ANALYSIS OF AN ASYMPTOTIC PRESERVING SCHEME FOR THE EULER-POISSON SYSTEM IN THE QUASINEUTRAL LIMIT*

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Abstract. In a previous work [9], a new numerical discretization of the Euler-Poisson system has been proposed. This scheme is 'Asymptotic Preserving' in the quasineutral limit (i.e. when the Debye length ε tends to zero), which means that it becomes consistent with the limit model when $\varepsilon \to 0$. In the present work, we show that the stability domain of the present scheme is independent of ε. This stability analysis is performed on the Fourier transformed (with respect to the space variable) linearized system. We show that the stability property is more robust when a space-decentered scheme is used (which brings in some numerical dissipation) rather than with a space-centered one. The linearization is first performed about a zero mean velocity, and then about a non-zero mean velocity. At the various stages of the analysis, our scheme is compared with more classical schemes and its improved stability property is outlined. The analysis of a fully discrete (in space and time) version of the scheme is also given. Finally, some considerations about a model nonlinear problem, the Burgers-Poisson problem, are also given.

Key words. stiffness, Debye length, electron plasma period, Burgers-Poisson, sheath problem, Klein-Gordon

AMS subject classifications. 82D10, 76W05, 76X05, 76N10, 76N20, 76L05

1. Introduction.

In a previous work [9] (see also [7]), a new numerical discretization of the Euler-Poisson system has been proposed. The Euler-Poisson system under consideration consists of the isentropic Euler equations for the particle and momentum densities coupled with the Poisson equation through a source term modeling the electrostatic force. In dimensionless units, the coupling constant can be expressed in terms of a parameter ε which represents the scaled Debye length. When ε is small, the coupling is strong. In this situation, the particle density is constrained to be close to the background density of the oppositely charged particle, which we suppose to be uniform and equal to 1 in scaled units. The velocity then evolves according to the incompressible Euler equation. The limit $\varepsilon \to 0$ is called the quasineutral limit, since the charge density almost vanishes identically. When two or more particle species are considered, the limit $\varepsilon \to 0$ leads to a more complex model usually referred to in the physics literature as a quasineutral model. In the present work, we restrict ourselves to the case of a single particle species as described above.

The scheme which has been proposed in [9] and [7] is 'Asymptotic Preserving' in the quasineutral limit, which means that it becomes consistent with the limit model when $\varepsilon \to 0$. The goal of the present work is to analyze the stability properties of this scheme in a one-dimensional framework, and to show that its stability domain is independent of ε. This stability analysis is performed on the Fourier transformed (with respect to the space variable) linearized system. We show that the stability property is more robust when a space-decentered scheme is used (which brings in some numerical

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dissipation) rather than with a space-centered one. The linearization is first performed about a zero mean velocity, and then about a non-zero mean velocity. At the various stages of the analysis, our scheme is compared with more classical schemes and its improved stability property is outlined. The analysis of a fully discrete (in space and time) version of the scheme is also given. Finally, some considerations about a model nonlinear problem, the Burgers-Poisson problem, are also given.

The Euler-Poisson model is one of the most widely used fluid models in plasma and semiconductor physics (see e.g. [21], [1] for plasmas, and [25] for semiconductors). It can be derived from a moment expansion of kinetic models such as the Vlasov or Boltzmann Poisson equations supplemented with a convenient closure assumption (see references above).

There are two important physical length and time scales associated with this model (see e.g. [1], [21]): the Debye length and the electron plasma period. These two scales are related one to each other by the thermal speed which is an order one quantity. We are interested in the quasineutral regime where both parameters can be very small compared with typical macroscopic length and time scales. A standard explicit scheme must resolve these micro-scale phenomena in order to remain stable. The satisfaction of these constraints requires huge computational resources which make the use of explicit methods almost impracticable.

The search for schemes free of such constraints has been the subject of a vast literature. A number of works deal with Particle-in-Cell (PIC) methods which are specific discretizations of the kinetic model, the Vlasov-Poisson system (see e.g. the direct implicit method [3], [23] and the implicit moment method [26], [27]). These methods have proved extremely efficient in number of situations but there are still regions where short time steps must be used (see e.g. the recent discussion in [19]). So the improvement of classical time-stepping strategies, if possible, would offer attractive perspectives. More references can be found in [9]. We also mention that the strategy developed in the present paper has been applied to PIC codes in [10]

The present work deals with fluid models. For fluid models the literature is comparatively less abundant. We can refer to the pioneering work [16], and more recently to [2], [4], [28], [29]. When the fluid models are drift-diffusion models, implicit strategies have been proposed in [31], [32], [24].

To cancel the fast scales associated with electron plasma frequency, quasineutral models have been very frequently considered [15] (see also references in [9]). Recently, two-fluid quasineutral models have been studied [6], [8], [11], [12], [13], [14].

However, in situations where quasineutral and non quasineutral regions coexist, a specific treatment is needed to connect the quasineutral model with a non quasineutral model across the interface. Such situations arise in sheath problems (see e.g. [18], [20], [12] ; other references can be found in [9]). In such problems, one has often to deal with a dynamic interface the tracking of which gives rise to a complex numerical problem. Additionally, the interface dynamics is not a priori known, and must either be derived from an asymptotic analysis like in [12], [13], or [20], or must be inferred from physical considerations. In both cases, great care is required to ensure that the proper dynamics is implemented. Another problem is related to the fact that the quasineutral to non-quasineutral transition may not be a sharp transition, but rather a fairly diffuse one, and its approximation into a sharp interface may actually lead to some unphysical behavior.

For these reasons, it is highly desirable to develop numerical methods which automatically shift from a quasineutral to a non-quasineutral model across the transition region when such a transition is encountered. The scheme proposed in [9] serves this purpose. Additionally, it has been shown that the numerical cost of this scheme is the same as the standard strategy (we refer the reader to [9] for more detail). The goal of the present work is to provide elements towards a rigorous analysis of its stability properties. We shall mostly restrict ourselves to a linearized stability analysis (except in section 5), although linearized stability analysis is insufficient when the solution is not close to the reference state.

The paper is organized as follows. In section 2, we present the model and our Asymptotic Preserving strategy. Our AP scheme consists in an implicit time differencing of both the flux term in the mass balance equation and the source term in the momentum equation. For this reason, we shall refer to it as an Implicit-Implicit (II) scheme. It will be compared to schemes in which either one or the other or both terms are treated explicitly (referred to as the (IE), (EI) or (EE) schemes). For instance, the (IE) scheme is implicit in the mass flux term and explicit in the momentum source term. The classical strategy it the (EI) scheme where the mass flux term is explicit and the momentum source term is implicit [16]. The (EE) strategy is unconditionnaly unstable as was shown by Fabre in [16] and reviewed below. We will always consider an explicit time differencing for the momentum flux term, so there will be no mention of it in our terminology.

In section 3, the Euler-Poisson system is linearized about zero mean velocity. The linearized Euler-Poisson (LEP) system has a wave equation-like formulation, which is indeed a Klein-Gordon equation with an $O(\varepsilon)$ mass term, as shown in section 3.1.

The Fourier transformed (with respect to the space variable) system is considered in section 3.2 and the stability of the time-stepping strategy is analyzed on this model system. This analysis is a model for the analysis of a fully discrete scheme (in space and time) using a centered space differencing. Indeed, the symbol of the continuous space derivative and that of a centered space-differencing are both pure imaginary and therefore behave in a similar way for stability considerations. Using the symbol of the continuous operator instead of the discrete one makes the computations a little bit simpler (we shall see that they are already quite complex) while still providing a significant insight into the properties of the fully discrete scheme. Of course, to mimic the fact that a fully discrete scheme cannot resolve large wave-numbers, the range of admissible wave-numbers is limited to a value $\pi/\Delta x$ where Δx is the value of the underlying space discretization. To support this approach, the fully discrete scheme will be analyzed in a particular situation.

In this section, we show that the AP strategy (i.e. the (II) scheme) is stable uniformly with respect to ε while the (IE) scheme is limited by a stability constraint of the form $\delta \leq O(\varepsilon)$ where δ is the time-step. The (IE) scheme corresponds to an explicit time discretization of the wave equation formulation of the (LEP) system while the (II) is an implicit one. In the case of the (II) scheme, a stability analysis of a fully discrete scheme is given. This analysis shows that the scheme is stable provided that ε is small enough (compared to Δx). This can be easily understood since, when ε is large the system is close to the standard Euler system for which a centered space-differencing is known to be unstable (even though the mass flux term is treated implicitly, the momentum flux is treated explicitly and the standard instability of the centered scheme manifests itself).

For this reason, in section 3.3, the influence of a decentered space-differencing is investigated. The numerical diffusion is modeled by a diffusion term added to the mass and momentum balance equations in the continuous model, leading to a Linearized Viscous Euler-Poisson (LVEP) model. Energy estimates for this system and for a space discretization of this system are given. The similarity of these estimates encourages us to discard the space discretization and instead, we consider the Fourier transform (with respect to the space variable) of the continuous system and we perform a stability analysis of the various time discrete schemes under considerations. The AP strategy (i.e. the (II) scheme) is compared with the (EI) scheme (which corresponds to the standard approach, see e.g. [16]) and to the (EE) scheme. Only the (II) scheme has a uniform stability with respect to ε. The (EI) scheme has a stability constraint of the form $\delta \leq O(\varepsilon)$ while the (EE) scheme is unconditionally unstable, as already noted by S. Fabre [16].

Only the (II) scheme has the desired asymptotic stability property. It remains to check if this property remains true when the system is linearized about a non-zero velocity u_0. This is performed in section 4. In this section, we restrict ourselves to the (II) scheme but we investigate the influence of the space discretization by looking first at the Fourier transformed non-viscous (LEP) system (mimicing a space centered differencing) and then, the viscous (LVEP) system. As already pointed out the (II) scheme for the non-viscous LEP system is only 'marginally' uniformly stable in the sense that it is stable if ε is small enough (depending on u_0 and other parameters such as Δx) while the (II) scheme for the viscous (LVEP) system is uniformly stable with respect to ε: there exists a stability region for δ which is independent of ε for ε ranging in the whole interval $[0, \infty)$. This result shows that the (II) scheme associated with a decentered space differencing has the requested uniform stability property with respect to ε. This has been observed numerically in the simulations shown in [9].

There remains the question of knowing whether these results which are proved for the linearized system translate to the fully nonlinear system. In section 5, we consider a toy model for the Euler-Poisson system: the Burgers-Poisson system. We show that this system has an entropy-entropy flux pair and that a space semi-discrete system shares this property with the continuous model. This leads to a stability result for this semi-discrete scheme which allows to prove a convergence estimate (at least as long as the solution of the Burgers-Poisson problem remains regular). The question whether an (II) time discretization of this model would show the same stability property is still an open question which will be investigated in future work. Another open problem is to show the stability of this scheme for weak solutions in the presence of shocks. This is a difficult question which has also been put aside in the present work (see e.g. [5, 22] for the convergence analysis of numerical schemes for entropy solutions in the case of a standard hyperbolic problem).

The extension of the present analysis to two or more space dimensions is still an open problem. For example, in section 4, we restrict ourselves to waves propagating in the same direction as the constant velocity u_0 about which the linearization is performed. A more general setting would be a propagation at an oblique angle with respect to u_0. However, the analysis of section 4 is already quite complicated and we shall leave this extension to future work. Let us just mention that the numerical simulations reported in [9] suggest that the scheme is actually uniformly stable also in the two-dimensional case.

A short conclusion is drawn in section 6. All technical proofs are deferred to an appendix (section 7).

A last short comment is in order to adress the issue of convergence. Here, we are seeking stability for large time steps compared with ε. Therefore, the scheme cannot be a good approximation of the original (ε perturbed) problem, but should be an approximation of the limit quasineutral model. In the present one-dimensional linearized setting, the quasineutral limit is trivial (the linearized solution is identically zero). It is readily seen that taking the limit ε and Δt to 0 keeping $\varepsilon \ll \Delta t$ in all the AP schemes developed below leads to this limit, showing that they are all consistent with the quasineutral model. Using this remark, and the Wendroff technique combining stability and consistency, it is certainly possible to show the convergence of the AP scheme towards the solution of the quasineutral model. However, we shall not pursue this route in this paper.

2. Quasi-neutral limit of the Euler-Poisson system and asymptotically stable schemes.

2.1. Quasi-neutral limit. In this work, we are mainly concerned with the Euler-Poisson model. For the sake of simplicity, we shall restrict ourselves to a one-species model, but our scheme construction can be easily extended to any number of charged-particle species (see [7], [9]). The scaled one-species Euler-Poisson model (EP) is written

$$\partial_t n + \nabla \cdot q = 0\,, \tag{2.1}$$

$$\partial_t q + \nabla \cdot (\frac{q \otimes q}{n}) + \nabla p(n) = n \nabla \phi\,, \tag{2.2}$$

$$\varepsilon^2 \Delta \phi = n - 1\,. \tag{2.3}$$

where $n = n(x,t)$ is the particle number density, $p(n)$ is the hydrodynamic pressure, $q = q(x,t)$ is the momentum (i.e. $q = nu$ where $u(x,t)$ is the average velocity), $p(n) = n^\gamma$ is the pressure law with $\gamma \geq 1$ and $\phi = \phi(x,t)$ is the potential, where $x \in \mathbb{R}^d$ is the position and $t > 0$ is the time. The scheme construction can be easily extended to a model including the full energy equation in place of the isentropic assumption on the pressure.

Here, we consider negatively charged electrons (with scaled charge equal to -1) and the right-hand side of the Poisson equation involves a uniform ion background density equal to 1 in the present scaled variables. Again, this simplification is for the sake of clarity only and a space-dependent ion background density or a model with simultaneously evolving electrons and ions could be considered as well (see [7], [9]).

The dimensionless parameter $\varepsilon = \lambda_D / L$ is the scaled Debye length, i.e. the ratio of the actual Debye length λ_D to the macroscopic length scale L. We recall that the Debye length has the following expression in physical variables:

$$\lambda_D = \left(\frac{\epsilon_0 k_B T}{e^2 n_0} \right)^{1/2},$$

where ϵ_0 is the vacuum electric permittivity, k_B the Boltzmann constant, T the average plasma temperature, e the absolute electron charge and n_0 the average plasma density.

