Global branches of travelling-waves to a Gross-Pitaevskii-Schrödinger system in one dimension

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Abstract

We are interested in the existence of travelling-wave solutions to a system which modelizes the motion of an uncharged impurity in a Bose condensate. We prove that in space dimension one, there exist travelling-waves moving with velocity c if and only if c is less than the sound velocity at infinity. In this case we investigate the structure of the set of travelling-waves and we show that it contains global subcontinua in appropriate Sobolev spaces.

Key words. Gross-Pitaevskii system, travelling-waves, global bifurcation.
AMS subject classifications. 35Q55, 35Q51, 37K50, 35P05, 35J10

1 Introduction.

This paper is devoted to the study of a special kind of solutions of a system describing the motion of an uncharged impurity in a Bose condensate. In dimensionless variables, the system reads

$$(1.1)\qquad \begin{cases} 2i\dfrac{\partial \psi}{\partial t} = -\Delta\psi + \frac{1}{\varepsilon^2}(|\psi|^2 + \frac{1}{\varepsilon^2}|\varphi|^2 - 1)\psi \\ 2i\delta\dfrac{\partial \varphi}{\partial t} = -\Delta\varphi + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k^2)\varphi. \end{cases}$$

Here ψ and φ are the wavefunctions for bosons, respectively for the impurity, $\delta = \frac{\mu}{M}$ where μ is the mass of impurity and M is the boson mass (δ is supposed to be small), $q^2 = \frac{l}{2d}$, l being the boson-impurity scattering length and d the boson diameter, k is a dimensionless measure for the single-particle impurity energy and ε is a dimensionless constant ($\varepsilon = (\frac{a\mu}{lM})^{\frac{1}{5}}$, where a is the "healing length"; in applications, $\varepsilon \cong 0.2$). Assuming that we are in a frame in which the condensate is at rest at infinity, the solutions must satisfy the "boundary conditions"

$$(1.2)\qquad |\psi| \longrightarrow 1, \quad \varphi \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty.$$

This system (originally introduced by Clark and Gross) was studied by J. Grant and P. H. Roberts (see [5]). Using formal asymptotic expansions and numerical experiments, they computed the effective radius and the induced mass of the uncharged impurity.

We consider here the system (1.1) in a one dimensional space and we look for solitary waves, that is for solutions of the form

$$\psi(x,t) = \tilde{\psi}(x-ct), \quad \varphi(x,t) = \tilde{\varphi}(x-ct). \tag{1.3}$$

This kind of solutions corresponds to the case where the only disturbance present in the condensate is that caused by the uniform motion of the impurity with velocity c. In view of the boundary conditions, we seek for solutions of the form

$$\tilde{\psi}(x) = (1+\tilde{r}(x))e^{i\psi_0(x)}, \quad \tilde{\varphi}(x) = \tilde{u}(x)e^{i\varphi_0(x)} \tag{1.4}$$

with $\tilde{r}(x) \longrightarrow 0$, $\tilde{u}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. By an easy computation we find that the real functions ψ_0, φ_0, $\tilde{r}$, $\tilde{u}$ must satisfy

$$\psi_0' = c(1 - \frac{1}{(1+\tilde{r})^2}), \tag{1.5}$$

$$\varphi_0' = c\delta, \tag{1.6}$$

$$\tilde{r}'' = c^2\Big(\frac{1}{(1+\tilde{r})^3} - (1+\tilde{r})\Big) + \frac{1}{\varepsilon^2}\Big((1+\tilde{r})^3 - (1+\tilde{r}) + \frac{1}{\varepsilon^2}(1+\tilde{r})\tilde{u}^2\Big), \tag{1.7}$$

$$\tilde{u}'' = \Big(\frac{q^2}{\varepsilon^2}(1+\tilde{r})^2 - c^2\delta^2 - k^2\Big)\tilde{u}. \tag{1.8}$$

From (1.6) we see that necessarily $\varphi_0(x) = c\delta x + C$. Note that the system is invariant under the transform $(\psi,\varphi) \longmapsto (e^{i\alpha}\psi, e^{i\beta}\varphi)$, so the integration constants in (1.5) and (1.6) are not important. Thus all we have to do is to solve the system (1.7)-(1.8). Thereafter it will be easy to find the corresponding phases from (1.5)-(1.6) and (1.4) will give a solitary-wave solution of (1.1).

