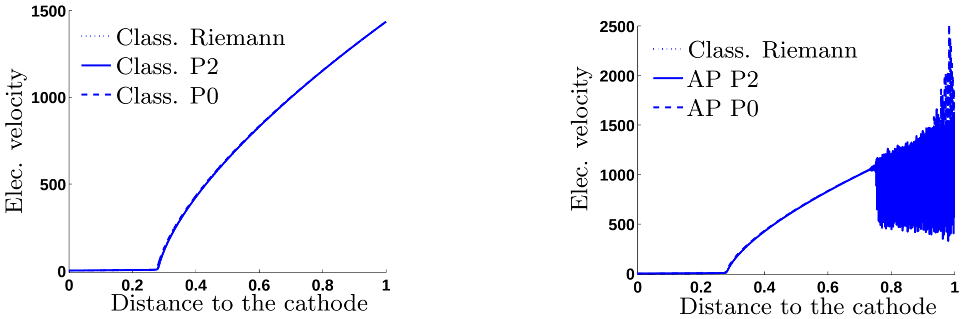

FIG. 3.2. *Resolved case: $\Delta x = 10^{-4} = \lambda$ and $\Delta t \leq \tau = 10^{-6}$. Electron velocity as a function of x, the distance to the cathode, at the rescaled time $t = 0.05$. On the left hand side, the results are computed with the classical scheme using the Riemann solver (dotted line), the degree 2 polynomial solver (solid line) and the degree 0 polynomial solver (dashed line). On the right hand side, the curve with dotted line is computed with the classical scheme using the Riemann solver and in solid and dashed lines the curves are computed with the asymptotic preserving scheme using the degree 2 polynomial solver (solid line) and the degree 0 polynomial solver (dashed line).*

On Figures 3.4, 3.5, 3.6 and 3.7, we present the electron density, the ion and electron velocities and the electric potential when the scaled Debye length is not resolved. We do not give the ion density because the curve is the same as the one of the electron density. On the left hand side of each figure we present results obtained in the resolved case ($\Delta x \leq \lambda$ and $\Delta t \leq \tau$) for the classical scheme using the Riemann

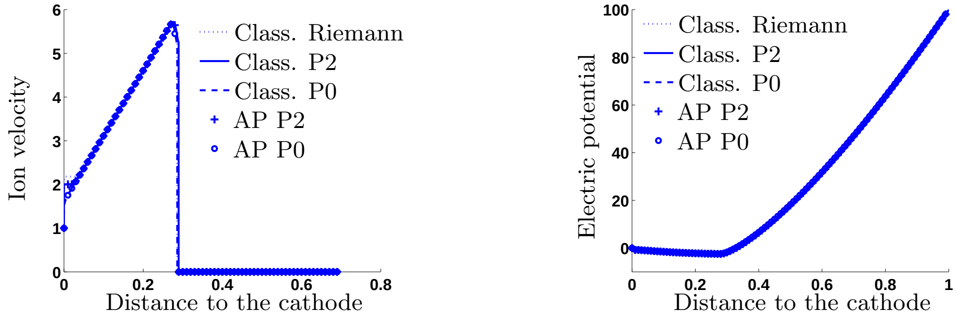

FIG. 3.3. *Resolved case:* $\Delta x = 10^{-4} = \lambda$ *and* $\Delta t \le \tau = 10^{-6}$. *Ion velocity (on the left hand side) and electric potential (on the right hand side) as functions of* x*, the distance to the cathode, at the rescaled time* $t = 0.05$. *The results are computed with the classical scheme using the Riemann solver (dotted line), the degree* 2 *polynomial solver (solid line) and the degree* 0 *polynomial solver (dashed line) and with the asymptotic preserving scheme using the degree* 2 *polynomial solver (cross markers) and the degree* 0 *polynomial solver (circle markers).*

solver. We recall that these curves are the reference curves. Furthermore, the results are computed for space step greater than the scaled Debye length λ with the classical scheme resolving the scaled plasma period τ and with the asymptotic preserving scheme non resolving the scaled plasma period τ. On the left hand side, we use the degree 0 polynomial solver and on the right hand side the degree 2 polynomial solver. We can see that for both schemes (classical and asymptotic preserving) the degree 0 polynomial solver gives stable results while the degree 2 polynomial solver gives unstable results.

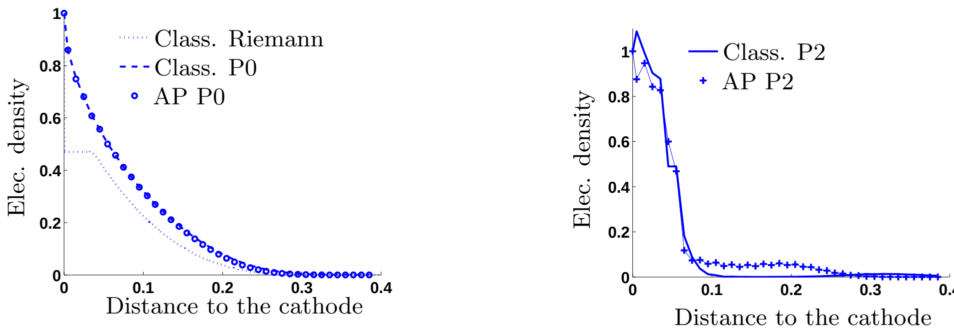

FIG. 3.4. *Electron density as a function of* x*, the distance to the cathode, at the rescaled time* $t = 0.05$*, in the unresolved in space case:* $\Delta x = 10^{-2} > \lambda = 10^{-4}$. *The classical scheme resolves the plasma period:* $\Delta t \le \tau$ *and the asymptotic preserving scheme does not:* $\Delta t > \tau$. *The results are computed with the classical scheme using the Riemann solver (left picture, dotted line) in the resolved case, this curve is the reference curve. On the left picture, we can see the results for the degree* 0 *polynomial solver with circle markers for the asymptotic preserving scheme and in dashed line for the classical scheme, on the right picture, the results for the degree* 2 *polynomial solver in solid line for the classical scheme and with cross markers for the asymptotic preserving scheme.*

On the left pictures of these figures, we can see that the classical and asymptotic preserving schemes with the degree 0 polynomial solver give diffusive but, stable results. The interface position is well predicted on the electron density curve. But, it is important to note that the results are computed with one hundred cells while the resolved reference curves are computed with ten thousands cells. This diffusion could

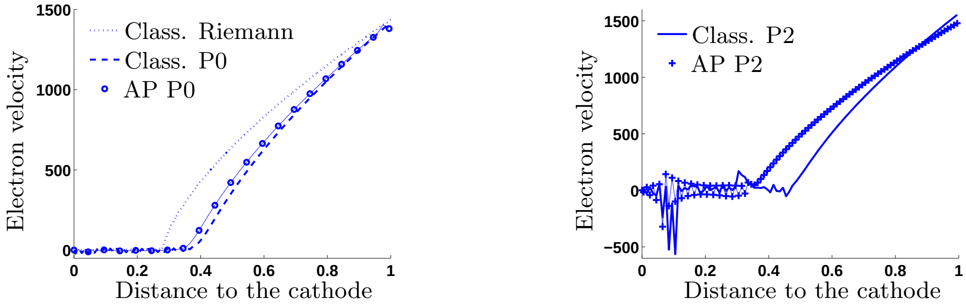

FIG. 3.5. *Electron velocity as a function of x, the distance to the cathode, at the rescaled time $t = 0.05$, in the unresolved in space case: $\Delta x = 10^{-2} > \lambda = 10^{-4}$. The classical scheme resolves the plasma period: $\Delta t \leq \tau$ and the asymptotic preserving scheme does not: $\Delta t > \tau$. The results are computed with the classical scheme using the Riemann solver (left picture, dotted line) in the resolved case, this curve is the reference curve. On the left picture, we can see the results for the degree 0 polynomial solver with circle markers for the asymptotic preserving scheme and in dashed line for the classical scheme, on the right picture, the results for the degree 2 polynomial solver in solid line for the classical scheme and with cross markers for the asymptotic preserving scheme.*

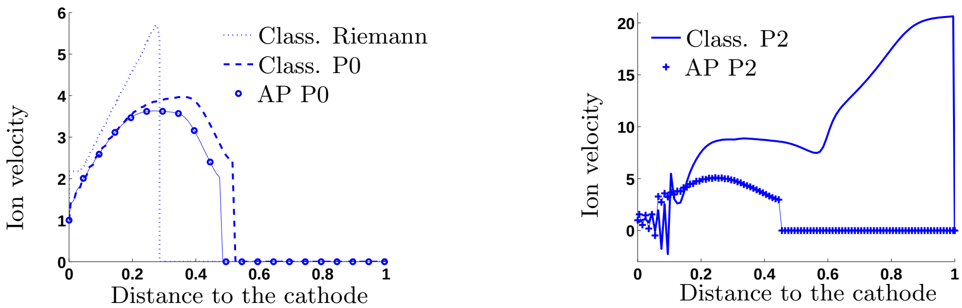

FIG. 3.6. *Ion velocity as a function of x, the distance to the cathode, at the rescaled time $t = 0.05$, in the unresolved in space case: $\Delta x = 10^{-2} > \lambda = 10^{-4}$. The classical scheme resolves the plasma period: $\Delta t \leq \tau$ and the asymptotic preserving scheme does not: $\Delta t > \tau$. The results are computed with the classical scheme using the Riemann solver (left picture, dotted line) in the resolved case, this curve is the reference curve. On the left picture, we can see the results for the degree 0 polynomial solver with circle markers for the asymptotic preserving scheme and in dashed line for the classical scheme, on the right picture, the results for the degree 2 polynomial solver in solid line for the classical scheme and with cross markers for the asymptotic preserving scheme.*

be certainly eliminated using order two schemes for example with a MUSCL method (see [33]), we defer it to a future work. Furthermore, it is important to note that the oscillations observed in the resolved case on the curve of the electron velocity for the asymptotic preserving scheme have disappeared in the unresolved in space case.