In many instances, the Debye length is very small compared to the macroscopic length and thus $\varepsilon \ll 1$. In this case, the quasineutral limit $\varepsilon \to 0$ of system (2.1)-(2.3) can be considered. It leads to $n = 1$, i.e. the electron density must be everywhere equal to the background ion density and to the following quasineutral model (QN) for the flux q:

$$\nabla \cdot q = 0\,, \tag{2.4}$$

$$\partial_t q + \nabla \cdot (q \otimes q) = \nabla \phi\,, \tag{2.5}$$

We note that ϕ is now the Lagrange multiplier of the incompressibility constraint (2.4) and upon changing ϕ into $-\pi$, that the quasineutral model is nothing but the Incompressible Euler equations with hydrostatic pressure π.

In the present paper, our goal is to analyze a class of numerical schemes for (EP) which remains uniformly stable in the limit $\varepsilon \to 0$. Such schemes have first been proposed and tested in [7], [9] where their uniform stability with respect to ε has been demonstrated.

The design of this class of scheme is based on the following remark. In the passage from (EP) to (QN), the equation for the electric potential ϕ changes dramatically, from the Poisson eq. (2.3) into

$$\Delta\phi = \nabla^2 : (q \otimes q)\,, \tag{2.6}$$

where ∇^2 denotes the Hessian matrix and : the contracted product of rank two tensors. Any attempt to design a uniformly stable scheme in the limit $\varepsilon \to 0$ requires to find a different (and equivalent) formulation of the Poisson equation (2.3) in which the transition towards eq. (2.6) when $\varepsilon \to 0$ appears explicitly.

Such a formulation is provided by the following equation:

$$-\nabla \cdot ((n + \varepsilon^2 \partial_{tt}^2)\nabla\phi) + \nabla^2 : f(n, q) = 0\,, \tag{2.7}$$

where the momentum flux tensor $f(n, q)$ is given by

$$f(n, q) = \frac{q \otimes q}{n} + p(n)\mathrm{Id}\,. \tag{2.8}$$

Eq. (2.7) is formally equivalent to the Poisson eq. provided (n, q) satisfies the mass and momentum eqs. (2.1), (2.2) and that the Poisson eq. and its time derivative are satisfied at $t = 0$:

$$(\varepsilon^2\Delta\phi - n + 1)|_{t=0} = 0\,, \quad (\partial_t(\varepsilon^2\Delta\phi - n + 1))|_{t=0} = 0\,.$$

We refer to [7], [9] for a proof of this equivalence. Essentially, it relies on a second order (wave-like) formulation of the Euler equations, and replacing the second time derivative of the density in this formulation by the second time derivative of the Laplacian of the potential (thanks to the Poisson eq.). We shall see below the importance of using a wave-like formulation of the Euler eqs in this matter.

We remark that solving the elliptic eq. (2.7) (after time discretization) is no more costly than solving the Poisson eq. (2.3) since both of them are elliptic equations. However, one important advantage of (2.7) over (2.3) is that it does not degenerate when $\varepsilon \to 0$ and moreover reduces to the (QN) elliptic eq. (2.6) for ϕ when $\varepsilon = 0$. Therefore, we can expect that the asymptotic stability with respect to ε is obtained if a suitable time discretization is used.

2.2. Asymptotically stable scheme in the quasineutral limit. The question of finding an asymptotically stable scheme in the QN limit is in the first place a question of time discretization. We propose a time-discretization which formally satisfies this asymptotic stability property. Below, we shall suppose that the symbol D stands for space discretizations of the ∇ operator and we shall postpone the question of finding suitable expressions of such operators to forthcoming sections.

First, we describe the most standard time marching procedure for solving the (EP) system. Suppose n^m, q^m, ϕ^m are approximations at time t^m of n, q and ϕ and let $\delta = t^{m+1} - t^m$. Then, the standard procedure consists in discretizing (2.1)-(2.3) by:

$$\delta^{-1}\left(n^{m+1} - n^m\right) + D \cdot q^m = 0\,, \tag{2.9}$$

$$\delta^{-1}\left(q^{m+1} - q^m\right) + Df(n^m, q^m) = n^{m+1} D\phi^{m+1}\,, \tag{2.10}$$

$$\varepsilon^2 D^2 \phi^{m+1} = n^{m+1} - 1\,. \tag{2.11}$$

We note that the scheme involves an implicit computation of the electric force in the momentum eq. but the cost stays the same as that of a fully explicit scheme. Indeed, n^{m+1} can be computed first by the mass eq. (2.9). Then, ϕ^{m+1} is obtained by the Poisson eq. (2.11). Finally, the momentum q^{m+1} can be advanced using (2.10). It is common wisdom in plasma physics that the stability of this scheme is constrained by the condition

$$\delta \lesssim \varepsilon\,. \tag{2.12}$$

which is very penalizing in the QN regime $\varepsilon \ll 1$.

Let us note that if an explicit electric force discretization is used at the r.h.s. of (2.10), the scheme is simply unconditionally unstable [16].

Our new Asymptotic Preserving (AP) strategy consists in computing the mass flux in (2.9) implicitly. More precisely, we consider the following scheme

$$\delta^{-1}\left(n^{m+1} - n^m\right) + D \cdot q^{m+1} = 0\,, \tag{2.13}$$

$$\delta^{-1}\left(q^{m+1} - q^m\right) + Df(n^m, q^m) = n^m D\phi^{m+1}\,, \tag{2.14}$$

$$\varepsilon^2 D^2 \phi^{m+1} = n^{m+1} - 1\,. \tag{2.15}$$

By reproducing the procedure which, in the continuous case, allowed to pass from the standard formulation of the Poisson eq. (2.3) to the reformulated one (2.7), we can transform the discrete Poisson eq. (2.15) into a reformulated one

$$-D \cdot ((\delta^{-2}\varepsilon^2 + n^m) D\phi^{m+1}) = \\ -D^2 : f(n^m, q^m) - \delta^{-2}(2n^m - n^{m-1} - 1) := G(n^m, q^m, n^{m-1})\,. \tag{2.16}$$

This is a new elliptic eq. for ϕ^{m+1} which is formally equivalent to the original one (2.15), is as easy to solve and which does not degenerate when $\varepsilon \to 0$.

The first important remark about this scheme is that it provides a discretization of the (QN) Euler system (2.4), (2.5) when $\varepsilon = 0$, i.e. it is an asymptotic preserving (AP) scheme when $\varepsilon \to 0$. To see this, we can use either of the formulations (2.15) or (2.16) of the Poisson equation, coupled with (2.13)-(2.14). Of course, we suppose that at step m, the data n^m and q^m satisfy $n^m = 1$ and $D \cdot q^m = 0$. If we let $\varepsilon \to 0$ in (2.15), we first find that $n^{m+1} = 1$. Then, given that $n^m = 1$, (2.13) yields $D \cdot q^{m+1} = 0$. Finally, (2.14) leads to

$$\delta^{-1}\left(q^{m+1} - q^m\right) + Df(1, q^m) = D\phi^{m+1}\,,$$

which clearly gives a discretization of the (QN) Euler system (2.4), (2.5). Note that the standard scheme (2.9)-(2.11), which has an explicit mass flux, would lead to $D \cdot q^m = 0$, the other limiting equations remaining unchanged, and that this would not allow us to compute ϕ^{m+1}.

If one uses (2.16) instead of (2.15), then one first lets $\varepsilon \to 0$ in (2.16) leading to

$$-D \cdot (D\phi^{m+1}) = -D^2 : f(1, q^m)\,,$$

(since $n^m = 1$), which is a discretization of the quasineutral potential equation (2.6). Then, by acting operator $D\cdot$ on (2.14) and using that $D \cdot q^m = 0$, we deduce that $D \cdot q^{m+1} = 0$. Finally, inserting this in (2.13), we get that $n^{m+1} = n^m = 1$ and again, we get a discretization of the (QN) Euler system (2.4), (2.5). Here, using the standard scheme (2.9), (2.10) (which has an explicit mass flux) in conjunction with the modified Poisson equation (2.16) would also lead to an AP scheme. However, it has been numerically experienced that this scheme is unstable and this possibility will be disregarded.

The second important remark is that the reformulated Poisson equation (2.16) allows us to compute ϕ^{m+1} in terms of known quantities $G(n^m, q^m, n^{m-1})$ from the previous time step. Indeed, once ϕ^{m+1} is known after solving (2.16), q^{m+1} can be computed by advancing the momentum eq. (2.14) and then, n^{m+1} in turn is obtained by using the density eq. (2.13). Therefore, the time marching procedure of the (AP) scheme (2.13)-(2.15) involves the same computational cost as that of the standard scheme (2.9)-(2.11).

Finally, even more importantly, we claim that, by contrast with the standard scheme, the (AP) scheme (2.13)-(2.15) is not constrained by the stability condition (2.12), but rather, has a stability condition of the form

$$\delta = O(1) \quad \text{as } \varepsilon \to 0 \,. \tag{2.17}$$

We stress however that the scheme is still constrained by the CFL condition of the hydrodynamic system (which is an $O(1)$ quantity in our scaled units) and this can be penalizing for electrons due to their very small mass. It is possible however to extend this strategy into a scheme which is asymptotically stable also in the small mass limit (see remark in [9]). This strategy will be developed in future works. We also note that similar ideas can be applied to the low Mach number limit of compressible Euler eqs. and to the drift approximation of highly magnetized plasmas (works also in progress).

The stability estimate (2.17) is so far formal. It has been verified in numerical computations using different types of space discretizations in [7], [9]. However, no rigorous arguments were available so far. The goal of the present paper is to provide some rigorous support to the estimate (2.17) by analyzing the linearized system.

Two types of analyses will be performed. A first one uses Fourier analysis and keeps the time semi-discrete framework, mimicking space discretization by suitable Fourier symbols. The second one considers both a space semi-discrete setting (the so-called method of lines) and a full time and space discretization and develops energy estimates. Both methods show that the AP strategy gives rise to a stability constraint of the type (2.17) while the standard strategy has the more severe stability constraint (2.12).

3. The linearized system about zero velocity.

3.1. The linearization of the Euler-Poisson system. We linearize system (2.1)-(2.3) about the stationary homogeneous solution $n_0 = 1$, $q_0 = 0$, $\partial_x \phi_0 = 0$. We denote by $\zeta \ll 1$ the size of the initial perturbation to this stationary state. Then, we can formally expand the solution as follows: $n = 1 + \zeta n' + O(\zeta^2)$, $q = \zeta u' + O(\zeta^2)$, $\phi = \zeta \phi' + O(\zeta^2)$. Retaining only terms of order ζ, we are led to the following Linearized Euler-Poisson (LEP) system (where the primes have been dropped for the sake of clarity):

$$\partial_t n + \partial_x u = 0\,, \tag{3.1}$$
$$\partial_t u + c_s^2 \partial_x n = \partial_x \phi\,, \tag{3.2}$$
$$\varepsilon^2 \partial_{xx}\phi = n, \tag{3.3}$$

where $c_s = \sqrt{p'(1)}$ is the speed of sound.

We note that both ϕ and u can be eliminated and the (LEP) system can be rephrased into a wave-like formulation (specifically a Klein-Gordon equation with mass equal to $1/\varepsilon^2$):

$$\partial_{tt}^2 n - c_s^2 \partial_{xx}^2 n + \frac{1}{\varepsilon^2} n = 0\,. \tag{3.4}$$

By a partial Fourier transform with respect to the x-variable, and denoting $\hat{n}(\xi, t)$, $\hat{u}(\xi, t)$, $\hat{\phi}(\xi, t)$ the transforms of n, u and ϕ, we obtain:

$$\partial_t \hat{n} + i\xi \hat{u} = 0\,, \tag{3.5}$$
$$\partial_t \hat{u} + i\xi c_s^2 \hat{n} = i\xi \hat{\phi}\,, \tag{3.6}$$
$$-\varepsilon^2 \xi^2 \hat{\phi} = \hat{n}. \tag{3.7}$$

After elimination of $\hat{\phi}$, the system reduces to a 2×2 system of the form:

$$\begin{pmatrix} \partial_t \hat{n} \\ \partial_t \hat{u} \end{pmatrix} + \begin{pmatrix} 0 & i\xi \\ i\xi c_s^2 + \frac{i}{\varepsilon^2 \xi} & 0 \end{pmatrix} \begin{pmatrix} \hat{n} \\ \hat{u} \end{pmatrix} = 0\,. \tag{3.8}$$

Its solutions are of the form

$$\begin{pmatrix} \hat{n}(\xi,t) \\ \hat{u}(\xi,t) \end{pmatrix} = \begin{pmatrix} \hat{n}_+(\xi) \\ \hat{u}_+(\xi) \end{pmatrix} e^{i\theta^\varepsilon(\xi)t} + \begin{pmatrix} \hat{n}_-(\xi) \\ \hat{u}_-(\xi) \end{pmatrix} e^{-i\theta^\varepsilon(\xi)t}\,, \tag{3.9}$$

where

$$\theta^\varepsilon(\xi) = \left(c_s^2 \xi^2 + \frac{1}{\varepsilon^2} \right)^{1/2}\,. \tag{3.10}$$

The quantities $s_\pm = \pm i\theta^\varepsilon(\xi)$ are the two eigenvalues of the matrix A at the left hand side of (3.8). The vectors $(\hat{n}_\pm(\xi), \hat{u}_\pm(\xi))^T$ are obtained as the projections of the initial data $(\hat{n}_0, \hat{u}_0)^T$ onto the associated eigenspaces of A (here the exponent T denotes the transpose).

From (3.9) and the fact that $|e^{\pm i\theta(\xi)t}| = 1$, we see that the L^2 norm of the solution in Fourier space stays bounded in time, which by Plancherel's identity shows that the corresponding L^2 norm in physical space also stays bounded in time. Additionally, when $\varepsilon \to 0$, we have

$$\theta^\varepsilon \sim \frac{1}{\varepsilon} \to \infty\,,$$

which shows that the solution has high frequency time oscillations of frequency $\sim \frac{1}{\varepsilon}$, irrespective of the value of the wave-number ξ.

Therefore, the quasineutral limit $\varepsilon \to 0$ gives rise to high frequency oscillations which are nothing but the ubiquitous plasma oscillations of the physics literature.

Explicit schemes are constrained to resolve these oscillations as a condition for their stability, as we shall see below.