After the scale change $\tilde{u}(x) = \frac{1}{\varepsilon}u(\frac{x}{\varepsilon})$, $\tilde{r}(x) = r(\frac{x}{\varepsilon})$, we find that the functions r and u satisfy

$$r'' = (1+r)^3 - (1+r) - c^2\varepsilon^2\Big(1 + r - \frac{1}{(1+r)^3}\Big) + (1+r)u^2, \tag{1.9}$$

$$u'' = (q^2(1+r)^2 - \lambda)u, \tag{1.10}$$

where

$$\lambda = \varepsilon^2(c^2\delta^2 + k^2). \tag{1.11}$$

The equation $r'' = (1+r)^3 - (1+r) - \frac{v^2}{4}\Big(1 + r - \frac{1}{(1+r)^3}\Big) + (1+r)U$, where U is a positive Borel measure, was studied in [7]. In the case $U \equiv 0$, it has been shown that this equation has only the trivial solution $r \equiv 0$ if $|v| \geq \sqrt{2}$; for $0 < |v| < \sqrt{2}$, it also admits the solution

$$r_v(x) = -1 + \sqrt{\frac{v^2}{2} + (1 - \frac{v^2}{2}) \tanh^2(\frac{\sqrt{2 - v^2}}{2} x)}. \tag{1.12}$$

Moreover, any other nontrivial solution is of the form $r_v(\cdot - x_0)$ for some $x_0 \in \mathbf{R}$. Equation (1.10) is linear in u ; more precisely, u must be an eigenvector of the linear operator $-\frac{d^2}{dx^2} + q^2(1+r)^2$ corresponding to the eigenvalue $\lambda = \varepsilon^2(c^2\delta^2 + k^2)$.

It is now clear that except for translations, the only solutions of (1.9)-(1.10) of the form $(r, 0)$ are $(0, 0)$ and $(r_{2c\varepsilon}, 0)$ (the latter one exists only for $|c\varepsilon| < \frac{1}{\sqrt{2}}$). We call these solutions the *trivial solutions* of (1.9)-(1.10). We will prove that there exist non-trivial solutions of (1.9)-(1.10) in a neighbourhood of $(r_{2c\varepsilon}, 0)$ (for suitable values of the parameter λ) and we will study the global structure of the set of non-trivial solutions.

It has been shown (see e.g. [7] and references therein) that using the Madelung's transform $\psi = \sqrt{\rho} e^{i\psi_0}$, the first equation in (1.1) can be put into a hydrodynamical form (i.e. it is equivalent to a system of Euler equations for a compressible inviscid fluid of density ρ and velocity $\nabla\psi_0$). In this context, $\frac{1}{\varepsilon\sqrt{2}}$ represents the sound velocity at infinity. It will be proved at the beginning of section 3 that (1.1) does not possess non-constant travelling-vaves moving with velocity $|c| \geq \frac{1}{\varepsilon\sqrt{2}}$. Hence we will assume throughout that $|c| < \frac{1}{\varepsilon\sqrt{2}}$.

Observe that the system (1.9)-(1.10) has a good variational formulation : its solutions are critical points of the "energy" functional. Indeed, since $1 + \tilde{r} = |\tilde{\psi}| \geq 0$, it is clear that we must have $\tilde{r} \geq -1$. Therefore we will seek for solutions r of (1.9) with $r > -1$. Let $V = \{r \in H^1(\mathbf{R}) \mid \inf_{x \in \mathbf{R}} r(x) > -1\}$. It is obvious that V is open in $H^1(\mathbf{R})$ because $H^1(\mathbf{R}) \subset C_b^0(\mathbf{R})$ by the Sobolev embedding. A pair $(r, u) \in V \times H^1(\mathbf{R})$ satisfy (1.9)-(1.10) if and only if (r,u) is a critical point of the C^∞ functional $E : V \times H^1(\mathbf{R}) \longrightarrow \mathbf{R}$,