On the left picture of Figure 3.7 the electric potential curve presents small oscillations near the cathode $x = 0$ for the degree 0 polynomial solver. They are larger for the classical scheme than for the asymptotic preserving scheme. These oscillations as well as the instability of the degree 2 polynomial solver are due to the boundary layer. Indeed, if we use for boundary conditions the values given by the classical scheme with the Riemann solver at the end of the boundary layer in the resolved case, we can see on Figure 3.8 that the degree 2 polynomial solver gives stable results for both classical and asymptotic preserving schemes. Then, in order to stabilize the schemes for general solvers we have to determine the values of the fluid quantities and

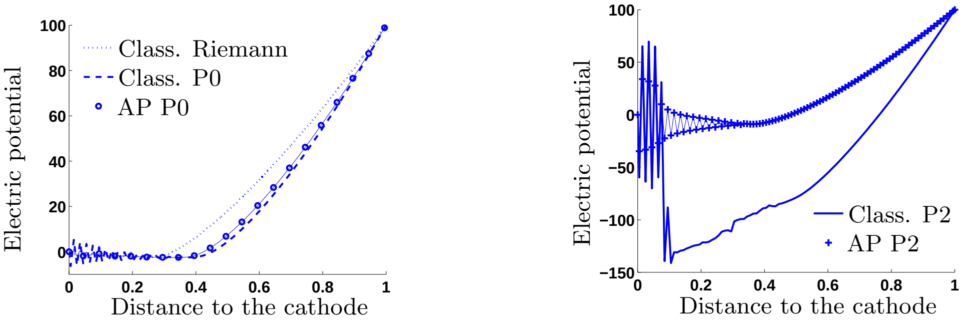

FIG. 3.7. *Electric potential as a function of x, the distance to the cathode, at the rescaled time $t = 0.05$, in the unresolved in space case: $\Delta x = 10^{-2} > \lambda = 10^{-4}$. The classical scheme resolves the plasma period: $\Delta t \leq \tau$ and the asymptotic preserving scheme does not: $\Delta t > \tau$. The results are computed with the classical scheme using the Riemann solver (left picture, dotted line) in the resolved case, this curve is the reference curve. On the left picture, we can see the results for the degree 0 polynomial solver with circle markers for the asymptotic preserving scheme and in dashed line for the classical scheme, on the right picture, the results for the degree 2 polynomial solver in solid line for the classical scheme and with cross markers for the asymptotic preserving scheme.*

electric potential at the end of the boundary layer. To do this, we study the boundary layer problem in the next section.

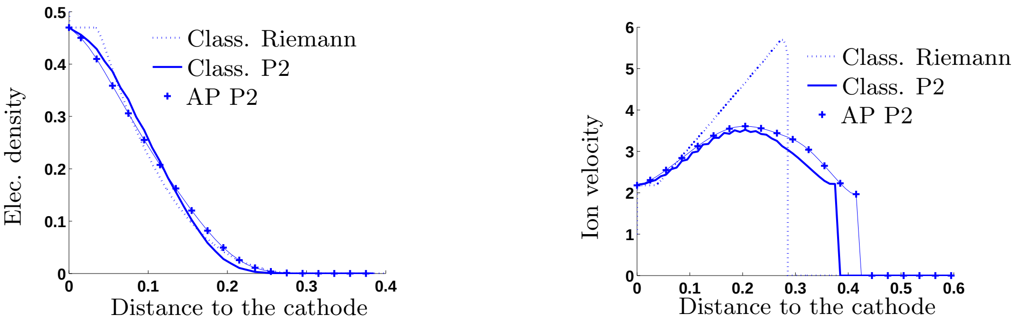

FIG. 3.8. *Electron density (left picture) and ion velocity (right picture) as functions of x, the distance to the cathode, at the rescaled time $t = 0.05$, in the unresolved in space case: $\Delta x = 10^{-2} > \lambda = 10^{-4}$. The classical scheme resolves the plasma period: $\Delta t \leq \tau$ and the asymptotic preserving scheme does not: $\Delta t > \tau$. The results are computed with the classical scheme using the Riemann solver (dotted line) in the resolved case, this curve is the reference curve. In solid line we can see the results for the classical scheme with the degree 2 polynomial solver and the curve with cross markers gives the results for the asymptotic preserving scheme with the degree 2 polynomial solver. In both cases, we use for boundary conditions the values given by the reference curves at the end of the boundary layer.*

4. Study of the boundary layer problem. We want to determine boundary conditions at equilibrium for the quasi-neutral regime, in order to use well-adapted boundary conditions for the classical and asymptotic preserving schemes. In this way, even with a mesh not resolving the Debye length, the schemes will remain stable. Thus, we introduce a boundary layer problem.

4.1. Presentation of the boundary layer problem. Assuming a boundary layer at the cathode $x = 0$, we write the following asymptotic expansion:

$$n^\lambda_{i,e}(x,t) = \bar n_{i,e}(x,t) + \tilde n_{i,e}(x/\lambda,t) + \lambda\,\hat n^\lambda_{i,e}(x,t), \tag{4.1}$$

$$q^\lambda_{i,e}(x,t) = \bar q_{i,e}(x,t) + \tilde q_{i,e}(x/\lambda,t) + \lambda\,\hat q^\lambda_{i,e}(x,t), \tag{4.2}$$

$$\phi^\lambda(x,t) = \bar\phi(x,t) + \phi(x/\lambda,t) + \lambda\,\hat\phi^\lambda(x,t), \tag{4.3}$$

where $\lim_{\lambda\to 0}\lambda\,(\hat n^\lambda_{i,e}, \hat q^\lambda_{i,e}, \hat\phi^\lambda) = (0,0,0)$ and $(\bar n_{i,e}, \bar q_{i,e}, \bar\phi)$ is solution of the quasi-neutral system (2.6). We prove the following formal result:

LEMMA 1 (Formal). *The boundary layer quantities $(\tilde n_{i,e}, \tilde q_{i,e}, \phi)$ are solutions of the system:*

$$n_{i,e}(y,t) = \tilde n_{i,e}(y,t) + \bar n_{i,e}(0,t) = \tilde n_{i,e}(y,t) + \bar n(0,t), \tag{4.4}$$

$$q_{i,e}(y,t) = \tilde q_{i,e}(y,t) + \bar q_{i,e}(0,t), \tag{4.5}$$

$$\partial_y q_i = 0, \tag{4.6}$$

$$\partial_y (q_i^2/n_i + p_i(n_i)) = -n_i\,\partial_y\phi, \tag{4.7}$$

$$\partial_y q_e = 0, \tag{4.8}$$

$$\partial_y (\varepsilon\, q_e^2/n_e + p_e(n_e)) = n_e\,\partial_y\phi, \tag{4.9}$$

$$-\partial^2_{yy}\phi = n_i - n_e, \tag{4.10}$$

for all $y > 0$ and all $t > 0$.

The boundary conditions are given by:

$$\phi(0,t) = -\bar\phi_0, \qquad \phi(+\infty,t) = 0, \tag{4.11}$$

$$(n_i,q_i)(0,t) = (n_{i0},q_{i0})(t), \qquad (n_i(+\infty), q_i(+\infty)) = (\bar n_0, \bar q_{i0}), \tag{4.12}$$

$$(n_e,q_e)(0,t) = (n_{e0},q_{e0})(t), \qquad (n_e(+\infty), q_e(+\infty)) = (\bar n_0, \bar q_{e0}), \tag{4.13}$$

with $(\bar\phi_0, \bar n_0, \bar q_{i0}, \bar q_{e0}) = (\bar\phi, \bar n, \bar q_i, \bar q_e)|_{(0,t)}$ and where (n_{i0}, q_{i0}) and (n_{e0}, q_{e0}) are the respective solutions at the point $x = 0$ of the following Riemann problems

$$\left\{\begin{array}{l} \partial_t n_i + \partial_x q_i = 0,\\ \partial_t q_i + \partial_x((q_i)^2/n_i + p_i(n_i)) = 0,\\ \begin{pmatrix} n_i\\ q_i\end{pmatrix}(x,0) = \left\{\begin{array}{ll} \begin{pmatrix} n_0\\ q_0\end{pmatrix}, & \textit{if } x<0,\\ \begin{pmatrix} n_i^+\\ q_i^+\end{pmatrix}, & \textit{if } x>0,\end{array}\right.\end{array}\right. \qquad \left\{\begin{array}{l} \partial_t n_e + \partial_x q_e = 0,\\ \varepsilon\partial_t q_e + \partial_x(\varepsilon\,(q_e)^2/n_e + p_e(n_e)) = 0,\\ \begin{pmatrix} n_e\\ q_e\end{pmatrix}(x,0) = \left\{\begin{array}{ll} \begin{pmatrix} n_0\\ q_0\end{pmatrix}, & \textit{if } x<0,\\ \begin{pmatrix} n_e^+\\ q_e^+\end{pmatrix}, & \textit{if } x>0,\end{array}\right.\end{array}\right. \tag{4.14}$$

where $(n^+_{i,e}, q^+_{i,e}) = \lim_{y\to 0}(n_{i,e}(y,t), q_{i,e}(y,t))$ for all $t > 0$.

Proof: We insert (4.1)-(4.3) in (2.1), (2.2), we multiply the fluid equations by λ and we write the system for $y = x/\lambda$. We obtain

$$\partial_y \tilde q_{i,e} = 0,$$

$$2\,\frac{q_i}{n_i}\,\partial_y\tilde q_i + \left(p_i'(n_i) - \left(\frac{q_i}{n_i}\right)^2\right)\partial_y\tilde n_i = -n_i\,\partial_y\phi,$$

$$2\,\varepsilon\,\frac{q_e}{n_e}\,\partial_y\tilde q_e + \left(p_e'(n_e) - \left(\frac{\varepsilon\,q_e}{n_e}\right)^2\right)\partial_y\tilde n_e = n_e\,\partial_y\phi,$$

$$-\partial^2_{yy}\phi = n_i - n_e,$$

for all $y > 0$ and all $t > 0$ and where we define $n_{i,e}(y,t) = \bar{n}_{i,e}(0,t) + \tilde{n}_{i,e}(y,t)$, $q_{i,e}(y,t) = \bar{q}_{i,e}(0,t) + \tilde{q}_{i,e}(y,t)$. Writing the quasi-neutrality, i.e. $\bar{n}_e = \bar{n}_i = \bar{n}$, gives (4.4)-(4.10).