Another view point which emphasizes the wave-like behaviour of this system is that of energy estimates. Multiplying (3.4) by $2n_t$, one has

$$\partial_t\Big((\partial_t n)^2 + c_s^2(\partial_x n)^2 + \frac{1}{\varepsilon^2}n^2\Big) = 2c_s^2\partial_x\Big(\partial_x n\partial_t n\Big)\,. \qquad (3.11)$$

Integrating over the computational domain say (a,b) to fix the ideas, we get:

$$\partial_t\int_a^b \Big((\partial_t n)^2 + c_s^2(\partial_x n)^2 + \frac{1}{\varepsilon^2}n^2\Big)\,dx = 2c_s^2\Big(\partial_x n\partial_t n\Big)\Big|_a^b\,. \qquad (3.12)$$

If periodic or homogeneous Dirichlet or homogeneous Neumann conditions are imposed at the boundary points, the right-hand side of the above equation vanishes and conservation of energy follows:

$$\partial_t\int_a^b \Big((\partial_t n)^2 + c_s^2(\partial_x^2 n)^2 + \frac{1}{\varepsilon^2}n^2\Big)\,dx = 0\,. \qquad (3.13)$$

This energy estimate indicates a wave-like behaviour. When $\varepsilon \to 0$, more and more energy is stored in the last term which can be interpreted as a potential.

Below, we state stability criteria for various kinds of time-discretizations of the (LEP) system. The proofs of all these facts are deferred to the appendix for the reader's convenience.

3.2. Asymptotic preserving strategy with centered-like space-differencing. For the (LEP) system in Fourier variables (3.5)-(3.7), our scheme (2.13)-(2.15) reads:

$$\frac{\hat n^{m+1}-\hat n^m}{\delta} + i\xi\hat u^{m+1} = 0\,, \qquad (3.14)$$

$$\frac{\hat u^{m+1}-\hat u^m}{\delta} + i\xi c_s^2\hat n^m + \frac{i}{\varepsilon^2\xi}\hat n^{m+1} = 0\,. \qquad (3.15)$$

It is easy to see that this scheme can be recast into an implicit scheme for the wave equation (3.4) expressed in Fourier variables, i.e.

$$\partial_{tt}^2\hat n + c_s^2\xi^2\hat n + \frac{1}{\varepsilon^2}\hat n = 0\,. \qquad (3.16)$$

Indeed, taking the time difference of (3.14) and combining with (3.15) to eliminate the velocity leads to:

$$\frac{\hat n^{m+1}-2\hat n^m+\hat n^{m-1}}{\delta^2} + c_s^2\xi^2\hat n^m + \frac{1}{\varepsilon^2}\hat n^{m+1} = 0\,. \qquad (3.17)$$

By contrast, let us consider the following explicit scheme for the wave-equation formulation:

$$\frac{\hat n^{m+1}-2\hat n^m+\hat n^{m-1}}{\delta^2} + c_s^2\xi^2\hat n^m + \frac{1}{\varepsilon^2}\hat n^{m} = 0\,. \qquad (3.18)$$

This scheme can be obtained by eliminating the velocity from the original formulation of the (LEP) system. Two schemes for the original (LEP) system lead to the same scheme (3.18). The first one is implicit in the mass flux and explicit in the momentum flux and in the source term:

$$\frac{\hat{n}^{m+1} - \hat{n}^m}{\delta} + i\xi \hat{u}^{m+1} = 0\,, \tag{3.19}$$

$$\frac{\hat{u}^{m+1} - \hat{u}^m}{\delta} + i\xi c_s^2 \hat{n}^m + \frac{i}{\varepsilon^2 \xi} \hat{n}^m = 0\,. \tag{3.20}$$

The second one is explicit in the mass flux and implicit in the momentum flux and the source term:

$$\frac{\hat{n}^{m+1} - \hat{n}^m}{\delta} + i\xi \hat{u}^{m} = 0\,, \tag{3.21}$$

$$\frac{\hat{u}^{m+1} - \hat{u}^m}{\delta} + i\xi c_s^2 \hat{n}^{m+1} + \frac{i}{\varepsilon^2 \xi} \hat{n}^{m+1} = 0\,. \tag{3.22}$$

We see that the schemes (3.14)-(3.15) on the one hand and (3.19)-(3.20) are both implicit in the mass flux. The first one is also implicit in the source term while the second one is explicit. For these reasons, we shall refer to the first scheme as the (II) scheme (for Implicit-Implicit) while the second one is the (IE) scheme (for Implicit-Explicit). What the equivalence of scheme (3.21), (3.22) with the scheme (3.18) shows is that if the mass flux is taken explicit, there is no way to obtain a uniformly stable scheme, no matter how implicit one takes the momentum flux or the source term.

If additionally the momentum flux is taken explicit and only the source term is taken implicit, one is led to the following scheme:

$$\frac{\hat{n}^{m+1} - \hat{n}^m}{\delta} + i\xi \hat{u}^{m} = 0\,, \tag{3.23}$$

$$\frac{\hat{u}^{m+1} - \hat{u}^m}{\delta} + i\xi c_s^2 \hat{n}^{m} + \frac{i}{\varepsilon^2 \xi} \hat{n}^{m+1} = 0\,. \tag{3.24}$$

which is equivelently written in the wave-equation formulation as follows:

$$\frac{\hat{n}^{m+1} - 2\hat{n}^m + \hat{n}^{m-1}}{\delta^2} + c_s^2 \xi^2 \hat{n}^{m-1} + \frac{1}{\varepsilon^2} \hat{n}^m = 0\,. \tag{3.25}$$

In this scheme the mass flux is taken explicit and the source term implicit. For this reason, we call this scheme the (EI) scheme. We shall analyze the stability of this scheme in the next section in relation with decentered space differencing. Indeed, we know already that with a centered space differencing, the hydrodynamic part of this scheme will be unconditionally unstable and that it is unlikely that the implicit source term restores stability and if it does so, it will do it only marginally.

For all the reasons previously exposed, in the remainder of this paper, we will always treat the momentum flux explicitly. We will consider various combinations of explicit or implicit schemes for the mass flux and the source term and the terminology will refer to these terms. For instance, the (IE) scheme refers to an implicit scheme in the mass flux but explicit in the source term.

Looking at the stability of a time-semidiscretized scheme in Fourier space gives a good indication of the behaviour of a fully discretized scheme associated with a space-centered discretization. Indeed, the symbol of a centered-space discretization is purely imaginary, like the symbol of the continuous space derivative. Of course, there might exist some quantitative differences between the fully discrete scheme and the semi-discrete scheme in Fourier space, but we believe that this analysis gives a correct rough picture. Using space continuous Fourier transform allows also to considerably simplify the computations and to express the results in a more compact form, which makes them easier to interpret.

In view of a space discretization of mesh size Δx, we assume that the range of admissible wave-numbers ξ is $|\xi| \leq \xi^* = \pi/\Delta x$.

We show in section 7.1 of the appendix that a sufficient stability condition for the (II) scheme for a given wave-number ξ is

$$\delta \leq \delta_{ii}(\xi) := \frac{2(\frac{1}{\varepsilon^2} + c_s^2\xi^2)^{1/2}}{c_s^2\xi^2} . \tag{3.26}$$

The (II) scheme is stable over the range of admissible wave-numbers ξ as soon as

$$\delta \leq \delta_{ii}^* := \delta_{ii}(\xi^*) = 2\left(\frac{\Delta x}{\pi c_s}\right)^2 \left(\frac{1}{\varepsilon^2} + \left(\frac{\pi c_s}{\Delta x}\right)^2\right)^{1/2} . \tag{3.27}$$

By contrast, the stability domain of the (IE) for a given wave-number ξ is given by:

$$\delta \leq \delta_{ie}(\xi) := \frac{2}{(\frac{1}{\varepsilon^2} + c_s^2\xi^2)^{1/2}} . \tag{3.28}$$

The (IE) scheme is stable over the range of admissible wave-numbers iff

$$\delta \leq \delta_{ie}^* := \delta_{ie}(\xi^*) = \frac{2}{(\frac{1}{\varepsilon^2} + \big(\frac{\pi c_s}{\Delta x}\big)^2)^{1/2}} . \tag{3.29}$$

The stability domain of the (IE) scheme shrinks down when $\varepsilon \to 0$, while that of the (II) scheme turns out to be uniformly stable with respect to ε. Furthermore, the (II) scheme is also consistent with the limit model when $\varepsilon \to 0$ (Asymptotic Preserving property) as we have seen in section 2.2 in the case of the fully nonlinear equation (the proof is the same for the linearized equation and is left to the reader).

Indeed, we see that $\delta_{ie}^* = O(\varepsilon)$ as $\varepsilon \to 0$ with a fixed Δx while $\delta_{ii}^* = O(\varepsilon^{-1})$, showing the announced behavior. More precisely, $\delta_{ii}^* \sim 2(\frac{\Delta x}{2\pi c_s})^2 \frac{1}{\varepsilon}$. We note that this increased stability with respect to ε is paid by a more restrictive CFL condition of diffusion type with respect to the acoustic waves. By contrast, in the absence of source term, or equivalently when $\Delta x \to 0$ for a fixed ε, the stability constraints for both the (IE) and (II) schemes reduce to $\delta \leq 2\frac{\Delta x}{\pi c_s}$, which is similar to a CFL condition. In section 7.1 we also show that the (II) scheme damps the plasma oscillations. At each time step, the amplitude of the oscillation is multiplied by a factor ε/δ (in the limit $\varepsilon \to 0$ for fixed δ). When ε is very small, the damping is very strong. At each time step, the phase of the solution is multiplied by the factor $e^{\pm i\theta}$ with

$$\theta = \cos^{-1} \frac{(1 - \frac{1}{2}\delta^2 c_s^2 \xi^2)}{(1 + \frac{\delta^2}{\varepsilon^2})^{1/2}} \quad \xrightarrow{\varepsilon \to 0} \quad \frac{\pi}{2} , \tag{3.30}$$

which means that, in the limit $\varepsilon \to 0$, a shift of $\pi/2$ is added to the phase of the oscillation at each time step. In this limit, the period of the plasma oscillations stays bounded, equal to 4δ.

To support the conclusions of this Fourier analysis of the scheme, we show below that the same stability and dissipation properties are still valid when we use a fully discretized scheme in both time and space.

We denote by $D^2 u = \frac{u_{i+1}-2u_i+u_{i-1}}{h^2}$ the standard second order difference operator and by $D_- u = \frac{u_i - u_{i-1}}{h}$ and $D_+ u = \frac{u_{i+1}-u_i}{h}$ the standard one-sided difference operator. Then $D^2 = D_+ D_- = D_- D_+$. Denote $\tilde{D} = (D_+ + D_-)/2$. Here h is the space step.

We consider a space discretization of (3.4) by replacing the continuous second order space derivative by the discrete operator D^2:

$$\partial^2_{tt} n - c_s^2 D^2 n + \frac{1}{\varepsilon^2} n = 0\,. \tag{3.31}$$

Applying the (II) time-difference scheme (3.17) to this equation leads to

$$\frac{n^{m+1} - 2n^m + n^{m-1}}{\delta^2} - c_s^2 D^2 n^m + \frac{1}{\varepsilon^2} n^{m+1} = 0. \tag{3.32}$$

This scheme can be viewed as a wave-type version of the following scheme for the original (LEP) system (3.1)-(3.3):

$$\frac{n^{m+1} - n^m}{\delta} + D_- u^{m+1} = 0\,, \tag{3.33}$$

$$\frac{u^{m+1} - u^m}{\delta} + c_s^2 D_+ n^m = D_+ \phi^{m+1}\,, \tag{3.34}$$

$$\varepsilon^2 D^2 \phi^{m+1} = n^{m+1}. \tag{3.35}$$

Indeed, eliminating u between (3.33) and (3.34), one has

$$\frac{n^{m+1} - 2n^m + n^{m-1}}{\delta^2} - c_s^2 D^2 n^m + D^2 \phi^{m+1} = 0, \tag{3.36}$$

and using (3.35) we get (3.32). We can also eliminate n^{m+1} from (3.36) using (3.35) and get a discretization of the reformulated Poisson eq. (2.7) (in linearized form):

$$\left(\frac{\varepsilon^2}{\delta^2} + 1\right) D^2 \phi^{m+1} = \frac{2n^m - n^{m-1}}{\delta^2} + c_s^2 D^2 n^m. \tag{3.37}$$

In section 7.2 of the appendix, we show that this scheme is stable provided that $2\varepsilon c_s \le h$. We observe that the scheme is asymptotically stable in the limit $\varepsilon \to 0$ but is not stable if ε is large. This feature can easily be explained. When $\varepsilon \to \infty$ the electric field term disappears and we are left with the standard (LEP) system discretized via centered differences. A time explicit discretization will be unconditionally unstable. Here, the time-differencing is implicit in the mass flux term but still explicit in the momentum flux. This semi-implicit discretization is not implicit enough to stabilize the space-centered discretization. It is a remarkable fact that the electric field contributes to stabilize the instability of the space-centered scheme. However, this scheme cannot be used in a far from quasineutrality regime which is a severe drawback in situations close to quasineutrality in some regions but far from it in other ones. This unpleasant feature will also be found via Fourier analysis when linearizing the Euler-Poisson system about a non-zero velocity (see section 4). This drawback will be overcome by means of a space-decentered discretization.

We also note that the implicit treatment of the last term in (3.32) gives a strong dissipation (like in the Fourier analysis case). This can be seen from the use of the so-called equivalent equation (i.e. the equation obtained by retaining the leading order term in the Taylor expansion of the time semi-discrete scheme). This equivalent equation is written

$$\partial^2_{tt} n - c_s^2 \partial^2_{xx} n + \frac{1}{\varepsilon^2} n + \frac{\delta}{\varepsilon^2} n_t = 0, \qquad (3.38)$$

which can be recast as a LEP system with a damping term

$$\partial_t n + \partial_x u = -\tfrac{\delta}{\varepsilon^2} n\,, \qquad (3.39)$$

$$\partial_t u + c_s^2 \partial_x n = \partial_x \phi\,, \qquad (3.40)$$

$$\varepsilon^2 \partial^2_{xx} \phi = n. \qquad (3.41)$$

The energy estimate (3.13) for (3.38) becomes

$$\partial_t \int_a^b \Big((\partial_t n)^2 + c_s^2 (\partial_x n)^2 + \frac{1}{\varepsilon^2} n^2 \Big)\, dx + \frac{2\delta}{\varepsilon^2} \int_a^b (\partial_t n)^2\, dx = 0, \qquad (3.42)$$

and we observe the strong damping of the energy with a rate which tends to infinity as $\varepsilon \to 0$ (for fixed δ) as fast as $1/\varepsilon^2$.