$$\begin{aligned} E(r,u) &= \int_{\mathbf{R}} |r'|^2 dx + \frac{1}{2}\int_{\mathbf{R}} \Big((1+r)^2 - 1\Big)^2 \Big(1 - \frac{2c^2\varepsilon^2}{(1+r)^2}\Big) dx \\ &\quad + \int_{\mathbf{R}} u^2(1+r)^2 dx + \frac{1}{q^2}\int_{\mathbf{R}} |u'|^2 dx - \frac{\lambda}{q^2}\int_{\mathbf{R}} u^2 dx. \end{aligned} \tag{1.13}$$

However, $E(r, \cdot)$ is quadratic in u for any fixed r and it would be very difficult to find critical points of E by using a classical topological argument.

In this paper we use bifurcation theory to show the existence of nontrivial solitary waves for the system (1.1). Note that this system (or equivalently (1.9)-(1.10)) is invariant by translations. To avoid the degeneracy of the linearized system due to this invariace, we work on symmetric function spaces. Consequently, the travelling-waves that we obtain will also present a symmetry. To be more precise, we will use the spaces

$$\mathbf{H} = H^2_{rad}(\mathbf{R}) = \{u \in H^2(\mathbf{R}) \mid u(x) = u(-x), \ \forall x \in \mathbf{R}\} \quad \text{and}$$

$$\mathbf{L} = L^2_{rad}(\mathbf{R}) = \{u \in L^2(\mathbf{R}) \mid u(x) = u(-x), \ \ a.e. \ \ x \in \mathbf{R}\}.$$

Clearly $\mathbf{H} \cap V$ is an open set of $\mathbf{H}$. We define $S : (\mathbf{H} \cap V) \times \mathbf{H} \longrightarrow \mathbf{L}$, $T : \mathbf{R} \times \mathbf{H} \times \mathbf{H} \longrightarrow \mathbf{L}$,

$$S(r,u) = -r'' + (1+r)^3 - (1+r) - c^2\varepsilon^2\Big(1 + r - \frac{1}{(1+r)^3}\Big) + (1+r)u^2, \tag{1.14}$$

$$T(\lambda, r, u) = -u'' + (q^2(1+r)^2 - \lambda)u. \tag{1.15}$$

It is obvious that S and T are well defined and of class C^∞ (recall that $\mathbf{H} \subset C_b^1(\mathbf{R})$ and $\mathbf{H}$ is an algebra). Clearly r and u satisfy the system (1.9)-(1.10) if and only if $S(r, u) = 0$ and $T(\lambda, r, u) = 0$.

In the next section, we will study the structure of the set of nontrivial solutions in a neighbourhood of the trivial ones. It follows easily from the Implicit Function Theorem that there are no nontrivial solutions of (1.9)-(1.10) in a neighbourhood of $(\lambda, 0, 0)$ for $\lambda < q^2$ (see the proof of Theorem 3.8). It is well-known that we may have nontrivial solutions arbitrarily close to $(\lambda, r_{2c\varepsilon}, 0)$ if and only if the differential $d_{(r,u)}(S, T)(\lambda, r_{2c\varepsilon}, 0)$ is not invertible. For $\lambda < q^2$, we will see that $d_{(r,u)}(S, T)(\lambda, r_{2c\varepsilon}, 0)$ is not invertible if and only if λ is an eigenvalue of the particular Schrödinger operator given by (1.10). In this case we show that all the nontrivial solutions in a neighbourhood of $(\lambda, r_{2c\varepsilon}, 0)$ form a smooth curve in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$.