Now, the formal convergence towards the quasi-neutral solution, gives, for all $x > 0$ and $t > 0$:

$$\lim_{\lambda\to 0} \begin{pmatrix} n^{\lambda}_{i,e}(x,t) \\ q^{\lambda}_{i,e}(x,t) \\ \phi^{\lambda}(x,t) \end{pmatrix} = \begin{pmatrix} \bar{n}_{i,e}(x,t) + \tilde{n}_{i,e}(+\infty,t) \\ \bar{q}_{i,e}(x,t) + \tilde{q}_{i,e}(+\infty,t) \\ \bar{\phi}(x,t) + \phi(+\infty,t) \end{pmatrix} = \begin{pmatrix} \bar{n}_{i,e}(x,t) \\ \bar{q}_{i,e}(x,t) \\ \bar{\phi}(x,t) \end{pmatrix}.$$

This gives the boundary conditions at $+\infty$.

In order to close the boundary layer problem (4.4)-(4.10) we have to determine the boundary conditions at the cathode $y = 0$. For the potential we have $\phi^{\lambda}(0,t) = 0$ for all $t > 0$ which gives, in the limit $\lambda \to 0$: $\bar{\phi}(0,t) + \phi(0,t) = 0$.

Passing to the formal limit $\lambda \to 0$ in (2.3), we obtain for all $t > 0$

$$\begin{aligned} (\bar{n}_i + \tilde{n}_i, \bar{q}_i + \tilde{q}_i)(0,t) &= (n_i, q_i)(0,t) = (n_{i0}, q_{i0})(t), \\ (\bar{n}_e + \tilde{n}_e, \bar{q}_e + \tilde{q}_e)(0,t) &= (n_e, q_e)(0,t) = (n_{e0}, q_{e0})(t), \end{aligned}$$

where (n_{i0}, q_{i0}) and (n_{e0}, q_{e0}) are the respective solutions at the point $x = 0$ of the Riemann problems (4.14), which are obtained passing to the limit $\lambda \to 0$, in (2.5).

4.2. Resolution of the boundary layer problem. In this section, we solve the boundary layer problem. The aim consists in finding $\bar{n}_0$, $\bar{q}_{i0}$, $\bar{q}_{e0}$ and $\bar{\phi}_0$, defined in Lemma 1, such that system (4.6), (4.10) has a solution.

Let us look at solutions to (4.6)-(4.10). First remark that (4.7), (4.9) give constant momentums. Thus, using boundary conditions (4.12), (4.13) we obtain

$$q_{i0} = \bar{q}_{i0}, \quad q_{e0} = \bar{q}_{e0}. \tag{4.15}$$

Now, for smooth solutions, we define the total ion and electon enthalpies k_i and k_e such that for all $n > 0$, $\partial_n k_i(n) = -(\bar{q}_{i0})^2/n^3 + \partial_n p_i(n)/n$ and $\partial_n k_e(n) = -\varepsilon\,(\bar{q}_{e0})^2/n^3 + \partial_n p_e(n)/n$, i.e. given by

$$k_i(n) = \frac{(\bar{q}_{i0})^2}{2\,n^2} + \frac{C_i\,\gamma_i}{\gamma_i - 1} n^{\gamma_i - 1}, \quad k_e(n) = \frac{\varepsilon(\bar{q}_{e0})^2}{2\,n^2} + \frac{C_e\,\gamma_e}{\gamma_e - 1} n^{\gamma_e - 1}. \tag{4.16}$$

Note that k_i and k_e are non monotonous functions. They decrease respectively on $(0, n_{iS})$ and on $(0, n_{eS})$, and increase respectively on $(n_{iS}, +\infty)$ and $(n_{eS}, +\infty)$, where n_{iS} and n_{eS} are the ion and electron sonic points defined by

$$n_{iS} = \left(\frac{(\bar{q}_{i0})^2}{C_i\,\gamma_i}\right)^{1/(\gamma_i+1)}, \quad n_{eS} = \left(\frac{\varepsilon(\bar{q}_{e0})^2}{C_e\,\gamma_e}\right)^{1/(\gamma_e+1)}. \tag{4.17}$$

We assume $n_{iS} > n_{eS}$, this is the case if ε is small, if $\bar{q}_{i0}$ and $\bar{q}_{e0}$ are of same order and if C_e, C_i, γ_i and γ_e are order 1 parameters. In the following, we denote by $k_{i,+}$ (respectively $k_{e,+}$) the increasing branch of k_i (respectively of k_e) i.e. its restriction to $(n_{iS}, +\infty)$ (respectively to $(n_{eS}, +\infty)$) and we denote by $k_{i,-}$ (respectively $k_{e,-}$) the decreasing branch of k_i (respectively of k_e) i.e. its restriction to $(0, n_{iS})$ (respectively to $(0, n_{eS})$).

From (4.7), (4.9), (4.15) and (4.11)-(4.13) we deduce:

$$k_i(n_i(y)) = -\phi(y) + k_i(\bar{n}_0), \tag{4.18}$$

$$k_e(n_e(y)) = \phi(y) + k_e(\bar{n}_0), \tag{4.19}$$

for all $y \geq 0$. The inversion of (4.18), (4.19) involves four solutions, using $k_{i,+}^{-1}$ or $k_{i,-}^{-1}$ and $k_{e,+}^{-1}$ or $k_{e,-}^{-1}$. For y close to $+\infty$, it depends on the position of $\bar{n}_0$ w.r.t. n_{iS} and n_{eS}. Furthermore, summing (4.18) and (4.19) gives

$$k_e(n_e(y)) + k_i(n_i(y)) = k_e(\bar{n}_0) + k_i(\bar{n}_0) = k(\bar{n}_0). \tag{4.20}$$

But, $k' = k_i' + k_e'$ is an increasing continuous function such that $lim_{n\to 0^+} k'(n) = -\infty$ and $lim_{n\to +\infty} k'(n) = +\infty$ then, there exists a unique $n_S > 0$ such that

$$k'(n_S) = 0, \tag{4.21}$$

and k is a decreasing function on $(0, n_S)$ and an increasing function on $(n_S, +\infty)$. Furthermore, $k'(n_{iS}) = k_e'(n_{iS}) > 0$ since $n_{iS} > n_{eS}$ and similarly, $k'(n_{eS}) < 0$ thus, we have:

$$n_{iS} > n_S > n_{eS}.$$

Regarding non smooth solutions, for electrons and ions, shocks are possible but, only $-$shocks, i.e. shocks associated to the eigen value $\lambda_{i,-} = u_i - c_i = q_i/n_i - \sqrt{p_i'(n_i)}$ for ions and to $\lambda_{e,-} = u_e - c_e = q_e/n_e - \sqrt{p_e'(n_e)/\varepsilon}$ for electrons. Indeed, a $+$shock (i.e. a shock associated to the eigen value $\lambda_{i,+} = u_i + c_i$ for ions and to $\lambda_{e,+} = u_e + c_e$ for electrons) between (n_l, q_l) and (n_r, q_r) is admissible if and only if $n_l > n_r$. Furthermore, Rankine-Hugoniot relations give $u_l = q_l/n_l > u_r = q_r/n_r$. But, equation (4.6) and (4.8) give a constant current $q_l = q_r$ then, $u_l > u_r$ yields $n_l < n_r$, which is in contradiction with the admissibility condition. Consequently, between a left state (n_l, q_l) and a right state (n_r, q_r), only $-$shocks are admissible solutions of (4.6), (4.7) (or (4.8), (4.9)). They are characterized by

For ions:

$$\begin{cases} n_l < n_{iS} < n_r, \quad q_l = q_r, \\ \dfrac{(q_l)^2}{n_l} + p_i(n_l) = \dfrac{(q_r)^2}{n_r} + p_i(n_r), \end{cases}$$

For electrons:

$$\begin{cases} n_l < n_{eS} < n_r, \quad q_l = q_r, \\ \dfrac{\varepsilon\,(q_l)^2}{n_l} + p_e(n_l) = \dfrac{\varepsilon\,(q_r)^2}{n_r} + p_e(n_r), \end{cases} \tag{4.22}$$

where n_{iS} and n_{eS} are the ion and electron sonic points, given by (4.17). Furthermore, it is important to note that each state (n_{i0}, q_{i0}) such that $n_{i0} > n_{iS}$ is subsonic (i.e. $q_{i0}/n_{i0} + \sqrt{p_i'(n_{i0})} > 0$ and $q_0/n_0 - \sqrt{p_i'(n_{i0})} < 0$) when (n_{i0}, q_{i0}), such that $n_{i0} < n_{iS}$, is supersonic (i.e. $q_{i0}/n_{i0} + \sqrt{p_i'(n_{i0})} > 0$ and $q_0/n_0 - \sqrt{p_i'(n_{i0})} > 0$). The same remarks hold for electrons.

Moreover, let us consider two states (n_{il}, q_{il}) and (n_{ir}, q_{ir}) related by a $-$shock. We want to compare $k_i(n_{il})$ and $k_i(n_{ir})$. We set $f_i(n) = (\bar{q}_{i0})^2/n + p_i(n)$ for all $n > 0$. Thanks to (4.22), we have $n_{il} < n_{iS} < n_{ir}$, $\bar{q}_{i0} = q_{il} = q_{ir}$ and $f_i(n_{il}) = f_i(n_{ir})$. Using the definition of k_i, we get

$$k_i(n_{il}) - k_i(n_{ir}) = \int_{n_{il}}^{n_{ir}} \frac{f_i'(s)}{s}\, ds = \int_{n_{il}}^{n_{iS}} \frac{f_i'(s)}{s}\, ds + \int_{n_{iS}}^{n_{ir}} \frac{f_i'(s)}{s}\, ds.$$

But, f_i' is negative on $(0, n_{iS})$ and positive on $(n_{iS}, +\infty)$, so

$$k_i(n_{il}) - k_i(n_{ir}) \leq \frac{1}{n_{iS}} \left(\int_{n_{il}}^{n_{iS}} f_i'(s)\, ds + \int_{n_{iS}}^{n_{ir}} f_i'(s)\, ds \right) = \frac{f_i(n_{ir}) - f_i(n_{il})}{n_{iS}} = 0.$$

The same result holds for electrons.