The use of centered differencing associated with only a partial implicit time-differencing may lead to stability problems when the coupling with the field term is weak, i.e. ε is large. This will be particularly clear with the analysis of the (LEP) system about non-zero velocity in section 4. For this reason, we now analyze schemes associated with decentered space differencing, which we shall mimic by looking at the (LEP) system with diffusion.

3.3. Asymptotic Preserving strategy associated with upwind space-decentered differencing. In order to mimic the influence of a decentered space difference, we consider the following two viscous perturbation of the (LEP) system. The linearized viscous Euler-Poisson (LVEP) system reads either as

$$\partial_t n + \partial_x u = 0\,, \qquad (3.43)$$

$$\partial_t u + c_s^2 \partial_x n = \nu \partial^2_{xx} u + \partial_x \phi\,, \qquad (3.44)$$

$$\varepsilon^2 \partial_{xx} \phi = n\,, \qquad (3.45)$$

if numerical diffusion is only added to the momentum balance equation or as

$$\partial_t n + \partial_x u = \beta \partial^2_{xx} n\,, \qquad (3.46)$$

$$\partial_t u + c_s^2 \partial_x n = \nu \partial^2_{xx} u + \partial_x \phi\,, \qquad (3.47)$$

$$\varepsilon^2 \partial_{xx} \phi = n\,, \qquad (3.48)$$

if we also consider numerical diffusion in the mass conservation equation. Here, β and ν are artificial viscosity coefficients. These systems will be referred to below respectively as the (LVEP-1) and (LVEP-2) systems. The (LVEP-1) system is of course a particular case of the (LVEP-2) system by taking $\beta = 0$.

Eliminating u and ϕ, the (LVEP-1) system reduces to the following Klein-Gordon equation with a viscoelastic term:

$$\partial^2_{tt} n = c_s^2 \partial^2_{xx} n - \frac{1}{\varepsilon^2} n + \nu \partial^3_{txx} n\ . \qquad (3.49)$$

For the (LVEP-2) system, following Slemrod, we introduce $v = u - \beta\partial_x n$. Then the above system is equivalent to:

$$\partial_t n + \partial_x v = 0\,, \tag{3.50}$$
$$\partial_t v + c_s^2\partial_x n = (\nu+\beta)\partial^2_{xx}u + \nu\beta\partial^3_{xxx}n + \partial_x\phi\,, \tag{3.51}$$
$$\varepsilon^2\partial^2_{xx}\phi = n\,. \tag{3.52}$$

Eliminating v and ϕ, we are led to the following Klein-Gordon equation with viscoelastic and capillary type terms:

$$\partial^2_{tt}n = c_s^2\partial^2_{xx}n - \frac{1}{\varepsilon^2}n + (\nu+\beta)\partial^3_{txx}n - \nu\beta\partial^4_{xxxx}n\,. \tag{3.53}$$

We now perform Fourier and energy analyses for the (LVEP-2) system which includes the (LVEP-1) system as a particular case. After a partial Fourier transform with respect to x, the system reads:

$$\partial_t\hat n + i\xi\hat u + \beta\xi^2\hat n = 0\,, \tag{3.54}$$
$$\partial_t\hat u + i\xi c_s^2\hat n + \nu\xi^2\hat u = i\xi\hat\phi\,, \tag{3.55}$$
$$-\varepsilon^2\xi^2\hat\phi = \hat n\,, \tag{3.56}$$

or equivalently, after elimination of $\hat\phi$:

$$\begin{pmatrix}\partial_t\hat n\\ \partial_t\hat u\end{pmatrix} + \begin{pmatrix}\beta\xi^2 & i\xi\\ i\xi c_s^2 + \frac{i}{\varepsilon^2\xi} & \nu\xi^2\end{pmatrix}\begin{pmatrix}\hat n\\ \hat u\end{pmatrix} = 0\,. \tag{3.57}$$

For $|\xi|$ not too large, the eigenmodes of the (LVEP-2) system are given by

$$s^\pm = r_{\beta,\nu}(\xi) \pm i\theta_\varepsilon(\xi) := \frac{\beta+\nu}{2}\xi^2 \pm i\left(\frac{1}{\varepsilon^2} + c_s^2\xi^2 - \frac{(\beta+\nu)^2}{4}\xi^4\right)^{1/2}. \tag{3.58}$$

Now the oscillations are damped by an exponential factor $e^{-r_{\beta,\nu}t}$ which is independent of ε and proportional to $\beta+\nu$. Moreover, the phase oscillates with a frequency $O(\frac{1}{\varepsilon})$ when $\varepsilon \to 0$, independently of ξ.

In order to derive energy estimates, we multiply (3.53) with $\partial_t n$ and get

$$\partial_t\Big(\tfrac12(\partial_t n)^2\Big) = c_s^2\,\partial_t n\,\partial^2_{xx}n - \frac{1}{\varepsilon^2}\partial_t\Big(\tfrac12 n^2\Big) + (\nu+\beta)\,\partial_t n\,\partial^3_{txx}n - \nu\beta\,\partial_t n\,\partial^4_{xxxx}n.$$

Simple computations give:

$$\partial_t n\,\partial^2_{xx}n = \partial_x(\partial_x n\,\partial_t n) - \partial_x n\,\partial^2_{tx}n = \partial_x(\partial_x n\,\partial_t n) - \partial_t\Big(\tfrac12(\partial_x n)^2\Big),$$

$$\partial_t n\,\partial^3_{txx}n = \partial_x(\partial^2_{tx}n\,\partial_t n) - (\partial^2_{tx}n)^2,$$

$$\partial_t n\,\partial^4_{xxxx}n = \partial_x(\partial^3_{xxx}n\,\partial_t n - \partial^2_{tx}n\,\partial^2_{xx}n) + \partial^3_{txx}n\,\partial^2_{xx}n = \partial_x(\partial^3_{xxx}n\,\partial_t n - \partial^2_{tx}n\,\partial^2_{xx}n) + \partial_t\Big(\tfrac12(\partial^2_{xx}n)^2\Big).$$

Summing up, we get:

$$\begin{aligned}&\partial_t\tfrac12\Big((\partial_t n)^2 + c_s^2(\partial_x n)^2 + \tfrac{1}{\varepsilon^2}n^2 + \nu\beta(\partial^2_{xx}n)^2\Big) + (\beta+\nu)(\partial^2_{tx}n)^2\\ &= \partial_x\Big(c_s^2 n_x\partial_t n + (\nu+\beta)\partial^2_{tx}n\,\partial_t n - \nu\beta(\partial^3_{xxx}n\,\partial_t n - \partial^2_{tx}n\,\partial^2_{xx}n)\Big).\end{aligned}$$

For periodic or homogeneous Neumann boundary conditions, we obtain:

$$\partial_t \tfrac{1}{2} \int_a^b \Big((\partial_t n)^2 + c_s^2 (\partial_x n)^2 + \tfrac{1}{\varepsilon^2} n^2 + \nu\beta (\partial_{xx}^2 n)^2 \Big)\, dx + (\beta+\nu) \int_a^b (\partial_{tx}^2 n)^2\, dx = 0.$$

Again we see that the second order dissipative coefficient is only dependent on $\nu+\beta$.

This energy estimate is still true for a space-discrete version of the (LVEP-2) system with $\beta = \nu$ (simply referred to by (LVEP) system). Indeed, the natural finite difference space discretization for the (LVEP) system is given by (with the same notations as above):

$$\partial_t n + \tilde{D} u = \beta D^2 n\,, \tag{3.59}$$

$$\partial_t u + c_s^2 \tilde{D} n = \beta D^2 u + \tilde{D}\phi\,, \tag{3.60}$$

$$\varepsilon^2 D^2 \phi = n\,. \tag{3.61}$$

where $\tilde{D} = (D_+ + D_-)/2$ and $D^2 = D_+ D_-$.

Both the upwinding and Lax-Friedrichs scheme for LEP (3.1)-(3.3) take the same form with $\beta = c_s h/2$ and $\beta = \frac{h^2}{2\delta}$ respectively, as we will explain below. This is also true for the Modified Lax-Friedrichs scheme used in [7] and [9], because in the linearized case, the Modified Lax-Friedrichs scheme and the upwind scheme coincide. Indeed, for the upwinding scheme, we first diagonalize the (LEP) system (3.1)-(3.3), then take upwind difference as given below,

$$\begin{aligned} \partial_t(u + c_s n) + c_s D_-(u + c_s n) &= \tilde{D}\phi, \\ \partial_t(u - c_s n) - c_s D_+(u - c_s n) &= \tilde{D}\phi, \\ \varepsilon^2 D^2 \phi &= n. \end{aligned}$$

Recast to the original form, it can be written

$$\begin{aligned} \partial_t n + \tilde{D} u &= \frac{c_s}{2}(D_+ - D_-) n, \\ \partial_t u + c_s^2 \tilde{D} n &= \frac{c_s}{2}(D_+ - D_-) u + \tilde{D}\phi, \\ \varepsilon^2 D^2 \phi &= n. \end{aligned}$$

Note that $D_+ - D_- = h D^2$. The above is same as (3.59-3.61) with $\beta = c_s h/2$. For the Lax-Friedrichs scheme we first write the scheme in its usual way:

$$\begin{aligned} (n^{m+1} - n^m)/\delta + \tilde{D} u^m &= \frac{h^2}{2\delta} D^2 n^m, \\ (u^{m+1} - u^m)/\delta + c_s^2 \tilde{D} n^m &= \frac{h^2}{2\delta} D^2 u^m + \tilde{D}\phi^{m+1}, \\ \varepsilon^2 D^2 \phi^{m+1} &= n^{m+1}. \end{aligned}$$

We can regard it as a time discretization scheme of (3.59)-(3.61) with $\beta = \frac{h^2}{2\delta}$.

In the appendix, section 7.3, we will show that the time continuous scheme (3.59)-(3.61) has the following energy estimate for the periodic boundary condition case.

$$\partial_t \tfrac{1}{2}\Big(c_s^2 \|D_- n\|^2 + \|D_- u\|^2 + \frac{1}{\varepsilon^2}\|n\|^2 \Big) + \beta(c_s^2 \|D^2 n\|^2 + \|D^2 u\|^2) + \frac{1}{\varepsilon^2}\|D_- n\|^2 = 0. \tag{3.62}$$

This indicates that the space discretization has the same behaviour as the space-continuous model. For this reason, from now on, we shall only consider the problem of the stability of the time discretization, and replace the influence of the space discretization by a Fourier transform with limitation of the admissible wave numbers to a bounded interval (the latter being related to the space step Δx).

In the remainder of this analysis, we will be using the (LVEP-2) system (3.46)-(3.48) with $\beta = \nu$ and will simply refer to it by (LVEP) system. The coefficient β is linked with numerical diffusion. For most schemes,

$$\beta = O(c_s \Delta x)\,. \tag{3.63}$$

Again, we suppose that $|\xi| \leq \xi^* = \pi/\Delta x$, having in mind that this mimics a space discretization of mesh step Δx. Therefore, assuming (3.63), there exists a constant $C^* > 0$ such that

$$\frac{\beta |\xi|}{c_s} \leq C^*\,. \tag{3.64}$$

The Asymptotic Preserving strategy ((II) scheme) for the (LVEP) system reads as follows:

$$\frac{\hat{n}^{m+1} - \hat{n}^m}{\delta} + i\xi \hat{u}^{m+1} + \beta \xi^2 \hat{n}^m = 0\,, \tag{3.65}$$

$$\frac{\hat{u}^{m+1} - \hat{u}^m}{\delta} + i\xi c_s^2 \hat{n}^m + \beta \xi^2 \hat{u}^m + \frac{i}{\varepsilon^2 \xi} \hat{n}^{m+1} = 0\,. \tag{3.66}$$

We shall compare this strategy with an explicit time-differencing of the mass flux term, but still implicit in the Poisson source term. Such a scheme ((EI) scheme) reads:

$$\frac{\hat{n}^{m+1} - \hat{n}^m}{\delta} + i\xi \hat{u}^{m} + \beta \xi^2 \hat{n}^m = 0\,, \tag{3.67}$$

$$\frac{\hat{u}^{m+1} - \hat{u}^m}{\delta} + i\xi c_s^2 \hat{n}^m + \beta \xi^2 \hat{u}^m + \frac{i}{\varepsilon^2 \xi} \hat{n}^{m+1} = 0\,. \tag{3.68}$$

Note that in section 3.2, we mentioned the converse strategy (implicit in the mass flux, and explicit in the Poisson source term, see system (3.19), (3.20)) called (IE) scheme and we showed that in the non-viscous case, the stability requirement scales down to 0 as $\varepsilon \to 0$. It is unlikely that the use of an artificial viscosity can cure this instability, since the cause of this instability is clearly linked to the coupling with the field (because of its dependence with respect to ε) while the motivation of the artificial viscosity is to compensate for the instability of the space-centered scheme of the purely hydrodynamic part of the model. For this reason and for the sake of conciseness, we do not consider the (IE) scheme in the viscous case.

For completeness, we mention the full explicit scheme, in both mass flux and Poisson source term. This scheme ((EE) scheme) reads as follows:

$$\frac{\hat{n}^{m+1} - \hat{n}^m}{\delta} + i\xi \hat{u}^{m} + \beta \xi^2 \hat{n}^m = 0\,, \tag{3.69}$$

$$\frac{\hat{u}^{m+1} - \hat{u}^m}{\delta} + i\xi c_s^2 \hat{n}^m + \beta \xi^2 \hat{u}^m + \frac{i}{\varepsilon^2 \xi} \hat{n}^{m} = 0\,. \tag{3.70}$$

We mention that the (EE) and (EI) schemes for the non-viscous system would be unconditionally unstable because of the well-known instability of space-centered schemes for hydrodynamics. It is easy to check this instability directly. The computation is left to the reader.

For these schemes, the stability analysis developed in the appendix (section 7.4) leads to the following conclusions. The Asymptotic Preserving (II) scheme shows a uniform stability condition with respect to the parameter ε, as expected. More precisely, a sufficient stability condition is

$$\delta \leq \frac{2\beta}{c_s^2} F(C^*), \tag{3.71}$$

where $F(C^*)$ is a convenient function of the constant C^* (see (3.64)). Again, this looks like a CFL condition, which only depends on the purely hydrodynamic part of the scheme and not on ε.