It is natural to ask how long such a branch of solutions exists. Recently, there were obtained general global bifurcation results for C^1 Fredholm mappings of index 0 which apply to a broad class of elliptic equations in $\mathbf{R}^N$ (see, e.g., [9], [10]). Using the ideas and techniques developed in [11] it can be proved that for any fixed $\lambda < q^2$, the mapping $(S, T(\lambda, \cdot, \cdot)) : (\mathbf{H} \cap V) \times \mathbf{H} \longrightarrow \mathbf{L} \times \mathbf{L}$ is Fredholm of index 0. By a general global bifurcation theorem (a variant of Theorem 6.1 in [9]) one can prove that either the branch of nontrivial solutions of (1.9)-(1.10) starting from a bifurcation point $(\lambda, r_{2c\varepsilon}, 0)$ is noncompact in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$ or it meets $[q^2, \infty) \times \mathbf{H} \times \mathbf{H}$ (note that $[q^2, \infty)$ is the essential spectrum of the linear Schrödinger operator appearing in (1.10)).

To obtain further information (such as unboundedness) about the branches of nontrivial solutions, a key ingredient would be the properness of the operator (S, T), at least on closed bounded sets. Unfortunately, it is easy to see that the operator (S, T) is not proper on closed bounded sets. Indeed, it suffices to take $r_n = r_{2c\varepsilon}(\cdot - n) + r_{2c\varepsilon}(\cdot + n)$ and to observe that $(S, T)(\lambda, r_n, 0) \longrightarrow (0, 0)$ as $n \longrightarrow \infty$, the sequence (r_n) is bounded in $\mathbf{H}$ but has no convergent subsequence.

In order to obtain a more precise description of the branches of nontrivial solutions and to avoid troubles due to the lack of properness, we choose a different approach : we reformulate the problem and we work on some weighted Sobolev space (which is a subspace of $\mathbf{H}$). In section 3, we use a variant of the Global Bifurcation Theorem of Rabinowitz ([12]) to obtain global branches of solutions of (1.9)-(1.10) in that space. Note that the use of a slowly increasing weight (for example, $(1 + x^2)^s$ for $s > 0$) is sufficient to eliminate the lack of properness and to obtain global branches of travelling-waves. It is worth to note that for $\lambda < q^2$, any nontrivial travelling-wave which is in $\mathbf{H}$ also belongs to the weighted space which is used (i.e., there is no loss of solutions). We show that there exists exactly one branch of nontrivial solutions bifurcating from the curve $(\lambda, r_{2c\varepsilon}, 0)$ if $q \leq \frac{1}{\sqrt{2 \ln 2}}$. The number of these branches is increasing with q and tends to infinity as $q \longrightarrow \infty$. We will prove that any of these branches is either unbounded (in the weighted space) or λ tends to q^2 along it. On the other hand, we prove that there are no nontrivial solutions of (1.9)-(1.10) for $\lambda > q^2$.

2 Local curves of solutions

In order to prove a local existence result of nontrivial solitary waves for the system (1.1), we have to study the properties of the linear operator $A = -\frac{d^2}{dx^2} + q^2(1 + r_{2c\varepsilon})^2$, which can be written as $A = -\frac{d^2}{dx^2} + q^2 r_{2c\varepsilon}(2 + r_{2c\varepsilon}) + q^2$. Since $-1 < r(x) < 0$ for any $x \in \mathbf{R}$, the function $r_{2c\varepsilon}(2 + r_{2c\varepsilon})$ is everywhere negative (and even). Actually, in a slightly more general framework, we will study the operator $L = -\frac{d^2}{dx^2} + V(x)$ for a negative potential V, the properties of A being then deduced from those of L by a shift. For any $\lambda \leq 0$, we also consider the Cauchy problem

$$\text{(2.1)} \qquad \begin{cases} -u''(x) + V(x)u(x) = \lambda u(x), \\ u(0) = 1, \quad u'(0) = 0. \end{cases}$$

If V is continuous and even (i.e., $V(x) = V(-x)$), it is clear that problem (2.1) has an unique *global* solution which is also even. We denote by u_λ this solution and by $n(\lambda)$ the number of zeroes of u_λ in $(0, \infty)$.