Now, in order to resolve the boundary layer problem, we have to characterize (n_{i0}, q_{i0}) and (n_{e0}, q_{e0}), the respective solutions at the point $x = 0$ of ion and electron Riemann problems defined in (4.14). It is well known that the solutions of (4.14), are constituted of three states, (n_0, q_0), (n_{iI}, q_{iI}) and (n_i^+, q_i^+) for ions and (n_0, q_0), (n_{eI}, q_{eI}) and (n_e^+, q_e^+) for electrons, separated by elementary waves. These elementary waves can be rarefaction waves or shocks associated to the eigen values $\lambda_{i,\pm} = u_i \pm c_i = q_i/n_i \pm \sqrt{p_i'(n_i)}$ for ions and to $\lambda_{e,\pm} = u_e \pm c_e = q_e/n_e \pm \sqrt{p_e'(n_e)/\varepsilon}$ for electrons. Note that n_i^+, q_i^+, n_e^+ and q_e^+ are unknown then, $n_{i0}, q_{i0}, n_{e0}, q_{e0}$ will not always fully determined. We call $-$rarefaction wave (respectively shock) a rarefaction wave (respectively shock) associated to $\lambda_{i,-}$ or $\lambda_{e,-}$. Let us look at the different cases according to the $-$elementary wave for ions. The same results will hold for electrons.

Let us first suppose that the $-$elementary wave is a rarefaction wave. In this case the intermediate state (n_{iI}, q_{iI}) satisfies

$$\begin{cases} n_0 \geq n_{iI} > n_i^+, \\ W_{i,-}(n_{iI}, q_{iI}) = W_{i,-}(n_0, q_0), \end{cases}$$

where $W_{i,-}$ is the Riemann invariant associated to $\lambda_{i,-}$. It is given by $W_{i,-}(n,q) = q/n + 2\sqrt{C_i\,\gamma_i}(\gamma_i - 1)^{-1}\, n^{(\gamma_i-1)/2}$. The left state (n_0, q_0) is assumed subsonic, i.e. such that $\lambda_{i,-}(n_0, q_0) < 0$ and $\lambda_{i,+}(n_0, q_0) > 0$. Thus, the solution at the point $x = 0$ cannot be given by the left state (n_0, q_0). Furthermore, we recall that we assume $q_{i0} > 0$. Then, if the $+$elementary wave is a $+$rarefaction wave, $\lambda_{i,+}(n_{i0}, q_{i0}) > 0$ and the solution at the point $x = 0$ cannot be neither in the $+$rarefaction waves neither given by the right state. Similarly, if the $+$elementary wave is a shock, the $+$shock velocity σ_{i+}, is positive and the solution at the point $x = 0$ cannot be given by the right state. Indeed, if $\sigma_{i+} < 0$, Rankine-Hugoniot relations give $q_i^+ < 0$ and so $q_{i0} = q_i^+ < 0$. Thus, there are only two possibilities: $\lambda_{i,-}(n_{iI}, q_{iI}) \geq 0$ and (n_{i0}, q_{i0}) is in the $-$rarefaction wave, or $\lambda_{i,-}(n_{iI}, q_{iI}) < 0$ and $(n_{i0}, q_{i0}) = (n_{iI}, q_{iI})$.

For the first possibility ($\lambda_{i,-}(n_{iI}, q_{iI}) \geq 0$), the state (n_{i0}, q_{i0}) is fully determined by $\lambda_{i,-}(n_{i0}, q_{i0}) = 0$ and $W_{i,-}(n_{i0}, q_{i0}) = W_{i,-}(n_0, q_0)$. It is given by

$$n_{i0} = n_{ic} \leq n_0, \quad q_{i0} = \sqrt{C_i\,\gamma_i}\, n_{i0}^{(\gamma_i+1)/2},$$

where n_{ic} is defined by

$$n_{lc} = \left(\left(\frac{q_0}{n_0} + \frac{2\sqrt{k_l\,\gamma_l}}{\gamma_l - 1}\, n_0^{(\gamma_l-1)/2} \right) \left(\frac{1}{\sqrt{k_l\,\gamma_l} + \frac{2\sqrt{k_l\,\gamma_l}}{\gamma_l-1}} \right) \right)^{2/(\gamma_l-1)}, \tag{4.23}$$

for $l = i$ or e and with $k_i = C_i$ and $k_e = C_e/\varepsilon$.

Using (4.17) and the definition of (n_{i0}, q_{i0}), we show that $n_{ic} = n_{iS}$. Thus, thanks to (4.22), no shock is possible with (n_{i0}, q_{i0}) for left state.

For the second possibility, $\lambda_{i,-}(n_{iI}, q_{iI}) < 0$, the state (n_{i0}, q_{i0}) is given by the intermediate state (n_{iI}, q_{iI}). It satisfies

$$n_0 \geq n_{i0} > n_{ic}, \quad q_{i0} = n_{i0} \left(\frac{q_0}{n_0} + \frac{2\sqrt{C_i\,\gamma_i}}{\gamma_i - 1} \left(n_0^{(\gamma_i-1)/2} - n_{i0}^{(\gamma_i-1)/2} \right) \right), \tag{4.24}$$

where n_{ic} is given by (4.23). An easy calculation shows that

$$n_{i0} \leq n_{iS} \Rightarrow n_{i0} \leq n_{ic}.$$

Indeed, if $n_{i0} \le n_{iS}$ then, $q_{i0} \ge \sqrt{C_i\,\gamma_i}\, n_{i0}^{(\gamma_i+1)/2}$ since q_{i0} is assumed positive. But, q_{i0} is given by (4.24), it yields

$$\left(\frac{q_0}{n_0} + \frac{2\sqrt{C_i\,\gamma_i}}{\gamma_i - 1}\left(n_0^{(\gamma_i-1)/2} - n_{i0}^{(\gamma_i-1)/2}\right)\right) \ge \sqrt{C_i\,\gamma_i}\, n_{i0}^{(\gamma_i-1)/2},$$

which gives $n_{i0} \le n_{ic}$.

Since $n_{i0} > n_{ic}$, we have $n_{i0} > n_{iS}$ and no shock, with left state (n_{i0}, q_{i0}), is possible for ions.

Now, let us suppose that the $-$elementary wave is a shock. We denote by σ_{i-} the shock velocity. The Lax entropy conditions gives $\lambda_{i,-}(n_0, q_0) > \sigma_{i-}$. Since $\lambda_{i,-}(n_0, q_0) < 0$, the left state (n_0, q_0) is assumed subsonic, we have $\sigma_{i-} < 0$ and the solution at the point $x = 0$ cannot be given by the left state (n_0, q_0). Furthermore, as previously, the assumption $q_{i0} > 0$ ensures that the solution at the point $x = 0$ can be only given by the intermediate state (n_{iI}, q_{iI}). Using Rankine-Hugoniot relations and Lax entropy conditions, we get

$$n_{i0} > n_0, \quad q_{i0} = n_{i0}\left(\frac{q_0}{n_0} - (n_{i0} - n_0)\sqrt{\frac{p_i(n_{i0}) - p_i(n_0)}{(n_{i0} - n_0)\, n_{i0}\, n_0}}\right). \tag{4.25}$$

We want to compare n_{i0} and the ion sonic point n_{iS} given by (4.17). Using (4.17), the positivity of q_{i0} and the definition of q_{i0}, we have

$$n_{iS}^{\gamma_i+1} \ge n_{i0}^{\gamma_i+1} \Longleftrightarrow \frac{q_0}{n_0} - (n_{i0} - n_0)\sqrt{\frac{p_i(n_{i0}) - p_i(n_0)}{(n_{i0} - n_0)\, n_{i0}\, n_0}} > \sqrt{C_i\,\gamma_i\, n_{i0}^{\gamma_i-1}},$$

but, $\lambda_{i,-}(n_0, q_0) < 0$ then, the previous result yields

$$0 \ge -(n_{i0} - n_0)\sqrt{\frac{p_i(n_{i0}) - p_i(n_0)}{(n_{i0} - n_0)\, n_{i0}\, n_0}} > \sqrt{C_i\,\gamma_i\, n_{i0}^{\gamma_i-1}} - \sqrt{C_i\,\gamma_i\, n_0^{\gamma_i-1}} > 0$$

since $n_{i0} > n_0$. Thus, $n_{i0} > n_{iS}$, and no shock, with left state (n_{i0}, q_{i0}), is possible for ions.