By contrast, the (EI) scheme has a stability domain which shrinks down to 0 as $\varepsilon \to 0$. The analysis is simplified by making the following assumption:

$$\delta \leq \frac{2\beta}{c_s^2} \frac{1}{1 + {C^*}^2} := \delta^* , \tag{3.72}$$

where C^* is given by (3.64). This condition is again similar to a CFL condition (see (3.63)). Under this condition, a sufficient stability constraint for this scheme is of the form

$$\delta \leq \frac{2}{\sqrt{1 + {C^*}^2}} \varepsilon , \quad \forall \varepsilon \in [0, \frac{\beta}{c_s^2 \sqrt{1 + {C^*}^2}}] . \tag{3.73}$$

Additionally if δ satisfying (3.72) is such that $\delta > 2\varepsilon$, the scheme is unstable. Therefore, the stability domain is an interval of size exactly $O(\varepsilon)$. This scheme is the linearized version of the standard scheme used for the Euler-Poisson problem. The conditional stability of this scheme in a fully discrete version was proved earlier by S. Fabre in [16].

Finally, the fully explicit (EE) scheme has a disastrous behaviour. Its stability condition for a given wave-number ξ is given by :

$$\delta \leq \delta_{ee}(\xi) := \frac{2\beta\xi^2}{\beta^2 \xi^4 + c_s^2 \xi^2 + \frac{1}{\varepsilon^2}} . \tag{3.74}$$

Over the range of admissible wave-numbers $[0, \xi^*]$, the minimal value of $\delta_{ee}(\xi)$ is zero, showing that the scheme is actually unstable. Therefore, numerical diffusion is not sufficient to stabilize the (EE) scheme. This was proved by S. Fabre for a fully discrete version of the scheme in [16].

To summarize, the (EE) scheme is unconditionally unstable, even used in conjunction with a decentered space discretization of the hydrodynamic fluxes. The (EI) is conditionally stable: the time-step must be kept proportional to ε. Finally, the (II) scheme is asymptotically stable when $\varepsilon \to 0$. Its stability requirement only involves a CFL condition related to the space discretization of the hydrodynamic part of the model. All the proofs are collected in the appendix, section 7.4, for the reader's convenience.

In the next section, we investigate if these conclusions remain true when the (LEP) system is linearized about a non-zero velocity. We show that the (II) scheme remains uniformly stable in the viscous case (i.e. for a space decentered discretization), but not in the non-viscous case (for a space-centered scheme).

4. The linearized Euler-Poisson system about non-zero velocity. Linearizing system (2.1)-(2.3) about the stationary homogeneous solution $n_0 = 1$, $u_0 = a$, a being constant (but not equal to 0) and $\partial_x \phi_0 = 0$, leads to the following Linearized Euler-Poisson (LEP) system:

$$\partial_t n + a\partial_x n + \partial_x u = 0\,, \tag{4.1}$$
$$\partial_t u + a\partial_x u + c_s^2 \partial_x n = \partial_x \phi\,, \tag{4.2}$$
$$\varepsilon^2 \partial^2_{xx}\phi = n. \tag{4.3}$$

When $|a| < |c_s|$, it can be reduced to a wave-like formulation (specifically a Klein-Gordon equation with mass equal to $1/\varepsilon^2$):

$$\partial^2_{tt} n + 2a\partial^2_{tx} n - (c_s^2 - a^2)\partial^2_{xx} n + \frac{1}{\varepsilon^2} n = 0\,. \tag{4.4}$$

We can get an energy estimate for this system by multiplying by $\partial_t n$. We have:

$$\partial_t\Big((\partial_t n)^2 + (c_s^2 - a^2)(\partial_x n)^2 + \frac{1}{\varepsilon^2}n^2\Big) = 2(c_s^2 - a^2)\partial_x\Big(\partial_x n\,\partial_t n\Big) - 2a\partial_x\Big((\partial_t n)^2\Big). \tag{4.5}$$

If periodic or homogeneous Dirichlet boundary conditions are imposed at the boundary points, conservation of energy follows:

$$\partial_t \int_a^b \Big((\partial_t n)^2 + (c_s^2 - a^2)(\partial_x n)^2 + \frac{1}{\varepsilon^2}n^2\Big)\,dx = 0\,. \tag{4.6}$$

We consider a space discretization of (4.4) by replacing the continuous second order space derivative by the discrete operator D^2, with the same meaning as above for $\tilde{D}$:

$$\partial^2_{tt} n + 2a\partial_t \tilde{D} n - (c_s^2 - a^2)D^2 n + \frac{1}{\varepsilon^2} n = 0\,. \tag{4.7}$$

Applying the (II) time-difference scheme (3.17) to this equation leads to

$$\frac{n^{m+1} - 2n^m + n^{m-1}}{\delta^2} + 2a\frac{\tilde{D}n^m - \tilde{D}n^{m-1}}{\delta} + a^2 D^2 n^{m-1} - c_s^2 D^2 n^m + \frac{1}{\varepsilon^2} n^{m+1} = 0. \tag{4.8}$$

This scheme can be viewed as a wave-type version of the following scheme for the original (LEP) system (4.1)-(4.3):

$$\frac{n^{m+1} - n^m}{\delta} + aD_- n^m + D_- u^{m+1} = 0\,, \tag{4.9}$$
$$\frac{u^{m+1} - u^m}{\delta} + aD_+ u^m + c_s^2 D_+ n^m = D_+ \phi^{m+1}\,, \tag{4.10}$$
$$\varepsilon^2 D^2 \phi^{m+1} = n^{m+1}. \tag{4.11}$$

Indeed, eliminating u between (4.9) and (4.10), one has

$$\frac{n^{m+1} - 2n^m + n^{m-1}}{\delta^2} + 2a\frac{\tilde{D}n^m - \tilde{D}n^{m-1}}{\delta} + a^2 D^2 n^{m-1} - c_s^2 D^2 n^m + D^2 \phi^{m+1} = 0, \tag{4.12}$$

and using (4.11), we get (4.8). We can also eliminate n^{m+1} from (4.12) using (4.11) and get a discretization of the reformulated Poisson eq. (2.7) (in linearized form):

$$(\frac{\varepsilon^2}{\delta^2}+1)D^2\phi^{m+1} = \frac{2n^m - n^{m-1}}{\delta^2} + c_s^2 D^2 n^m - 2a\frac{\tilde{D}n^m - \tilde{D}n^{m-1}}{\delta} - a^2 D^2 n^{m-1}. \quad (4.13)$$

We also note that the implicit treatment of the last term in (4.8) gives a strong dissipation (like in the Fourier analysis case). This can be seen from the use of the so-called equivalent equation (i.e. the equation obtained by retaining the leading order term in the Taylor expansion of the time semi-discrete scheme). This equivalent equation is written

$$\partial_{tt}^2 n + 2a\partial_{tx}^2 n - (c_s^2 - a^2)\partial_{xx}^2 n + \frac{1}{\varepsilon^2}n = -\frac{\delta}{\varepsilon^2}\partial_t n + a^2\delta\partial_{txx}^3 n + a\delta\partial_{ttx}^3 n\,. \quad (4.14)$$

The first two terms on the right hand side are dissipative terms, while the nature of the last term is less clear.

By examining the (LEP) system about zero velocity, we have shown that schemes that are implicit in both the mass flux term and the electric field source term are asymptotically stable in the small Debye length limit $\varepsilon \to 0$.

Two schemes have been shown to have this property. A first scheme is associated with centered space-differencing: the (II) scheme for the (LEP) system without diffusion. This scheme is equivalently the (WEI) implicit scheme for the wave-equation formulation of the (LEP) system (see above and section 3.2). A second scheme scheme is associated with a decentered shock capturing type space-discretization: the (II) scheme for the viscous linearized Euler-Poisson system or (LVEP) system (see section 3.3). We have seen in section 3.2 that the (II) scheme for the inviscid (LEP) system is not uniformly stable with respect to ε when $\varepsilon \to \infty$ and that this lack of uniformity is related with the space centered-differencing.

We are now concerned with the stability of these two schemes for the linearized Euler-Poisson problem about non-zero velocity. This analysis will confirm that the (II) scheme without diffusion is uniformly stable in the limit $\varepsilon \to 0$ but not in the limit $\varepsilon \to \infty$ while the (II) scheme with diffusion is uniformly stable for all values of ε whatever small or large.

We linearize system (2.1)-(2.3) about the stationary homogeneous solution $n_0 = 1$, $q_0 =$ Constant (but not equal to 0) and $\partial_x\phi_0 = 0$. We denote by $\zeta \ll 1$ the size of the initial perturbation to this stationary state. Then, we can formally expand the solution as follows: $n = 1+\zeta n' + O(\zeta^2)$, $q = q_0 + \zeta(q_0 n' + u') + O(\zeta^2)$, $\phi = \zeta\phi' + O(\zeta^2)$. Retaining only terms of order ζ, we are led to the following Linearized Euler-Poisson (LEP) system (where the primes have been dropped for the sake of clarity):

$$\partial_t n + \partial_x u = 0\,, \quad (4.15)$$

$$\partial_t u + \partial_x(2q_0 u - q_0^2 n) + c_s^2\partial_x n = \partial_x\phi\,, \quad (4.16)$$

$$\varepsilon^2\partial_{xx}\phi = n. \quad (4.17)$$

After Fourier transform in space, the system reads

$$\partial_t\hat{n} + i\xi\hat{u} = 0\,, \quad (4.18)$$

$$\partial_t\hat{u} + i\xi(2q_0 u - q_0^2 n) + i\xi c_s^2\hat{n} = i\xi\hat{\phi}\,, \quad (4.19)$$

$$-\varepsilon^2\xi^2\hat{\phi} = \hat{n}. \quad (4.20)$$

After elimination of $\hat\phi$, the system reduces to a 2×2 system of the form:

$$\begin{pmatrix} \partial_t \hat n \\ \partial_t \hat u \end{pmatrix} + \begin{pmatrix} 0 & i\xi \\ -iq_0^2\xi + i\xi c_s^2 + \frac{i}{\varepsilon^2 \xi} & 2iq_0\xi \end{pmatrix} \begin{pmatrix} \hat n \\ \hat u \end{pmatrix} = 0\,. \tag{4.21}$$

Its solutions are of the form

$$\begin{pmatrix} \hat n(\xi,t) \\ \hat u(\xi,t) \end{pmatrix} = \begin{pmatrix} \hat n_+(\xi) \\ \hat u_+(\xi) \end{pmatrix} e^{i\theta_+^\varepsilon(\xi)t} + \begin{pmatrix} \hat n_-(\xi) \\ \hat u_-(\xi) \end{pmatrix} e^{i\theta_-^\varepsilon(\xi)t}\,, \tag{4.22}$$

where

$$\theta_\pm^\varepsilon(\xi) = -q_0\xi \pm \left(c_s^2\xi^2 + \frac{1}{\varepsilon^2} \right)^{1/2}\,. \tag{4.23}$$

These eigenmodes are obtained by shifting the phase by the quantity $-q_0\xi$. Indeed, the linearization about a uniform non-zero velocity can be deduced from a linearization about zero velocity by a Galilean transformation $x' = x + q_0 t$, $v' = v$ thanks to the Galilean invariance of the nonlinear Euler problem. A Galilean shift translates in Fourier space into a constant phase shift.

It should not be inferred however that the stability of the schemes for the linearization about non-zero velocity can be deduced from considerations concerning the linearized system about zero velocity. Indeed, the mass and momentum flux terms are treated differently (one being implicit while the other one is explicit) which breaks this Galilean invariance.

Now, the (II) scheme for the inviscid Euler-Poisson system linearized about non-zero velocity reads:

$$\frac{\hat n^{m+1} - \hat n^m}{\delta} + i\xi \hat u^{m+1} = 0\,, \tag{4.24}$$

$$\frac{\hat u^{m+1} - \hat u^m}{\delta} + i\xi(2q_0 u^m - q_0^2 n^m) + i\xi c_s^2 \hat n^m + \frac{i}{\varepsilon^2\xi}\hat n^{m+1} = 0\,, \tag{4.25}$$

while the (II) scheme for the viscous system is written:

$$\frac{\hat n^{m+1} - \hat n^m}{\delta} + i\xi \hat u^{m+1} + \beta\xi^2 \hat n^m = 0\,, \tag{4.26}$$

$$\frac{\hat u^{m+1} - \hat u^m}{\delta} + i\xi(2q_0 u^m - q_0^2 n^m) + i\xi c_s^2 \hat n^m + \beta\xi^2\hat u^m + \frac{i}{\varepsilon^2\xi}\hat n^{m+1} = 0\,. \tag{4.27}$$

We recall that the inviscid linearized system in Fourier space mimics a centered spatial discretization while the viscous system mimics a decentered spatial discretization associated with a numerical diffusion equal to β.

In section 7.5, we prove that the (II) scheme for the inviscid system is uniformly stable in the limit $\varepsilon \to 0$. But ε must be large enough. To state a more precise result, we recall that the range of admissible wave numbers ξ is limited to the interval $|\xi| \le \xi^* = \frac{\pi}{\Delta x}$ in order to mimic the influence of a space discretization Δx. Then, for all $K < 1$, there exists a $\delta^* > 0$ such that for all $\delta < \delta^*$ and all ε such that $\varepsilon < K\delta$, the (II) scheme for the inviscid system is stable. We note that as K gets closer to 1, δ^* tends to 0. In particular, the scheme is not stable when $\varepsilon = O(1)$ and $\delta \to 0$, which would be the situation if we wanted to resolve the plasma period scale. We show in section 7.5 that if $\varepsilon = 0$, the scheme is unstable. Therefore, the (II) scheme for the inviscid system cannot be of practical usefulness.

We now turn to the (II) scheme for the viscous system. We assume that the diffusion constant β is proportional to the time step: $\beta = B\delta$. Then, the result shown in in section 7.5 is that if B is large enough (depending only on q_0 and c_s), there exists an interval of time steps $[0, \delta^*]$ (again, only depending on q_0 and c_s) such that for all δ's in this interval, all admissible wave-numbers ξ (i.e. such that $|\xi| \leq \xi^*$) and all $\varepsilon \geq 0$, the scheme is stable. Usually, the diffusion constant is taken proportional to Δx (or to $\Delta x^2/\delta$ in the case of the Lax-Friedrichs scheme) rather than to δ. The result would also remain true in this case but the proof would be more complex. Indeed, supposing that $\beta = O(\delta)$ allows to express the problem in terms of the 'CFL variable' $X = \xi\delta$ only, which simplifies the (already quite technical) proof. Also, the result in itself is interesting as it shows that the scheme is stable even with a vanishingly small diffusion proportional to δ.