Proposition 2.1 *Let $V \in L^2 \cap L^\infty(\mathbf{R}^N)$, $V \not\equiv 0$ be continuous, less than or equal to zero, even, and satisfy $\lim\limits_{x \longrightarrow \pm\infty} V(x) = 0$. The operator $L = -\frac{d^2}{dx^2} + V(x) : \mathbf{H} \longrightarrow \mathbf{L}$ is self-adjoint and has the following properties :*

i) $\sigma_{ess}(L) = [0, \infty)$.

ii) L has at least one negative eigenvalue.

iii) Any eigenvalue of L is simple.

iv) For any $\lambda < 0$ and $\varepsilon > 0$, there exists $C > 0$ such that

$$\text{(2.2)} \qquad |u_\lambda^{(m)}(x)| \leq Ce^{\sqrt{-\lambda+\varepsilon}|x|}, \qquad m = 0, 1, 2.$$

If $\lambda < 0$ is an eigenvalue and $0 < \varepsilon < -\lambda$, there exist $C_1, C_2, M > 0$ such that

$$\text{(2.3)} \qquad C_1 e^{-\sqrt{-\lambda+\varepsilon}|x|} \leq |u_\lambda^{(m)}(x)| \leq C_2 e^{-\sqrt{-\lambda-\varepsilon}|x|} \qquad \text{on} \quad [M, \infty), \qquad m = 0, 1, 2.$$

v) For any $\lambda \leq 0$, the number of eigenvalues of L in $(-\infty, \lambda)$ is exactly $n(\lambda)$, the number of zeroes of u_λ in $(0, \infty)$.

vi) If $\int_0^\infty x|V(x)|dx < \infty$, then L has at most $1 + \int_0^\infty x|V(x)|dx$ negative eigenvalues.

Proof. i) The operator $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbf{R})$ (with domain $H^2(\mathbf{R})$) is self-adjoint, so it is easy to see that L is self-adjoint. Multiplication by V is a relatively compact perturbation of $-\Delta$ and it follows from a classical theorem of Weyl that $\sigma_{ess}(L) = \sigma_{ess}(-\Delta) = [0, \infty)$.

ii) It suffices to show that there exists $u \in \mathbf{H}$ such that $\langle Lu, u \rangle_{\mathbf{L}} < 0$ and it will follow from the Min-Max Principle (see [13], Theorem XIII.1, p.76) that L has negative eigenvalues. Consider an even function $u \in C_0^\infty$ such that $u \equiv 1$ on $[-1, 1]$ and u is non-increasing on $[0, \infty)$. Let $u_n(x) = u(\frac{x}{n})$. Then

$$\langle Lu_n, u_n \rangle_{\mathbf{L}} = \frac{1}{n} \int_{\mathbf{R}} |u'(x)|^2 dx + \int_{\mathbf{R}} |u(\frac{x}{n})|^2 V(x) dx \longrightarrow \int_{\mathbf{R}} V(x) dx < 0$$

as $n \longrightarrow \infty$, so $\langle Lu_n, u_n \rangle_{\mathbf{L}} < 0$ for n sufficiently large.

iii) Clearly, λ is an eigenvalue of L if and only if $u_\lambda \in \mathbf{H}$. If this is the case, it is obvious that $Ker(L-\lambda) = Span\{u_\lambda\}$. Since L is self-adjoint, we have $Ker(L-\lambda) \cap Im(L-\lambda) = \{0\}$, so for any $n \in \mathbf{N}^*$ we have $Ker(L-\lambda)^n = Ker(L-\lambda) = Span\{u_\lambda\}$, that is λ is a simple eigenvalue.

iv) By (2.1), u_λ and u'_λ cannot vanish simultaneously, so u_λ must change sign any time it vanishes and u_λ has only isolated zeroes. There exists $d > 0$ such that $V(x) - \lambda > -\frac{\lambda}{2} > 0$ on $[d, \infty)$ because $V(x) \longrightarrow 0$ as $x \longrightarrow \infty$. Two situations may occur :