It is important to note that in all cases $n_{i0} \ge n_{iS}$. The same results hold for electrons. Now, let us remark that ion or electron shocks, with right state $(\bar{n}_0, \bar{q}_{i0})$ or $(\bar{n}_0, \bar{q}_{e0})$, are possible. It depends on the position of $\bar{n}_0$ w.r.t. n_{iS} and n_{eS}. We look at the solution considering the different cases and we prove the following result

Theorem 4.1. *We consider the boundary layer problem (4.4)-(4.14), where (n_0, q_0) is a subsonic state for ions and electrons, i.e. such that*

$$q_0/n_0 + \sqrt{p_i'(n_0)} > 0, \qquad q_0/n_0 + \sqrt{p_e'(n_0)/\varepsilon} > 0,$$
$$q_0/n_0 - \sqrt{p_i'(n_0)} < 0, \qquad q_0/n_0 - \sqrt{p_e'(n_0)/\varepsilon} < 0.$$

Furthermore, we suppose that $q_{i0}, q_{e0} > 0$. We define the ion and electron sonic points, n_{iS} and n_{eS}, by (4.17). Then:

1. *If $\bar{n}_0 > n_{iS} > n_{eS}$ we have:*

1.1 If (n_{i0}, q_{i0}) *is given by* $n_{i0} = n_{ic} \le n_0$*, with* n_{ic} *given by (4.23), and* $q_{i0} = \sqrt{C_i\,\gamma_i}\, n_{i0}^{(\gamma_i+1)/2}$ *and if* (n_{e0}, q_{e0}) *satisfies*

$$n_0 \ge n_{e0} > n_{ec}, \quad q_{e0} = n_{e0}\left(\frac{q_0}{n_0} + \frac{2\sqrt{\gamma_e\,C_e/\varepsilon}}{\gamma_e - 1}\left(n_0^{\frac{\gamma_e-1}{2}} - n_{e0}^{\frac{\gamma_e-1}{2}}\right)\right), \tag{4.26}$$

or (n_{e0}, q_{e0}) *satisfies*

$$n_{e0} > n_0, \quad q_{e0} = n_{e0}\left(\frac{q_0}{n_0} - (n_{e0} - n_0)\sqrt{\frac{p_e(n_{e0}) - p_e(n_0)}{\varepsilon\,(n_{e0} - n_0)\,n_{e0}\,n_0}}\right), \tag{4.27}$$

problem (4.4)-(4.14) has a solution. Three cases are possible:

- *The solution exists if* n_{i0} *and* n_{e0} *satisfy*

$$k_i(n_{i0}) + k_e(n_{e0}) \ge k_i(n_S) + k_e(n_S), \tag{4.28}$$

where n_S *is defined by (4.21). The solution is continuous with increasing potential* ϕ *and electron density* n_e*, and a decreasing ion density* n_i*. It satisfies*

$$\bar\phi_0 = k_i(n_{i0}) - k_i(\bar n_0), \tag{4.29}$$
$$\bar q_{i0} = q_{i0}, \tag{4.30}$$
$$\bar q_{e0} = q_{e0}, \tag{4.31}$$
$$k_i(n_{i0}) + k_e(n_{e0}) = k_i(\bar n_0) + k_e(\bar n_0). \tag{4.32}$$

- *The solution exists if* n_{i0} *and* n_{e0} *satisfy (4.28). It is continuous with decreasing potential* ϕ *and electron density* n_e*, and an increasing ion density* n_i *and it satisfies (4.29)-(4.32).*

- *The solution is unsmooth with a jump of* n_i *from a left state* $n_i^{\star,-}$ *to a right state* $n_i^{\star,+}$ *satisfying (4.38). The ion density,* n_i*, is a decreasing function before the shock and an increasing function after the shock while the potential,* ϕ*, and the electron density,* n_e*, are both decreasing functions. This solution exists if and only if the following property is satisfied*

$$k_e(n_{e0}) > k_e(n_S) + k_i(n_S) - k_i(n_i^{\star,+}) + k_i(n_i^{\star,-}) - k_i(n_{iS}), \tag{4.33}$$

1.2 If (n_{i0}, q_{i0}) *satisfies (4.24) or (4.25)*

1.2.1 If (n_{e0}, q_{e0}) *is given by* $n_{e0} = n_{ec} \le n_0$*, with* n_{ec} *given by (4.23), and* $q_{e0} = \sqrt{\gamma_e\,C_e/\varepsilon}\, n_{e0}^{(\gamma_e+1)/2}$*, problem (4.4)-(4.14) has a solution. Two cases are possible*

- *The solution exists if* n_{i0} *and* n_{e0} *satisfy (4.28) where* n_S *is defined by (4.21). It is continuous with increasing potential* ϕ *and electron density* n_e*, and a decreasing ion density* n_i *and it satisfies (4.29)-(4.32).*

- *The solution is unsmooth with a jump of* n_e *from a left state* $n_e^{\star,1,-}$ *to a right state* $n_e^{\star,1,+}$ *satisfying (4.37). The electron density,*

n_e, is a decreasing function before the shock and an increasing function after the shock while the potential, ϕ, and the ion density, n_i, are respectively increasing and decreasing functions. This solution exists if and only if the following property is satisfied

$$k_i(n_{i0}) > k_i(n_S)+k_e(n_S)-k_e(n_e^{\star,1,+})+k_e(n_e^{\star,1,-})-k_e(n_{eS}). \quad (4.34)$$

1.2.2 *If (n_{e0}, q_{e0}) satisfy (4.26) or (4.27), problem (4.4)-(4.14) has a solution. Two cases are possible*

- *The solution exists if $n_{e0} \le \bar{n}_0$ and n_{i0} and n_{e0} satisfy (4.28) where n_S is defined by (4.21). It is continuous with increasing potential ϕ and electron density n_e, and a decreasing ion density n_i. It satisfies (4.29)-(4.32).*

- *The solution is unsmooth with a jump of n_i from a left state $n_i^{\star,-}$ to a right state $n_i^{\star,+}$ satisfying (4.38). The ion density, n_i, is a decreasing function before the shock and an increasing function after the shock while the potential, ϕ, and the electron density, n_e, are both decreasing functions. This solution exists if and only if property (4.33) is satisfied.*

2. *If $n_{iS} > n_S > \bar{n}_0 > n_{eS}$, if (n_{i0}, q_{i0}) is given by $n_{i0} = n_{ic} \le n_0$ and $q_{i0} = \sqrt{C_i\,\gamma_i}\, n_{i0}^{(\gamma_i+1)/2}$, and if (n_{e0}, q_{e0}) satisfies (4.26) or (4.27) then, problem (4.4)-(4.14) has a solution. This solution exists if (4.28) is satisfied. It is continuous and the potential, ϕ, the ion and electron densities, n_i and n_e, are decreasing functions. It satisfies (4.29)-(4.32).*

Proof

If $n_{iS} > n_{eS} > \bar{n}_0$, a shock, with right state $(\bar{n}_0, \bar{q}_{i0})$ or $(\bar{n}_0, \bar{q}_{e0})$, is not possible for ions and electrons (see (4.22)). For smooth solutions, we invert (4.18) and (4.19) in a neighborhood of $y = +\infty$ using $k_{i,-}^{-1}$ and $k_{e,-}^{-1}$, it yields

$$n_i(y) = n_i[\phi(y)] = k_{i,-}^{-1}(-\phi(y) + k_i(\bar{n}_0)),$$
$$n_e(y) = n_e[\phi(y)] = k_{e,-}^{-1}(\phi(y) + k_e(\bar{n}_0)).$$

Then, in a neighborhood of $y = +\infty$, the Poisson equation (4.10) is written

$$\partial_{yy}^2\phi = n_e[\phi] - n_i[\phi],$$

with the boundary conditions $\phi(0) = -\bar{\phi}_0$ and ϕ tends to 0 as $y \to +\infty$. Note that ϕ and $\partial_y\phi$ are continuous functions. We write the previous differential equation as a first-order differential system

$$\partial_y\phi = E, \qquad \partial_y E = n_e[\phi] - n_i[\phi]. \quad (4.35)$$

Since $k_{i,-}^{-1}$ is a decreasing function, $n_i[\cdot]$ is an increasing function. Similarly, $n_e[\cdot]$ is a decreasing function and, $n_e[\cdot]-n_i[\cdot]$ is a decreasing function. The point $(\phi, E) = (0, 0)$ is an elliptic stationary point of (4.35). There is no solution to (4.35) with $(0, 0)$ as final condition other than the constant solution $(\phi, E) = (0, 0)$ itself.

If $\bar{n}_0 > n_{iS} > n_{eS}$, a shock is possible for ions and for electrons (see (4.22)). For smooth solutions, we invert (4.18) and (4.19) in a neighborhood of $y = +\infty$ using $k_{i,+}^{-1}$ and $k_{e,+}^{-1}$, we get

$$n_i(y) = n_i[\phi(y)] = k_{i,+}^{-1}(-\phi(y) + k_i(\bar{n}_0)),$$
$$n_e(y) = n_e[\phi(y)] = k_{e,+}^{-1}(\phi(y) + k_e(\bar{n}_0)). \quad (4.36)$$

We insert the results in (4.35). The point $(\phi, E) = (0,0)$ is a stationary point of (4.35) which is hyperbolic since $n_e[\cdot] - n_i[\cdot]$ is now an increasing function.

There are two branches of solutions arriving on $(0,0)$ (see Figure 4.1). On the first branch, ϕ increases towards 0 and E decreases towards 0, and on the second branch, ϕ decreases towards 0 and E increases towards 0.

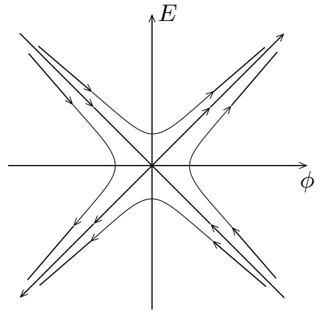

Fig. 4.1. *Phase-portrait of the solution of system (4.35) near the hyperbolic stationary point* $(0,0)$ *in the case* $\bar{n}_0 > n_{iS} > n_{eS}$.

Let us look at the first branch. We consider the different positions of n_{e0} and n_{i0} w.r.t. n_{iS} and n_{eS}.