These result show that, at least for the linearized problem about any uniform state (of zero or non-zero velocity), the (II) scheme associated with a decentered space-differencing is uniformly stable with respect to all values of ε be it small or large.

5. Burgers-Poisson equation. In a preliminary attempt to analyze the nonlinear effects, we consider the following model consisting of a nonlinear Burgers-Poisson system:

$$\partial_t u + u\partial_x u = -\partial_x \phi, \tag{5.1}$$

$$\varepsilon^2 \partial^2_{xx} \phi = u. \tag{5.2}$$

This equation appears as a model describing high-frequency wave propagation in relaxing media [30, 17]. In [30], special soliton solutions called 'loop solitons' are constructed using inverse scattering theory. However, these solutions are multivalued and cannot be computed by the method developed below. Note that the quasineutral limit has been investigated theoretically in [17].

This system owns the following entropy pair $(\frac{u^2}{2}, \frac{u^3}{3} + \frac{\varepsilon^2 \phi_x^2}{2})$. Indeed, simple computations show that

$$\partial_t \left(\frac{u^2}{2}\right) + \partial_x \left(\frac{u^3}{3} + \frac{\varepsilon^2 (\partial_x \phi)^2}{2}\right) = 0\,. \tag{5.3}$$

The theoretical study of this equation will be reported in a future paper.

We consider the following conservative scheme,

$$\partial_t u_j + \frac{u_{j+1} + u_j + u_{j-1}}{3} \frac{u_{j+1} - u_{j-1}}{2h} = -\frac{\phi_{j+1} - \phi_{j-1}}{2h}\,, \tag{5.4}$$

$$\varepsilon^2 \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{h^2} = u_j, \tag{5.5}$$

where $h = \Delta x$ is the grid size. The above scheme may generate some oscillations near shocks due to the dispersive nature of the scheme. We can recast the above scheme in a conservative form,

$$\partial_t u_j + \frac{f_{j+1/2} - f_{j-1/2}}{h} = -\frac{\phi_{j+1} - \phi_{j-1}}{2h}\,, \tag{5.6}$$

$$\varepsilon^2 \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{h^2} = u_j, \tag{5.7}$$

where $f_{j+1/2} = \frac{1}{6}(u_j^2 + u_j u_{j+1} + u_{j+1}^2)$. We have a discrete entropy pair

$$\partial_t \left(\frac{u_j^2}{2} \right) + \frac{g_{j+1/2} - g_{j-1/2}}{h} = 0\,, \tag{5.8}$$

where $g_{j+1/2} = \frac{1}{6}(u_j^2 u_{j+1} + u_j u_{j+1}^2) + \frac{\varepsilon^2}{2}\left(\frac{\phi_{j+1}-\phi_j}{h}\right)^2$. As a direct consequence, we can prove a stability result for the space semi-discretization of this model. We can also get some convergence results assuming the solution is smooth.

More precisely, we only consider the periodic boundary condition case for simplicity. Let $\phi(x), u(x)$ be the exact solution. Let $\Phi_j = \phi(x_j)$, $U_j = \varepsilon^2 \frac{\Phi_{j+1}-2\Phi_j+\Phi_{j-1}}{h^2}$. Assuming that the initial data are approximated up to second order, we have

$$\|u_h - u\| = \sqrt{\sum_j (u_j - u(x_j))^2 h} \leq Ch^2,$$

and

$$\varepsilon\|\phi_h - \phi\| = \varepsilon\sqrt{\sum_j (\phi_j - \phi(x_j))^2 h} \leq Ch^2,$$

for all $\varepsilon \leq \varepsilon_0$, with arbitrary $\varepsilon_0 > 0$ and where $C = C(u, \phi, \varepsilon_0)$. We refer to section 7.6 in the appendix for the proof.

This analysis indicates that a similar implicit time-discretization as in the (LVEP) system will provide asymptotic stability with respect to $\varepsilon \to 0$. The investigation of this problem will be the topic of future work.

6. Conclusion. In this paper, we have analyzed the stability property of a recently proposed numerical scheme for the Euler-Poisson system [9]. For this analysis, we have considered the linearized Euler-Poisson system about both zero and non-zero velocities, and, after Fourier transform in space, we have applied the time discretization under scrutiny. We have mimiced the influence of a space decentered discretization by adding viscosity terms in both the mass and momentum conservation equations. We have shown that the resulting scheme is uniformly stable with respect to the Debye parameter ε. By contrast, other schemes with less implicit treatment or with centered space differencing are shown to lose the uniform stability property. Elements of a nonlinear stability analysis have been given by considering a model Burgers-Poisson problem. The analysis of this strategy for the kinetic version of this model [10] is in progress.

7. Appendix: proofs of statements.

7.1. AP strategy with centered space differencing: Fourier analysis. (II) scheme: The characteristics equation for the (II) scheme (3.14)-(3.15) for the non-viscous (LEP) system is written:

$$(1 + \frac{\delta^2}{\varepsilon^2})\lambda^2 - 2(1 - \frac{1}{2}\delta^2 c_s^2 \xi^2)\lambda + 1 = 0\,. \tag{7.1}$$

Obviously, the product of the two roots is smaller than unity. Therefore, a sufficient condition for stability is that the discriminant is negative. In this case, the two roots are complex conjugate numbers of module less than unity. Of course, it is not a necessary stability condition since, in the case of a positive discriminant, there may exist real roots which are both less than unity. However, any sufficient condition which leads to an ε-independent stability requirement is enough for our purpose.

The discriminant of this polynomial is

$$D' = \delta^2 (\frac{1}{4}\delta^2 c_s^4 \xi^4 - (\frac{1}{\varepsilon^2} + c_s^2 \xi^2)) . \tag{7.2}$$

Therefore, we have $D' \leq 0$ iff condition (3.26) is satisfied. $\delta_{ii}(\xi)$ is a monotonously decreasing function of ξ, so that condition (3.27) must be satisfied in order for the scheme to be stable over the range of admissible wave numbers. The eigenvalues are given by:

$$\lambda_\pm = \frac{1}{1+\frac{\delta^2}{\varepsilon^2}} \left(1 - \frac{1}{2} c_s^2 \xi^2 \delta^2 \pm i \sqrt{|D'|} \right) = |\lambda_\pm| e^{\pm i\theta} ,$$

with

$$\cos\theta = \frac{1 - \frac{1}{2} c_s^2 \xi^2 \delta^2}{\sqrt{1+\frac{\delta^2}{\varepsilon^2}}} , \quad |\lambda_\pm| = \left(1 + \frac{\delta^2}{\varepsilon^2} \right)^{-1/2} .$$

We note that $|\lambda_\pm| \sim \frac{\varepsilon}{\delta}$ as $\varepsilon \to 0$ and that the behaviour of the phase is given by (3.30).

(IE) scheme: The characteristics equation of the (IE) scheme (3.19)-(3.20) for the non-viscous (LEP) system is written:

$$\lambda^2 - 2(1 - \frac{\delta^2}{2}(\frac{1}{\varepsilon^2} + c_s^2 \xi^2))\lambda + 1 = 0 . \tag{7.3}$$

Obviously, the product of the two roots is 1. So, the stability condition is equivalent to saying that the discriminant of the polynomial is negative. Indeed, in this case, the two roots are complex conjugate and their product is also the square of their module and is equal to one. In the converse case, the two roots are real and one is necessarily above unity, showing instability. The discriminant is

$$D' = \delta^2 (\frac{1}{\varepsilon^2} + c_s^2 \xi^2)(\frac{1}{4}\delta^2 (\frac{1}{\varepsilon^2} + c_s^2 \xi^2) - 1) . \tag{7.4}$$

Therefore, $D' \leq 0$ iff condition (3.28) is satisfied. We note that $\delta_{ie}(\xi)$ is a monotonously decreasing function of ξ, so that condition (3.29) must be satisfied in order for the scheme to be stable over the range of admissible wave numbers. The eigenvalues $\lambda_\pm$ are given by

$$\lambda_\pm = 1 - \frac{\delta^2}{2}(\frac{1}{\varepsilon^2} + c_s^2 \xi^2) \pm i \sqrt{|D'|} = e^{\pm i\theta} , \quad \cos\theta = 1 - \frac{1}{2} \left(\delta (\frac{1}{\varepsilon^2} + c_s^2 \xi^2)^{1/2} \right)^2 .$$

In the limit $\delta \to 0$ with fixed ε, we recover the fact that θ is an approximation of the phase of the continuous problem given by (3.10) but the stability constraint prevents us from investigating the limit $\varepsilon \to 0$ with fixed δ.

7.2. AP strategy with centered space differencing: stability analysis of the fully discrete scheme. We prove the asymptotic stability of the scheme (3.33)-(3.35). We assume periodic boundary conditions. The approximation n^m is a discrete sequence n_j^m for $j \in \mathbb{Z}/N\mathbb{Z}$ if N is the number of points (such that $Nh = L$ and L is the interval length). We view n^m as an element of a discrete ℓ^2 space, provided with the inner product $\langle a, b\rangle = \sum a_i b_i h$ and norm $\|a\|^2 = \langle a, a\rangle$. By discrete summation by parts, one can easily check the following formulas:

$$\langle D^2 a, b\rangle = -\langle D_- a, D_- b\rangle, \quad \langle \tilde{D} a, b\rangle = -\langle a, \tilde{D} b\rangle.$$

Denote $D_t^2 n^m = \frac{n^{m+1}-2n^m+n^{m-1}}{\delta^2}$. Using these formulae, we have:

$$2\langle n^{m+1} - n^m, D_t^2 n^m\rangle = \|\frac{n^{m+1} - n^m}{\delta}\|^2 - \|\frac{n^m - n^{m-1}}{\delta}\|^2 + \delta^2 \|D_t^2 n^m\|^2,$$

$$\begin{aligned} 2\langle n^{m+1} - n^m, -c_s^2 D^2 n^m\rangle &= c_s^2\Big(\|D_- n^{m+1}\|^2 - \|D_- n^m\|^2 - \|D_-(n^{m+1} - n^m)\|^2\Big) \\ &\geq c_s^2\Big(\|D_- n^{m+1}\|^2 - \|D_- n^m\|^2\Big) - \frac{4c_s^2}{h^2}\|n^{m+1} - n^m\|^2, \end{aligned}$$

$$2\langle n^{m+1} - n^m, \frac{1}{\varepsilon^2} n^{m+1}\rangle = \frac{\|n^{m+1}\|^2 - \|n^m\|^2 + \|n^{m+1} - n^m\|^2}{\varepsilon^2}.$$

Adding up these formulas leads to:

$$\begin{aligned} &\|\frac{n^{m+1} - n^m}{\delta}\|^2 + c_s^2\|D_- n^{m+1}\|^2 + \tfrac{1}{\varepsilon^2}\|n^{m+1}\|^2 + \delta^2\|D_t^2 n^m\|^2 + \Big(\frac{1}{\varepsilon^2} - \frac{4c_s^2}{h^2}\Big)\|n^{m+1} - n^m\|^2 \\ &\leq \|\frac{n^m - n^{m-1}}{\delta}\|^2 + c_s^2\|D_- n^m\|^2 + \tfrac{1}{\varepsilon^2}\|n^m\|^2. \end{aligned}$$

If $2\varepsilon c_s \leq h$, the scheme is stable.

7.3. Time-continuous, space-discrete viscous (LVEP) system: Energy estimate. In this section, we prove (3.62). We use the same framework as in section 7.2. Multiplying (3.59) by $D^2 n$ one has

$$\langle D^2 n, \partial_t n\rangle + \langle D^2 n, \tilde{D} u\rangle = \beta\|D^2 n\|^2.$$

A simple computation gives

$$\langle D^2 n, \partial_t n\rangle = -\partial_t\Big(\tfrac{1}{2}\|D_- n\|^2\Big).$$

We recast it in the conservative form

$$\partial_t\Big(\tfrac{1}{2}\|D_- n\|^2\Big) + \beta\|D^2 n\|^2 = \langle D^2 n, \tilde{D} u\rangle. \tag{7.5}$$

Similarly, we multiply (3.60) by $D^2 u$ and get

$$\partial_t\Big(\tfrac{1}{2}\|D_- u\|^2\Big) + \beta\|D^2 u\|^2 + \langle D^2 u, \tilde{D}\phi\rangle = c_s^2\langle D^2 u, \tilde{D} n\rangle. \tag{7.6}$$

In the same way, using (3.59) and (3.61), we obtain

$$\langle D^2 u, \tilde{D}\phi\rangle = -\langle \tilde{D} u, D^2\phi\rangle = \tfrac{1}{\varepsilon^2}\langle \partial_t n - \beta D^2 n, n\rangle = \tfrac{1}{\varepsilon^2}\partial_t(\tfrac{1}{2}\|n\|^2) + \frac{\beta}{\varepsilon^2}\|D_- n\|^2. \tag{7.7}$$

Note that

$$\langle D^2 n, \tilde{D} u\rangle + \langle D^2 u, \tilde{D} n\rangle = 0. \tag{7.8}$$

We multiply (3.59) by c_s^2 and combine with (7.6) to (7.8) and get:

$$\partial_t \tfrac{1}{2}\Big(c_s^2\|D_- n\|^2 + \|D_- u\|^2 + \tfrac{1}{\varepsilon^2}\|n\|^2\Big) + \beta\Big(c_s^2\|D^2 n\|^2 + \|D^2 u\|^2 + \tfrac{1}{\varepsilon^2}\|D_- n\|^2\Big) = 0,$$

which is the announced energy estimate.

7.4. AP strategy with decentered space-differencing: Fourier analysis of the linearized system about zero mean velocity. (II) scheme: The characteristics equation of the (II) scheme (3.65), (3.66) for the viscous (LVEP) system reads:

$$(1+\frac{\delta^2}{\varepsilon^2})\lambda^2 - 2\lambda\left(-\beta\xi^2\delta + 1 - \frac{1}{2}c_s^2\xi^2\delta^2\right) + (\beta\xi^2\delta - 1)^2 = 0. \tag{7.9}$$

We assume that $(\beta\xi^2\delta - 1)^2 \le 1$. This requires that $\delta \le 2/\beta|\xi|^2$. Over the range of admissible wave-numbers ξ, the r.h.s. of this inequality takes its minimal value for $\xi = C^* c_s/\beta$ (with C^* defined by (3.64)) which gives the following CFL-like condition:

$$\delta \le \frac{2\beta}{c_s^2}\,\frac{1}{{C^*}^2}\,. \tag{7.10}$$

Under this condition, the product of the two roots of (7.9) is less than unity. Therefore, a sufficient stability condition is that the discriminant of the polynomial is negative.