If $n_{i0} = n_{iS}$ and $n_{e0} \geq n_{eS}$, for continuous solutions, the potential ϕ increases towards 0 while E decreases towards 0. Thanks to (4.36), n_e increases towards $\bar{n}_0 > n_{eS}$ and n_i decreases towards $\bar{n}_0 > n_{iS}$. Suppose that at some point $y_\star$, $n_e(y_\star) = n_{eS}$. Then, if $n_{e0} = n_{eS}$ we have $n_e(y_\star) = n_{e0}$ but, $n_i(y_\star) > \bar{n}_0 > n_{iS} = n_{i0}$. Otherwise if $n_{eS} < n_{e0} \leq \bar{n}_0$, there exists $y_{\star,0} > y_\star$ such that $n_e(y_{\star,0}) = n_{e0}$ but, $n_i(y_{\star,0}) > \bar{n}_0 > n_{iS} = n_{i0}$. Finally if $n_{eS} < \bar{n}_0 < n_{e0}$, n_e has not reached n_{e0}. At the point $n_e(y_\star) = n_{eS}$, k_e has reached its minimum and thanks to (4.19), before $y_\star$, ϕ must decrease. But, $n_i(y_\star) > \bar{n}_0 > n_{iS} > n_{eS}$, and near $y_\star$, $\partial_y E = n_e - n_i < 0$ then, $E > 0$. But, ϕ decreases and so $\partial_y \phi = E < 0$. Therefore, the solution cannot be extended any further back and, n_i cannot come from $n_{i0} = n_{iS}$. For unsmooth solutions with a jump of n_e, at some point $y_{\star,0} > y_\star$, from the left electron density $n_e^{\star,-} < n_{eS}$ to the right electron density $n_e^{\star,+} > n_{eS}$, ϕ shall continuously increases. But, now n_e is constrained to come from the supersonic branch ($k_{e,-}$) and so decreases, from n_{eS} towards $n_e^{\star,-}$. For a continuous ion density, n_i still decreases towards $\bar{n}_0$. The same arguments as previously hold, if n_e reaches n_{e0}, we have $n_i > n_{iS}$ and when n_e reaches n_{eS}, the solution cannot be extended any further back, so n_i cannot come from $n_{i0} = n_{iS}$. For an unsmooth ion density with, at some point $y_{\star,1} > y_\star$, a jump of n_i (before n_e has reached n_{eS}), n_i is constrained to come from the supersonic branch on $(0, y_{\star,1})$ and so increases from 0. When n_e reaches n_{e0}, we have $n_i < n_{iS}$ and when n_e reaches n_{eS} the solution cannot be extended any further back, so n_i cannot come from $n_{i0} = n_{iS}$.

If $n_{i0} > n_{iS}$ and $n_{e0} \geq n_{eS}$, for continuous solutions, we still have an increasing potential ϕ towards 0 and a decreasing function E towards 0. Thanks to (4.36), n_e increases towards $\bar{n}_0 > n_{eS}$ and n_i decreases towards $\bar{n}_0 > n_{iS}$. If $n_{e0} \leq \bar{n}_0$ there exists $y_\star > 0$ such that $n_e(y_\star) = n_{e0}$ and we can have $n_i(y_\star) = n_{i0}$ if and only if $n_{i0} > \bar{n}_0$. Now, writing (4.18) and (4.20) for $y = y_\star$, we get (4.29) and (4.32). In order to have a solution $\bar{n}_0$ of this equation we must have $k_i(n_{i0}) + k_e(n_{e0}) \geq k_i(n_S) + k_e(n_S)$. Note that $\bar{n}_0 \geq n_{e0} \geq n_{eS}$ gives $k_e(\bar{n}_0) - k_e(n_{e0}) \geq 0$ and $k_i(n_{i0}) - k_i(\bar{n}_0) \geq 0$ and so $n_{i0} \geq \bar{n}_0 > n_{iS}$. Then, if $n_{e0} \leq \bar{n}_0$ and n_{i0} and n_{e0} satisfy the conditions given in (4.28), it is possible to connect from $y = 0$ to $y = +\infty$, the states $(n_{i0}, q_{i0}, n_{e0}, q_{e0}, -\bar{\phi}_0)$ and $(n_i, q_i, n_e, q_e, \phi) = (\bar{n}_0, \bar{q}_0, \bar{n}_0, \bar{q}_0, 0)$ with the boundary layer problem. Remark that this solution does not give decreasing densities since n_e is an increasing function.

We consider unsmooth solutions, with, at some point $y_{\star,0} > y_\star$, a jump of n_i from the left ion density $n_i^{\star,0,-} < n_{iS}$ to the right density $n_i^{\star,0,+} > n_{iS}$. Before $y_{\star,0}$, the ion density follows the supersonic branch of k_i and n_i increases towards $n_i^{\star,0} < n_{iS}$ then, it cannot come from $\bar{n}_0 > n_{iS}$ except if the potential changes its monotony which is not possible.

For an unsmooth electron density, with at some point $y_{\star,1} > y_\star$, a jump of n_e from the left electron density $n_e^{\star,1,-} < n_{eS}$ to the right density $n_e^{\star,1,+} > n_{eS}$, the potential ϕ continuously increases. Before $y_{\star,1}$, the electron density follows the supersonic branch of k_e and n_e decreases towards $n_e^{\star,1} < n_{eS}$. Before and after $y_{\star,1}$, n_i continuously decreases towards $\bar{n}_0$. As previously, when n_e reaches n_{eS} at some point $y_{\star,2}$, the solution cannot be extended any further back, and so a solution is possible if and only if $n_{e0} = n_{eS} = n_e(y_{\star,2})$ but, we can have $n_i(y_{\star,2}) = n_{i0}$ if and only if $n_{i0} > \bar{n}_0$. In this case, we have $n_{i0} > n_i(y_\star) > \bar{n}_0 > n_{iS}$, $\bar{n}_0 > n_e^{\star,+} > n_{eS} = n_{e0} > n_e^{\star,-}$ and using (4.22) and (4.20) yields

$$\begin{aligned}
&k_i(n_i(y_{\star,1}) + k_e(n_e^{\star,1,+}) = k_i(\bar{n}_0) + k_e(\bar{n}_0),\\
&\frac{\varepsilon\,(q_{e0})^2}{n_e^{\star,1,+}} + p_e(n_e^{\star,1,+}) = \frac{\varepsilon\,(q_{e0})^2}{n_e^{\star,1,-}} + p_e(n_e^{\star,1,-}),\\
&k_i(n_i(y_{\star,1}) + k_e(n_e^{\star,1,-}) = k_i(n_{i0}) + k_e(n_{e0}) = k_i(n_{i0}) + k_e(n_{eS}).
\end{aligned} \tag{4.37}$$

This gives $k_i(n_{i0}) = k_i(\bar{n}_0) + k_e(\bar{n}_0) - k_e(n_e^{\star,1,+}) + k_e(n_e^{\star,1,-}) - k_e(n_{eS}) > k_i(\bar{n}_0)$ since $\bar{n}_0 > n_e^{\star,1,+}$ and $k_e(n_{eS}) = \min k_e$. If $\bar{n}_0$ exists, it satisfies $\bar{n}_0 < n_{i0}$. In order to have a solution $\bar{n}_0$, n_{i0} and n_{e0} must satisfy (4.34) and $n_{e0} = n_{eS}$. This gives a condition on n_{i0}. In conclusion if $n_{e0} = n_{eS}$ and n_{i0} satisfies (4.34), it is possible to find a solution of the boundary layer problem (4.4)-(4.14) with an unsmooth electron density. Note that in this case, the electron density is non monotonous.

Now, let us look at the second branch of solutions of (4.35), arriving on $(\phi, E) = (0, 0)$. We still consider the different positions of n_{e0} and n_{i0} w.r.t. n_{iS} and n_{eS}.

If $n_{i0} \geq n_{iS}$ and $n_{e0} = n_{eS}$, for continuous or unsmooth solutions, the same arguments as previously used for the first branch in the case $n_{e0} \geq n_{eS}$ and $n_{i0} = n_{iS}$, give that it is not possible to find a solution of the boundary layer problem.

If $n_{i0} \geq n_{iS}$ and $n_{e0} > n_{eS}$, for continuous or unsmooth solutions We use the same arguments as those used for the first branch in the case $n_{e0} \geq n_{eS}$ and $n_{i0} > n_{iS}$. The conclusion is the following. For continuous solutions, if $n_{i0} \leq \bar{n}_0$ and n_{i0} and n_{e0} satisfy (4.28), it is possible to connect from $y = 0$ to $y = +\infty$, the states $(n_{i0}, q_{i0}, n_{e0}, q_{e0}, -\bar{\phi}_0)$ and $(n_i, q_i, n_e, q_e, \phi) = (\bar{n}_0, \bar{q}_0, \bar{n}_0, \bar{q}_0, 0)$ with the boundary layer problem. The resulting solution gives a decreasing electron density but, an increasing ion density.

For an unsmooth electron density it is not possible to find a solution of the boundary layer problem. And, for a continuous electron density and an unsmooth ion density with at some point $y_\star$, a jump of n_i from the left ion density $n_i^{\star,-}$ to the right density $n_i^{\star,+}$. We introduce assumption (4.33) on n_{i0}, n_{e0}, $n_i^{\star,-}$ and $n_i^{\star,+}$ where

$$\begin{aligned}
&k_e(n_e(y_{\star,1}) + k_i(n_i\star,+) = k_e(\bar{n}_0) + k_i(\bar{n}_0),\\
&\frac{(q_{i0})^2}{n_i^{\star,+}} + p_i(n_i^{\star,+}) = \frac{(q_{i0})^2}{n_i^{\star,-}} + p_i(n_i^{\star,-}),\\
&k_e(n_e(y_\star) + k_i(n_i^{\star,-}) = k_e(n_{e0}) + k_i(n_{i0}) = k_e(n_{e0}) + k_i(n_{iS}).
\end{aligned} \tag{4.38}$$

The conclusion is the following. If n_{i0}, n_{e0}, $n_i^{\star,-}$ and $n_i^{\star,+}$ satisfy (4.33)and $n_{i0} = n_{iS}$, it is possible to find a solution of the boundary layer problem. This solution gives a non monotonous ion density. This concludes the case $\bar{n}_0 > n_{iS} > n_{eS}$.

Let us finish with the case $n_{iS} > \bar{n}_0 > n_{eS}$. In this case, a shock is possible for electrons but, not for ions (see (4.22)). For smooth solutions, we invert (4.18) and (4.19) in a neighborhood of $y = +\infty$ using $k_{i,-}^{-1}$ and $k_{e,+}^{-1}$, it yields

$$\begin{aligned} n_i(y) &= n_i[\phi(y)] = k_{i,-}^{-1}(-\phi(y) + k_i(\bar{n}_0)), \\ n_e(y) &= n_e[\phi(y)] = k_{e,+}^{-1}(\phi(y) + k_e(\bar{n}_0)). \end{aligned} \tag{4.39}$$

Therefore, $n_e[\cdot] - n_i[\cdot]$ is the difference of two increasing functions. And, we have $d/d\phi(n_e[\phi] - n_i[\phi])|_{\phi=0} = 1/\partial_n k_{e,+}(\bar{n}_0) + 1/\partial_n k_{i,-}(\bar{n}_0) > 0$ if and only if $\bar{n}_0 < n_S$, where n_S is the plasma sonic point defined by (4.21). If $n_{iS} > \bar{n}_0 > n_S$, the point $(\phi, E) = (0, 0)$ is an elliptic stationary point of system (4.35) and there is no solution to (4.35) with $(0, 0)$ as final condition except the constant solution $(\phi, E) = (0, 0)$ itself.

Thus, we consider $n_{iS} > n_S > \bar{n}_0 > n_{eS}$. In this case, there are two branches of solutions arriving on $(0, 0)$ (see Figure 4.1).

Let us look at the first branch. We recall that $n_{i0} \geq n_{iS}$ and $n_{e0} \geq n_{eS}$. For continuous solutions, the potential ϕ increases towards 0 and E decreases towards 0. Thanks to (4.39), n_i and n_e increases towards $\bar{n}_0$. Suppose that at some point $y_\star > 0$, $n_e(y_\star) = n_{eS}$. If $n_{e0} = n_{eS}$, we have $n_e(y_\star) = n_{e0}$ but, $n_i(y_\star) \leq \bar{n}_0 < n_{iS} \leq n_{i0}$ and, n_i cannot come from n_{i0}. If $n_{e0} > n_{eS}$, there exists $y_{\star,0} > y_\star$ such that $n_e(y_{\star,0}) = n_{e0}$ but, $n_i(y_{\star,0}) \leq \bar{n}_0 < n_{iS} \leq n_{i0}$ and, n_i cannot come from n_{i0}. Furthermore, when n_e has reached n_{eS}, the solution cannot be extended any further back. We consider unsmooth solutions with at some point $y_{\star,1} > y_\star$, a jump of n_e from a left value $n_e^{\star,1,-} < n_{eS}$ to a right value $n_e^{\star,1,+} > n_{eS}$. For $y < y_{\star,1}$, the potential increases continuously but, n_e is constrained to follow the supersonic branch of k_e, i.e. $k_{e,-}$ then, it decreases towards $n_e^{\star,1,-} < n_{eS}$. At some point $y_{\star,2} < y_{\star,1}$, $n_e(y_{\star,2}) = n_{eS}$. If $n_{e0} = n_{eS}$, we have $n_e(y_\star) = n_{e0}$ but, $n_i(y_\star) \leq \bar{n}_0 < n_{iS} \leq n_{i0}$ and, n_i cannot come from n_{i0}. If $n_{e0} > n_{eS}$, n_e cannot come from n_{e0}.

We look at the second branch and consider different cases according to the positions of n_{i0} and n_{e0} w.r.t. n_{is} and n_{eS}.

If $n_{i0} \geq n_{iS}$ and $n_{e0} \geq n_{eS}$, continuous solutions give a decreasing potential, ϕ, towards 0 and an increasing function E towards 0. Thus, n_i and n_e decreases towards $n_{iS} > n_S > \bar{n}_0 > n_{eS}$. Suppose that at some point $y_\star > 0$, $n_i(y_\star) = n_{iS}$ then, using the same arguments as previously, we show that the solution cannot be extended any further back. Since $n_{i0} \geq n_{iS}$, n_i can come from n_{i0} if and only if $n_{i0} = n_{iS} = n_i(y_\star)$. Moreover, we can have $n_e(y_\star) = n_{e0}$ if and only if $n_{e0} \geq \bar{n}_0 > n_{eS}$. In this case, writing (4.20) for $y = y_\star$, we get $k_i(n_{iS}) + k_e(n_{e0}) = k_i(\bar{n}_0) + k_e(\bar{n}_0)$. In order to have a solution $\bar{n}_0$ of this equation, we must have $k_i(n_{iS}) + k_e(n_{e0}) \geq k_i(n_S) + k_e(n_S)$. Note that $k_i(\bar{n}_0) - k_i(n_{iS}) > 0$ and $k_e(n_{e0}) - k_e(\bar{n}_0) > 0$ and so, $n_{e0} > \bar{n}_0 > n_{eS}$. Then, if $n_{i0} = n_{iS}$ and if n_{i0} and n_{e0} satisfy (4.28), it is possible to connect from $y = 0$ to $y = +\infty$, the states $(n_{i0}, q_{i0}, n_{e0}, q_{e0}, -\bar{\phi}_0)$ and $(n_i, q_i, n_e, q_e, \phi) = (\bar{n}_0, \bar{q}_0, \bar{n}_0, \bar{q}_0, 0)$ with the boundary layer problem. Remark that this solution give continuous decreasing electron and ion densities. We finish with unsmooth solutions. If, at some point $y_{\star,0} > y_\star$, n_e jumps from a left value $n_e^{\star,-}$ to a right value $n_e^{\star,+}$, then, before $y_{\star,0}$, n_e increases towards $n_e^{\star,-}$, ϕ and, n_i continuously decrease respectively towards 0 and $\bar{n}_0$. At some point $y_{\star,1} < y_{\star,0}$, we have $n_i(y_{\star,1}) = n_{iS}$ and, the solution cannot be extended any further back. Since $n_{i0} \geq n_{iS}$, n_i can come from n_{i0}, if and only if

$n_{i0} = n_{iS} = n_i(y_{\star,2})$ but, now $n_e(y_{\star,2}) < n_e^{\star,-} < n_{eS} \leq n_{e0}$ and, n_e cannot come from n_{e0}.

This concludes the proof of Theorem 4.1.

5. Numerical results using the solution of the boundary layer problem. We want to use the results on the boundary layer given in Theorem 4.1 to determine well-adapted boundary conditions for the one dimensional plasma expansion test case presented in section 3. We want boundary conditions at the quasi-neutral equilibrium, $\bar{n}_0$, $\bar{q}_{e0}$, $\bar{q}_{i0}, \bar{\phi}_0$, to stabilize the classical scheme with general solvers, when the mesh does not resolve the scaled Debye length but, resolves the plasma period and for the asymptotic preserving scheme when it does not resolved the scaled Debye length and the scaled plasma period. We recall that for this test case, we have $\gamma_i = \gamma_e = 5/3$, $C_i = C_e = 1$, $\varepsilon = 10^{-4}$, $\lambda = 10^{-4}$, $\phi_A = 100$ and $(n_0, q_0) = (1, 1)$. Following numerical results in the resolved case given in section 3, we look for solutions of the boundary layer problem with decreasing electron and ion densities. Thanks to Theorem 4.1, there are several solutions to the boundary layer problem but, only one solution gives decreasing densities. It is characterized by $n_{iS} > n_S > \bar{n}_0 > n_{eS}$ where n_{iS} and n_{eS} are given by (4.17), (n_{i0}, q_{i0}) is given by $n_{i0} = n_{ic} \leq n_0$, where n_{ic} is defined by (4.23), and $q_{i0} = \sqrt{C_i\,\gamma_i}\, n_{i0}^{(\gamma_i+1)/2}$, and (n_{e0}, q_{e0}) satisfies (4.26) or (4.27). This solution exists if and only if (4.28) is satisfied. Note that (n_{i0}, q_{i0}) is fully determined by (n_0, q_0). Thus, using the boundary layer system (4.6)-(4.10), we have $\bar{q}_{i0} = q_{i0}$ and $\bar{q}_{i0}$ is determined. On the contrary, (n_{e0}, q_{e0}) is not fully determined by (n_0, q_0) since (4.26), or (4.27), only gives one equation for two unknowns. But, this is not surprising. Indeed, $\bar{n}_0 > n_{eS}$ then, $(\bar{n}_0, \bar{q}_{e0})$ is a subsonic electron state and $(\bar{n}_0, \bar{q}_{e0})$ can not fully determined by the information coming from the left hand side. If (n_{e0}, q_{e0}) was fully determined by (n_0, q_0) then, using the boundary layer system (4.6)-(4.10) and equation (4.32), we would have $(\bar{n}_0, \bar{q}_{e0})$ fully determined by the left state (n_0, q_0). Then, (n_{e0}, q_{e0}) is unknown and depends on the information coming from the right hand side. But, ε is a small parameter thus, multiplying (4.26) or (4.27) by $\sqrt{\varepsilon}$ and letting ε tend to 0 give $n_{e0} = n_0$. Similarly, letting ε tend to 0 in (4.32) yields

$$h_e(n_0) + k_i(n_{i0}) = h_e(\bar{n}_0) + k_i(\bar{n}_0), \tag{5.1}$$

where, for all $n > 0$, $h_e(n) = C_e \frac{\gamma_e}{\gamma_e - 1}\, n^{\gamma_e - 1}$. Note that (5.1) and (4.29) give $\bar{n}_0$ and $\bar{\phi}_0$ as functions of known data.

It remains to determine $\bar{q}_{e0}$. We use the right information and we set $\bar{q}_{e0} = \lim_{x\to 0} \bar{q}_e(x,t)$ where $\bar{q}_e$ is the quasi-neutral solution since in the boundary layer problem, we have $\partial_y q_e = 0$.

So, in order to take into account the boundary layer in the numerical simulations of the plasma expansion test case, we discretize the Euler-Poisson system (2.1)-(2.2) with the classical or asymptotic preserving scheme, using the boundary conditions for the potential

$$\phi(x=0) = \bar{\phi}_0, \quad \phi(x=1) = \phi_A,$$

where $\bar{\phi}_0$ is given by (4.29), and using the boundary conditions for the fluid quantities (2.3), where $(n_{i0}^\lambda, q_{i0}^\lambda)$ and $(n_{e0}^\lambda, q_{e0}^\lambda)$ are the solutions at the point $x = 0$ of the Riemann problems (2.5) changing, in the ion problem, (n_0, q_0) by $(\bar{n}_0, \bar{q}_{i0})$ given by (5.1) and $\bar{q}_{i0} = \sqrt{C_i\,\gamma_i}\, n_{i0}^{(\gamma_i+1)/2}$ with $n_{i0} = n_{ic}$, n_{ic} given by (4.23). In the electron Riemann problem, we change (n_0, q_0) by $(\bar{n}_0, q_e^\lambda(0^+, t))$.