The discriminant is:

$$\begin{aligned} D' &= (\frac{1}{2}c_s^2\xi^2\delta^2)^2 + c_s^2\xi^2\delta^2(\beta\xi^2\delta - 1) - \frac{\delta^2}{\varepsilon^2}(\beta\xi^2\delta - 1)^2 \\ &\le (\frac{1}{2}c_s^2\xi^2)^2\,\delta^2\,P(\delta) \\ P(\delta) &:= \delta^2 + \frac{4\beta\delta}{c_s^2} - \frac{4}{c_s^2\xi^2}\,. \end{aligned} \tag{7.11}$$

It is readily seen that P has only one positive root δ_0 and is negative for $\delta < \delta_0$. An easy computation shows that

$$\delta_0 = \frac{2\beta}{c_s^2}\left(\left(1 + \frac{c_s^2}{\beta^2\xi^2}\right)^{1/2} - 1\right)\,.$$

The minimal value of this expression over the range of admissible values of ξ leads to the following stability requirement:

$$\delta \le \frac{2\beta}{c_s^2}\left(\left(1 + {C^*}^2\right)^{1/2} - 1\right)\,. \tag{7.12}$$

Collecting (7.10) and (7.12) leads to (3.71) with

$$F(C^*) = \min\left(\frac{1}{{C^*}^2}, \left(1 + {C^*}^2\right)^{1/2} - 1\right)\,.$$

(EI) scheme: The characteristics equation of the (EI) scheme (3.67), (3.68) for the viscous (LVEP) system reads:

$$\lambda^2 + 2\lambda\left(\beta\xi^2\delta - 1 + \frac{\delta^2}{2\varepsilon^2}\right) + (\beta\xi^2\delta - 1)^2 + c_s^2\xi^2\delta^2 = 0. \tag{7.13}$$

We assume that the product of the two roots is lower than unity. This imposes

$$(\beta\xi^2\delta - 1)^2 + c_s^2\xi^2\delta^2 \leq 1\,,$$

i.e.

$$\delta \leq \frac{2\beta}{c_s^2}\frac{1}{1+\frac{\beta^2\xi^2}{c_s^2}}\,. \tag{7.14}$$

The right-hand side is a decreasing function of ξ, and for the condition to be valid over the range of admissible wave-numbers, we request (3.72), which can be viewed as a CFL condition, using (3.63). Again, we wish to make clear that our goal is to find the ε-dependence of the stability condition, provided a CFL-like condition for the hydrodynamic part of the system is satisfied.

Under this condition, a sufficient condition for stability is that the discriminant of this polynomial is negative. The discriminant is given by

$$D' = \frac{1}{4}\left(\frac{\delta}{\varepsilon}\right)^4 + \left(\frac{\delta}{\varepsilon}\right)^2(\beta\xi^2\delta - 1) - c_s^2\xi^2\delta^2\,, \tag{7.15}$$

or, in terms of the variable $X = \delta/\varepsilon$:

$$D' = X^2P(X)\,, \quad P(X) := \frac{1}{4}X^2 + \varepsilon\beta\xi^2X - 1 - \varepsilon^2c_s^2\xi^2\,. \tag{7.16}$$

It is readily seen that $P(X)$ has only one positive root X_0 and that $P(X)$ is negative for $X < X_0$. For ε bounded by a constant depending on c_s and β only, namely $\varepsilon \leq \beta(c_s^2\sqrt{1+C^{*2}})^{-1}$, we can estimate that

$$\frac{2}{\sqrt{1+C^{*2}}} \leq X_0 < 2\,. \tag{7.17}$$

The stability requirement is satisfied as soon as $X \leq X_0$, or $\delta \leq X_0\varepsilon$. Therefore, the method is stable as soon as $\delta \leq 2(1+C^{*2})^{-1/2}\varepsilon$, which is condition (3.73).

To prove (7.17), we emphasize the ε-dependence of X_0 by writing $X_0(\varepsilon)$ and we compute it easily:

$$X_0(\varepsilon) = 2(-\beta\xi^2\varepsilon + \sqrt{1+\varepsilon^2(c_s^2\xi^2+\beta^2\xi^4)})\,.$$

To determine the monotonicity of X_0 with respect to ε, we compute

$$\frac{1}{2}\frac{dX_0}{d\varepsilon} = -\beta\xi^2 + \frac{\varepsilon(c_s^2\xi^2+\beta^2\xi^4)}{\sqrt{1+\varepsilon^2(c_s^2\xi^2+\beta^2\xi^4)}}\,.$$

We note that $dX_0/d\varepsilon(0) < 0$ and $X_0(0) = 2$. The unique positive root ε^* of $dX_0/d\varepsilon(\varepsilon) = 0$ is given by

$$\varepsilon^*(\xi) = \frac{\beta}{c_s\sqrt{c_s^2+\beta^2\xi^2}}\,.$$

Over the range of admissible wave numbers $|\xi| \leq c_s C^* \beta^{-1}$, we have:

$$\min_{|\xi| \leq c_s C^* \beta^{-1}} \varepsilon^*(\xi) = \frac{\beta}{c_s^2 \sqrt{1 + {C^*}^2}} := \bar{\varepsilon}^* .$$

Therefore, $X_0(\varepsilon)$ is a decreasing function of ε for all $\varepsilon \leq \bar{\varepsilon}^*$ and all admissible wave-numbers ξ. Now, for $\varepsilon = \varepsilon^*(\xi)$, we have

$$X_0(\varepsilon^*) = \frac{2c_s^2 \varepsilon^*}{\beta} = \frac{2c_s}{\sqrt{c_s^2 + \beta^2 \xi^2}} ,$$

and consequently $X_0(\varepsilon^*) \geq 2(1 + {C^*}^2)^{-1/2}$ for all admissible wave-numbers ξ. Therefore, we have the bound (7.17) for all $\varepsilon \leq \beta(c_s^2 \sqrt{1 + {C^*}^2})^{-1}$ and all admissible wave numbers ξ.

We now show that if δ/ε is too large, the scheme is unstable. First, we assume that (3.72) is satisfied. Now, in view of (7.17), if $\delta > 2\varepsilon$, the discriminant D' is positive. Therefore, the two roots are real and of product less that unity. Now, we consider the sum S of these roots, equal to

$$S = -2 \left(\beta \xi^2 \delta - 1 + \frac{\delta^2}{2\varepsilon^2} \right) .$$

By (7.14), we have $|\beta \xi^2 \delta - 1| \leq 1$, and therefore,

$$\frac{|S|}{2} > \frac{\delta^2}{2\varepsilon^2} - 1.$$

Then $|S|/2 > 1$ as soon as $\delta > 2\varepsilon$, in which case, the absolute value of one of the two real roots must be larger than unity, showing instability.

(EE) scheme: The characteristics equation of the (EE) scheme (3.69), (3.70) for the viscous (LVEP) system leads to the following expression of the eigenvalues:

$$\lambda = 1 - \beta \xi^2 \delta \pm i \left(c_s^2 \xi^2 + \frac{1}{\varepsilon^2} \right)^{1/2} \delta . \qquad (7.18)$$

Then

$$1 - |\lambda|^2 = 2\beta \xi^2 \delta - \left(\beta^2 \xi^4 + c_s^2 \xi^2 + \frac{1}{\varepsilon^2} \right) \delta^2 , \qquad (7.19)$$

is positive if and only if condition (3.74) is satisfied.

7.5. Linearized schemes about non-zero velocity: Fourier analysis. (II) scheme for the non-viscous (LEP) model: The characteristic equation for the (II) scheme for the non-viscous (LEP) model (4.24)-(4.25) is written:

$$\lambda^2 (1 + \frac{\delta^2}{\varepsilon^2}) - 2\lambda (1 - i q_0 \xi \delta + \frac{1}{2} q_0^2 \xi^2 \delta^2 - \frac{1}{2} c_s^2 \xi^2 \delta^2) + 1 - 2 i q_0 \xi \delta = 0 . \qquad (7.20)$$

To make the analysis simpler, we introduce the following notations: $X = \xi \delta$, $u = q_0$, $\Omega = \delta^2/\varepsilon^2$, $c^2 = c_s^2$. In these notations, the polynomial (7.20) is written

$$\lambda^2 (1 + \Omega) - 2\lambda (1 - iuX + \frac{1}{2} u^2 X^2 - \frac{1}{2} c^2 X^2) + 1 - 2iuX = 0 . \qquad (7.21)$$

The two roots of this polynomial are given by

$$\lambda = \frac{1}{1+\Omega}(1 - iuX + \frac{1}{2}(u^2 - c^2)X^2 \pm \sqrt{\Delta}), \tag{7.22}$$

where the discriminant is given by:

$$\Delta = -c^2X^2 - iu(u^2 - c^2)X^3 + \frac{1}{4}(u^2 - c^2)^2X^4 - \Omega(1 - 2iuX). \tag{7.23}$$

Now, in what follows, C will denote generic constants and $\mathcal{U}$ generic neighbourhoods of $X = 0$ which are independent of Ω. We first note that

$$|1 - 2iuX| = (1 + 4u^2X^2)^{1/2} \le 1 + CX^2, \quad \text{for} \quad C > 2u^2,$$
$$| - c^2X^2 - iu(u^2 - c^2)X^3 + \frac{1}{4}(u^2 - c^2)^2X^4| \le CX^2,$$

for X in $\mathcal{U}$. Then

$$|\Delta| \le (1 + CX^2)\Omega + CX^2 \le (1 + CX^2)\Omega,$$

for all $\Omega \ge \Omega_0$. Therefore,

$$\sqrt{|\Delta|} \le (1 + CX^2)\sqrt{\Omega}.$$

Similarly, we have

$$|1 - iuX + \frac{1}{2}(u^2 - c^2)X^2| = ((1 + \frac{1}{2}(u^2 - c^2)X^2)^2 + u^2X^2)^{1/2} \le 1 + CX^2,$$

for $X \in \mathcal{U}$.

Then, summing up all these estimates, we deduce that

$$|\lambda| = (1 + CX^2)\frac{1 + \sqrt{\Omega}}{1 + \Omega},$$

for $X \in \mathcal{U}$ and $\Omega \ge \Omega_0$. Let $\Omega_0 > 1$. There exists $C > 0$ such that for all $\Omega \ge \Omega_0$, $\frac{1+\sqrt{\Omega}}{1+\Omega} < 1 - C$. Then, for all X in a neighbourhood $\mathcal{U}$ of 0, we have $|\lambda| < 1$, showing the stability of the method.

We recover the result stated in section 4, by taking $K = 1/\Omega_0$ and $\xi^*\delta \in \mathcal{U}$ where ξ^* is the maximal range of admissible wave-numbers.

The condition that Ω should stay away from 0 is needed. This is clear since in this case, the scheme reduces to a semi-implicit centered scheme for the Linearized Euler problem, which cannot be stable. Let us verify this fact by setting $\Omega = 0$. Then, we get

$$\lambda = 1 - iuX + \frac{1}{2}(u^2 - c^2)X^2 \pm \sqrt{\Delta}, \tag{7.24}$$

where the discriminant is given by:

$$\Delta = -c^2X^2 - iu(u^2 - c^2)X^3 + \frac{1}{4}(u^2 - c^2)^2X^4. \tag{7.25}$$

We can write

$$\Delta = -c^2X^2(1 + iu\frac{u^2 - c^2}{c^2}X + O(X^2)).$$

Thus,

$$\sqrt{\Delta} = icX(1 + iu\frac{u^2 - c^2}{2c^2}X + O(X^2))\,,$$

and

$$\lambda = 1 + (\frac{1}{2}(u^2 - c^2) \mp u\frac{u^2 - c^2}{2c})X^2 + i(-u \pm c)X + O(X^3).$$

Then,

$$\begin{aligned}|\lambda|^2 &= (1 + (\frac{1}{2}(u^2 - c^2) \mp u\frac{u^2 - c^2}{2c})X^2 + O(X^3))^2 + ((-u \pm c)X + O(X^3))^2 \\ &= 1 + ((u^2 - c^2)(1 \mp \frac{u}{c}) + (-u \pm c)^2)X^2 + O(X^3) \\ &= 1 \mp \frac{u}{c}(c \mp u)^2)X^2 + O(X^3).\end{aligned}$$

One of the roots is of module strictly bigger than unity for $X \in \mathcal{U}$ as soon as $u \neq 0$ and therefore, the scheme is unstable.

(II) scheme for the viscous (LVEP) model: The characteristic equation for the (II) scheme for the viscous (LVEP) model (4.26)-(4.27) is written:

$$\begin{aligned}&\lambda^2(1 + \Omega) - 2\lambda(1 - BX^2 - iuX + \frac{1}{2}(u^2 - c^2)X^2) + \\ &\qquad +(1 - BX^2)(1 - BX^2 - 2iuX) = 0 \qquad (7.26)\end{aligned}$$

We have used the same notations as previously: $X = \xi\delta$, $u = q_0$, $\Omega = \delta^2/\varepsilon^2$, $c^2 = c_s^2$ and we have set $B = \beta/\delta$.

The two roots of this polynomial are given by

$$\lambda = \frac{1}{1 + \Omega}(1 - BX^2 - iuX + \frac{1}{2}(u^2 - c^2)X^2 \pm \sqrt{\Delta}), \qquad (7.27)$$

where the discriminant is given by:

$$\begin{aligned}\Delta = &-(1 - BX^2)c^2X^2 - iu(u^2 - c^2)X^3 + \frac{1}{4}(u^2 - c^2)^2X^4 - Bu^2X^4 \\ &-\Omega(1 - BX^2)(1 - BX^2 - 2iuX). \qquad (7.28)\end{aligned}$$

We are now going to treat the two cases Ω small and Ω large differently. Basically, in the small Ω case, stability will result from numerical diffusion (i.e. the term linked with B), while in the large Ω case, stability will be gained from the field term (i.e. the term linked with Ω). Again, in this proof, C will denote generic constants and $\mathcal{U}$ generic neighbourhoods of $X = 0$ which are independent of Ω.