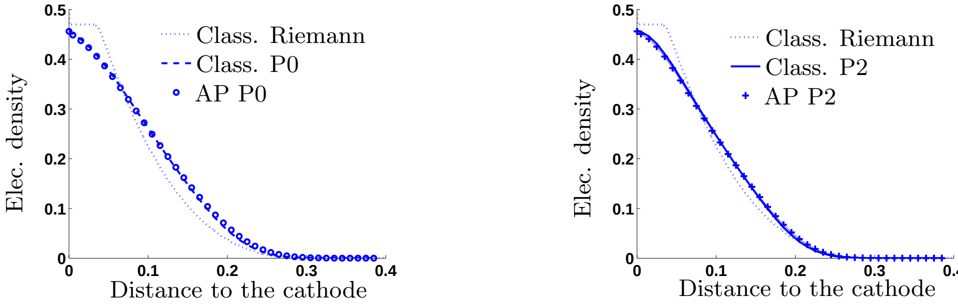

FIG. 5.1. *Electron density as a function of x, the distance to the cathode, at the rescaled time $t = 0.05$, in the unresolved in space case: $\Delta x = 10^{-2} > \lambda = 10^{-4}$. The classical scheme resolves the plasma period: $\Delta t \leq \tau$ and the asymptotic preserving scheme does not: $\Delta t > \tau$. The results are computed with the classical scheme using the Riemann solver (dotted line) in the resolved case with boundary conditions at $x = 0$ given by $n_e = n_i = n_0$, $q_e = q_i = q_0$ and $\phi = 0$. This curve is the reference curve. On the left picture, in dashed line we have the result for the classical scheme with the degree 0 polynomial solver, on the right picture, in solid line for the classical scheme with the degree 2 polynomial solver, on the left picture with circle markers for the asymptotic preserving scheme with the degree 0 polynomial solver and on the right picture with cross markers for the asymptotic preserving scheme with the degree 2 polynomial solver. In all cases, we use for boundary conditions the values given by the boundary layer problem: $n_e = n_i = \bar{n}_0$, $q_e = \bar{q}_{e0}$, $q_i = \bar{q}_{i0}$ and $\phi = \bar{\phi}_0$.*

We perform the same simulations as in section 3, i.e. we use for reference curves the results given by the classical scheme using the Riemann solver in the resolved case ($\Delta x = \lambda$ and $\Delta t^m \leq \tau$ for all $m \geq 0$). In this resolved simulation we do not use the well-adapted boundary conditions but (2.3), (2.4) and (2.5).

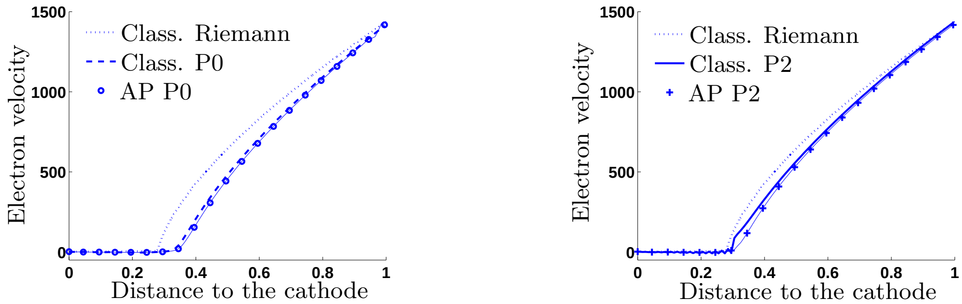

FIG. 5.2. *Electron velocity as a function of x, the distance to the cathode, at the rescaled time $t = 0.05$, in the unresolved in space case: $\Delta x = 10^{-2} > \lambda = 10^{-4}$. The classical scheme resolves the plasma period: $\Delta t \leq \tau$ and the asymptotic preserving scheme does not: $\Delta t > \tau$. The results are computed with the classical scheme using the Riemann solver (dotted line) in the resolved case with not well-adapted the boundary conditions at $x = 0$. This curve is the reference curve. On the left picture, in dashed line we have the result for the classical scheme with the degree 0 polynomial solver, on the right picture, in solid line for the classical scheme with the degree 2 polynomial solver, on the left picture with circle markers for the asymptotic preserving scheme with the degree 0 polynomial solver and on the right picture with cross markers for the asymptotic preserving scheme with the degree 2 polynomial solver. In all cases, we use the well-adapted boundary conditions.*

We compare the reference curves to the results given by the classical and asymptotic preserving schemes using the degree 0 or 2 polynomial solvers when they do not resolve the scaled Debye length ($\Delta x = 10^{-2} > \lambda = 10^{-4}$) and with the well-adapted

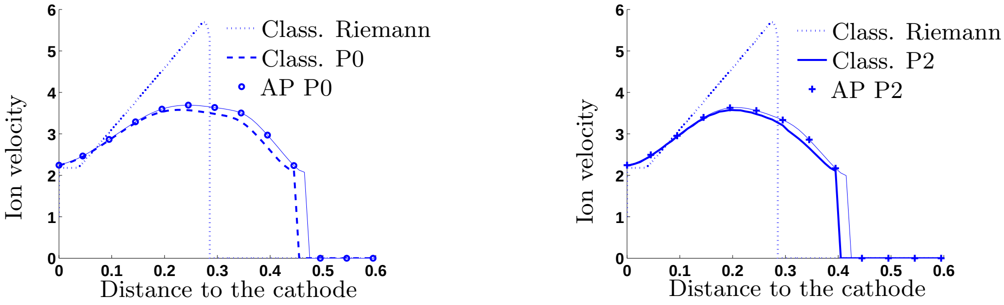

FIG. 5.3. *Ion velocity as a function of x, the distance to the cathode, at the rescaled time $t = 0.05$, in the unresolved in space case: $\Delta x = 10^{-2} > \lambda = 10^{-4}$. The classical scheme resolves the plasma period: $\Delta t \leq \tau$ and the asymptotic preserving scheme does not: $\Delta t > \tau$. The results are computed with the classical scheme using the Riemann solver (dotted line) in the resolved case with not well-adapted the boundary conditions at $x = 0$. This curve is the reference curve. On the left picture, in dashed line we have the result for the classical scheme with the degree 0 polynomial solver, on the right picture, in solid line for the classical scheme with the degree 2 polynomial solver, on the left picture with circle markers for the asymptotic preserving scheme with the degree 0 polynomial solver and on the right picture with cross markers for the asymptotic preserving scheme with the degree 2 polynomial solver. In all cases, we use the well-adapted boundary conditions.*

boundary conditions. The classical scheme resolves the plasma period ($\Delta t^m \leq \tau$ for all $m \geq 0$) and the asymptotic preserving scheme does not ($\Delta t^m > \tau$ for all $m \geq 0$). The results are given in Figures 5.1-5.4. We can see that all schemes give results close to the reference curves with a bigger diffusion for the degree 0 polynomial solver. The degree 2 polynomial solver is now stable, remind it was not without the boundary layer resolution, see Figure 3.4. There are still small oscillations on the electron velocity and on the potential but, only for the classical scheme with the degree 2 polynomial solver. We believe that these oscilations are due to the approximation $\varepsilon = 0$ in the determination of the well-adapted boundary data. We defer to a future work the determination of these boundary data without this approximation.

6. Conclusion. In this paper we have presented a boundary layer analysis. This analysis allows to determine boundary conditions well-adapted to the quasi-neutral regime. Using these boundary conditions, we numerically showed that the classical and the asymptotic preserving schemes remains stable for general Roe type solvers when the mesh does not resolve the small scale of the Debye length.

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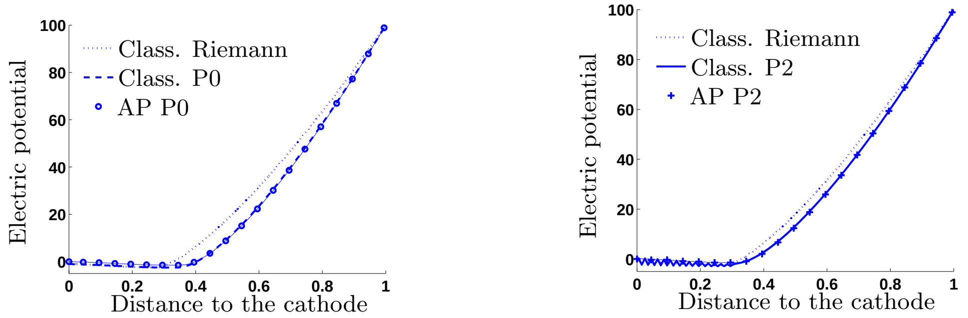

FIG. 5.4. *Electric potential as a function of x, the distance to the cathode, at the rescaled time $t = 0.05$, in the unresolved in space case: $\Delta x = 10^{-2} > \lambda = 10^{-4}$. The classical scheme resolves the plasma period: $\Delta t \leq \tau$ and the asymptotic preserving scheme does not: $\Delta t > \tau$. The results are computed with the classical scheme using the Riemann solver (dotted line) in the resolved case with not well-adapted the boundary conditions at $x = 0$. This curve is the reference curve. On the left picture, in dashed line we have the result for the classical scheme with the degree 0 polynomial solver, on the right picture, in solid line for the classical scheme with the degree 2 polynomial solver, on the left picture with circle markers for the asymptotic preserving scheme with the degree 0 polynomial solver and on the right picture with cross markers for the asymptotic preserving scheme with the degree 2 polynomial solver. In all cases, we use the well-adapted boundary conditions.*

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