First case: Ω large. We suppose that $\Omega \geq \Omega_0$ where Ω_0 will be specified later on. We first write

$$\mathrm{Re}(\Delta) = -(1 - BX^2)c^2X^2 + \frac{1}{4}(u^2 - c^2)^2X^4 - Bu^2X^4 - \Omega(1 - BX^2)^2, \qquad (7.29)$$

$$\mathrm{Im}(\Delta) = -u(u^2 - c^2)X^3 + \Omega(1 - BX^2)2uX. \qquad (7.30)$$

Clearly, there exists a neighbourhood $\mathcal{U}$ and constants C such that

$$\begin{aligned}|-(1-BX^2)c^2X^2+\frac{1}{4}(u^2-c^2)^2X^4-Bu^2X^4| &\le CX^2(1-BX^2)\\ &\le CX^2(1-2BX^2)\\ &\le CX^2.\end{aligned}$$

On the other hand, for $X\in\mathcal{U}$, we have $0\le 1-2BX^2\le(1-BX^2)^2\le 1$. Therefore, $\mathrm{Re}(\Delta)<0$ and

$$(\Omega-CX^2)(1-2BX^2)\le|\mathrm{Re}(\Delta)|\le\Omega+CX^2,$$

(with possibly different C's at the left and right hand sides of this inequality). In a similar way, for $X\in\mathcal{U}$,

$$|\mathrm{Im}(\Delta)|\le C|X|(1-BX^2)\Omega\le C|X|(1-2BX^2)\Omega\le C|X|\Omega,$$

again, with possibly different C's.

We denote $\Delta=|\Delta|(\cos\alpha+i\sin\alpha)$. We have

$$|\tan\alpha|=\frac{|\mathrm{Im}(\Delta)|}{|\mathrm{Re}(\Delta)|}\le\frac{C|X|\Omega}{\Omega-CX^2}\le\frac{C|X|\Omega_0}{\Omega_0-CX^2}\le C|X|,$$

because $\Omega/(\Omega-CX^2)$ is a decreasing function of Ω. Since $\mathrm{Re}(\Delta)<0$, this means $|\pi-\alpha|\le C|X|$ and $|\frac{\pi}{2}-\frac{\alpha}{2}|\le C|X|$. Then,

$$|\cos\frac{\alpha}{2}|=|\sin(\frac{\pi}{2}-\frac{\alpha}{2})|\le C|X|.$$

We also have, in view of (7.28):

$$|\Delta|\le\Omega(1+C|X|)+CX^2\le\Omega(1+C|X|).$$

Now, we write

$$\lambda=\frac{1}{1+\Omega}(1-(B-\frac{1}{2}(u^2-c^2))X^2\pm\sqrt{|\Delta|}\cos\frac{\alpha}{2}+i(-uX\pm\sqrt{|\Delta|}\sin\frac{\alpha}{2})). \quad (7.31)$$

If B is large enough depending on u and c, then $0\le 1-(B-\frac{1}{2}(u^2-c^2))X^2\le 1-\frac{B}{2}X^2\le 1$. On the other hand,

$$\sqrt{|\Delta|}\le\sqrt{\Omega}\,(1+C|X|).$$

Therefore,

$$\begin{aligned}|\mathrm{Re}((1+\Omega)\lambda)|&\le 1+\sqrt{\Omega}\,C|X|\,(1+C|X|),\\ |\mathrm{Im}((1+\Omega)\lambda)|&\le C|X|+\sqrt{\Omega}\,(1+C|X|).\end{aligned}$$

It follows that

$$\begin{aligned}|\lambda|^2&\le\frac{1}{(1+\Omega)^2}((1+\sqrt{\Omega}\,C|X|\,(1+C|X|))^2+(C|X|+\sqrt{\Omega}\,(1+C|X|))^2)\\ &\le F(\Omega,\Xi),\end{aligned}$$

where $\Xi = \max_{X\in\mathcal{U}}(C|X|)$ and

$$F(\Omega,\Xi) = \frac{1}{(1+\Omega)^2}((1+\sqrt{\Omega}\,\Xi\,(1+\Xi))^2 + (\Xi+\sqrt{\Omega}\,(1+\Xi))^2).$$

We note that F is C^∞ in the domain $(\Omega,\Xi) \in [\Omega_0,\infty) \times [0,\Xi_0]$ for any $\Omega_0 > 0$, $\Xi_0 > 0$. Furthermore, $F(\Omega,\Xi) \to 0$ as $\Omega \to \infty$ uniformly for $\Xi \in [0,\Xi_0]$. Therefore, there exists Ω^* such that $F(\Omega,\Xi) \leq \frac{1}{2}$ for all $(\Omega,\Xi) \in [\Omega^*,\infty) \times [0,\Xi_0]$. Then $F(\Omega,\Xi)$ is uniformly continuous on the bounded domain $[\Omega_0,\Omega^*] \times [0,\Xi_0]$. Now, $F(\Omega,0) = \frac{1}{1+\Omega} \leq \frac{1}{1+\Omega_0} < 1$. Let η be an arbitrary number such that $0 < \eta < \min(\frac{1}{2}, \frac{\Omega_0}{1+\Omega_0})$. Then, by uniform continuity, there exists a $\Xi^* \in [0,\Xi_0]$ such that $F(\Omega,\Xi) \leq 1-\eta$ for all $(\Omega,\Xi) \in [\Omega_0,\Omega^*] \times [0,\Xi^*]$ and consequently for all $(\Omega,\Xi) \in [\Omega_0,\infty) \times [0,\Xi^*]$. It follows that $|\lambda| < 1$ for all $\Omega > \Omega_0$ and all X in a neighbourhood $\mathcal{U}$ of 0. To obtain this result, we only had to suppose that $\Omega_0 > 0$ and that B is large enough depending on u and c;

Second case: Ω small. We now suppose that $\Omega < \Omega_0$. In view of (7.29), we can write:

$$\mathrm{Re}(\Delta) = -\Omega(1-BX^2)^2 - X^2(u^2 + (1-BX^2)(c^2-u^2)) + O(X^4),$$

where the $O(X^4) = \frac{1}{4}(u^2-c^2)^2X^4$ term is independent of Ω and B. Noting that we can take $\mathcal{U}$ (depending on B) such that for $X \in \mathcal{U}$, $\frac{1}{2} \leq 1-BX^2 \leq 1$, we easily show that

$$C_1 := \min(c^2, \frac{u^2+c^2}{2}) \leq u^2 + (1-BX^2)(c^2-u^2) \leq C_2 := \max(c^2, \frac{u^2+c^2}{2}).$$

We deduce that for X in such a neighbourhood, $\mathrm{Re}(\Delta) < 0$ and

$$C_1'(\Omega + X^2) \leq |\mathrm{Re}(\Delta)| \leq \Omega + C_2'X^2.$$

Similarly, with (7.30) we obtain:

$$|\mathrm{Im}(\Delta)| \leq C|X|(\Omega + X^2).$$

Therefore, α being defined as above, we get $|\tan\alpha| \leq C|X|$. Since $\mathrm{Re}(\Delta) < 0$, this means $|\pi - \alpha| \leq C|X|$ and $|\frac{\pi}{2} - \frac{\alpha}{2}| \leq C|X|$. Then,

$$|\cos\frac{\alpha}{2}| = |\sin(\frac{\pi}{2} - \frac{\alpha}{2})| \leq C|X|.$$

Choosing the neighbourhood $\mathcal{U}$ such that $C|X| \leq 1$ for $X \in \mathcal{U}$ in (7.28), we also deduce that

$$|\Delta| \leq (\Omega + CX^2).$$

Again, using that B is large enough such that $0 \leq 1 - (B - \frac{1}{2}(u^2-c^2))X^2 \leq 1 - \frac{B}{2}X^2$, and with the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ we get

$$|\mathrm{Re}((1+\Omega)\lambda)| \leq 1 - \frac{B}{2}X^2 + C|X|(\Omega + CX^2)^{1/2} \leq 1 - (\frac{B}{2} - C)X^2 + C|X|\sqrt{\Omega},$$
$$|\mathrm{Im}((1+\Omega)\lambda)| \leq C|X| + (\Omega + |X|^2)^{1/2} \leq C|X| + \sqrt{\Omega}.$$

We note that none of the constants in the formula above depend on B. Finally, we have $(1+\Omega)^{-2} = 1 - 2\Omega + O(\Omega^2)$. Therefore, for Ω_0 small enough and $\Omega \leq \Omega_0$, we have $(1+\Omega)^{-2} \leq 1 - \frac{3}{2}\Omega$. Then,

$$
\begin{aligned}
|\lambda|^2 &\leq \frac{1}{(1+\Omega)^2}((1-(\frac{B}{2}-C)X^2 + C|X|\sqrt{\Omega})^2 + (C|X| + \sqrt{\Omega})^2) \\
&\leq (1-\frac{3}{2}\Omega)(1-(B-C)X^2 + C|X|\sqrt{\Omega} + \Omega + O((|X|+\sqrt{\Omega})^4)) \\
&\leq 1-(B-C)X^2 + C|X|\sqrt{\Omega} - \frac{1}{2}\Omega + O((|X|+\sqrt{\Omega})^4),
\end{aligned}
$$

where the passage from the first to the second line is by expanding the squares and from the second to the third is by keeping only terms of order $O((|X|+\sqrt{\Omega})^2)$. Now, writing the last line as a sum of squares, we get:

$$
|\lambda|^2 \leq 1-(B-C)(|X| - \frac{C}{2(B-C)}\sqrt{\Omega})^2 + (\frac{C^2}{4(B-C)} - \frac{1}{2})\Omega + O((|X|+\sqrt{\Omega})^4),
$$

Now, for B large enough depending on C, we can make both $B-C > \frac{1}{4}$ and $\frac{C^2}{4(B-C)} - \frac{1}{2} < -\frac{1}{4}$. Therefore, for such B, we have

$$
|\lambda|^2 \leq 1 - \frac{1}{4}((|X| - \frac{C}{2(B-C)}\sqrt{\Omega})^2 + \Omega) + O((|X|+\sqrt{\Omega})^4),
$$

and for the pair $(|X|, \sqrt{\Omega})$ in a small neighbourhood of the origin, we deduce that $|\lambda|^2 < 1$, which ends the proof of the case Ω small.

So, we have shown that B can be chosen large enough (depending on u and c) such that there exists a neighbourhood $\mathcal{U}$ of $X = 0$ only depending on u and c and $|\lambda| < 1$ for all $X \in \mathcal{U}$ and all $\Omega \in \mathbb{R}$. Back to the original variables, this is the result stated in section 4.

7.6. Convergence of a space semi-discrete scheme for the Burgers-Poisson problem. We have the following truncation error

$$
\partial_t U_j + \frac{U_{j+1}+U_j+U_{j-1}}{3}\frac{U_{j+1}-U_{j-1}}{2h} = -\frac{\Phi_{j+1}-\Phi_{j-1}}{2h} + O(h^2)\,, \tag{7.32}
$$

$$
\varepsilon^2 \frac{\Phi_{j+1} - 2\Phi_j + \Phi_{j-1}}{h^2} = U_j, \tag{7.33}
$$

for all $\varepsilon \leq \varepsilon_0$ and where $O(h^2)$ depends only on u, ϕ and ε_0. Denoting by $v_j = U_j - u_j$, $\psi_j = \Phi_j - \phi_j$, we have

$$
\begin{aligned}
\partial_t v_j + \frac{U_{j+1}+U_j+U_{j-1}}{3}\frac{v_{j+1}-v_{j-1}}{2h} + \frac{v_{j+1}+v_j+v_{j-1}}{3}\frac{U_{j+1}-U_{j-1}}{2h} \\
-\frac{v_{j+1}+v_j+v_{j-1}}{3}\frac{v_{j+1}-v_{j-1}}{2h} = -\frac{\psi_{j+1}-\psi_{j-1}}{2h} + O(h^2)\,,
\end{aligned} \tag{7.34}
$$

$$
\varepsilon^2 \frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{h^2} = v_j. \tag{7.35}
$$

We deduce that

$$
\begin{aligned}
\partial_t \left(\frac{v_j^2}{2}\right) + \frac{\bar{g}_{j+1/2} - \bar{g}_{j-1/2}}{h} + \frac{U_{j+1}+U_j+U_{j-1}}{3} v_j \frac{v_{j+1}-v_{j-1}}{2h} \\
+ v_j \frac{v_{j+1}+v_j+v_{j-1}}{3}\frac{U_{j+1}-U_{j-1}}{2h} = O(h^2) v_j\,,
\end{aligned} \tag{7.36}
$$

where $\bar{g}_{j+1/2} = -\frac{1}{6}(v_j^2 v_{j+1} + v_j v_{j+1}^2) + \frac{\varepsilon^2}{2}\left(\frac{\psi_{j+1}-\psi_j}{h}\right)^2$. Using summation by parts, we have

$$\partial_t \left(\sum_j \frac{v_j^2}{2} \right) - \sum_j v_j v_{j+1} \frac{U_{j+2} - U_{j-1}}{6h} + \sum_j v_j \frac{v_{j+1} + v_j + v_{j-1}}{3} \frac{U_{j+1} - U_{j-1}}{2h} = \sum_j O(h^2) v_j \,. \tag{7.37}$$

Hence,

$$\partial_t \left(\sum_j \frac{v_j^2}{2} h \right) \le C \left(\sum_j \frac{v_j^2}{2} h \right) + O(h^4)\,. \tag{7.38}$$

Therefore we have:

$$\left(\sum_j \frac{v_j^2}{2} h \right)(t) \le C \left(\sum_j \frac{v_j^2}{2} h \right)(0) + O(h^4)\,. \tag{7.39}$$

It is easy to verify that

$$\left(\sum_j (U_j - u(x_j))^2 h \right)(t) \le O(h^4),$$

assuming that we also have a second order approximation of the initial data. Then we have the following error estimate.

$$\|u_h - u\| = \sqrt{\sum_j (u_j - u(x_j))^2 h} \le Ch^2,$$

for all $\varepsilon \le \varepsilon_0$ and where C depends only on u, ϕ and ε_0. We also have directly from (7.35) that

$$\varepsilon \|\phi_h - \phi\| = \varepsilon \sqrt{\sum_j (\phi_j - \phi(x_j))^2 h} \le Ch^2,$$

for all $\varepsilon \le \varepsilon_0$ and where C depends only on u, ϕ and ε_0. This proves the result.

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