1°. There exists $x_0 > d$ such that $u_\lambda(x_0)$ and $u'_\lambda(x_0)$ have the same sign, say, they are positive. Then $u''_\lambda = (V(x) - \lambda)u_\lambda$, so u''_λ will remain positive after x_0 as long as $u_\lambda > 0$, which implies that u'_λ is increasing, hence it remains positive as long as $u_\lambda > 0$. Consequently, u_λ is increasing after x_0 as long as it remains positive, which implies that u_λ is positive and increasing on $[x_0, \infty)$. Since $u'_\lambda(x) \geq u'_\lambda(x_0) > 0$ for any $x > x_0$, we have necessarily $\lim_{x\to\infty} u_\lambda(x) = \infty$. By (2.1) we find that $\lim_{x\to\infty} u''_\lambda(x) = \infty$, so we have also $\lim_{x\to\infty} u'_\lambda(x) = \infty$. Let $f(x) = (u'_\lambda(x))^2$ and $g(x) = u^2_\lambda(x)$. Clearly, $f(x) \longrightarrow \infty$, $g(x) \longrightarrow \infty$ as $x \longrightarrow \infty$ and

$$\frac{f'(x)}{g'(x)} = \frac{u''_\lambda(x)}{u_\lambda(x)} = V(x) - \lambda \longrightarrow -\lambda \qquad \text{as } x \longrightarrow \infty.$$

L'Hôspital's rule implies that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = -\lambda$, which gives $\lim_{x\to\infty} \frac{u'_\lambda(x)}{u_\lambda(x)} = \sqrt{-\lambda}$. Thus for any $\epsilon > 0$, there exists $x_\epsilon > 0$ such that

$$\sqrt{-\lambda-\epsilon} < \frac{u'_\lambda(x)}{u_\lambda(x)} < \sqrt{-\lambda+\epsilon} \qquad \text{on } [x_\epsilon, \infty). \tag{2.4}$$

Integrating (2.4) from x_ϵ to x we get for any $x > x_\epsilon$,

$$\sqrt{-\lambda-\epsilon}(x - x_\epsilon) < \ln u_\lambda(x) - \ln u_\lambda(x_\epsilon) < \sqrt{-\lambda+\epsilon}(x - x_\epsilon),$$

that is

$$u_\lambda(x_\epsilon)e^{\sqrt{-\lambda-\epsilon}(x-x_\epsilon)} < u_\lambda(x) < u_\lambda(x_\epsilon)e^{\sqrt{-\lambda+\epsilon}(x-x_\epsilon)} \qquad \text{for any } x > x_\epsilon. \tag{2.5}$$

Note that the above situation always occurs if u_λ has a zero in (d, ∞). Indeed, if $u_\lambda(x_0) = 0$, then necessarily $u_\lambda(x)$ and $u'_\lambda(x)$ have opposite signs for $x < x_0$ and x close to x_0 (because if u_λ and u'_λ have the same sign at some $x_1 \in (d, x_0)$, we have just seen that u_λ cannot vanish in after x_1). But u_λ changes sign at x_0 and $u'_\lambda(x_0) \neq 0$, hence u_λ and u'_λ have the same sign just after x_0.

2°. The functions u_λ and u'_λ have opposite sign in (d, ∞). Replacing u_λ by $-u_\lambda$ if necessary, we may suppose that $u_\lambda > 0$ and $u'_\lambda < 0$ in (d, ∞) (observe that u'_λ cannot vanish because it also changes sign at any zero and we would be in case 1°). So u_λ is decreasing and positive on (d, ∞). Let $l = \lim_{x\to\infty} u_\lambda(x)$. Clearly, $l \geq 0$. If $l > 0$, then $u''_\lambda(x) \longrightarrow -\lambda l > 0$ as $x \longrightarrow \infty$ by (2.1), which implies $u'_\lambda(x) \longrightarrow \infty$ as $x \longrightarrow \infty$, a contradiction. Thus necessarily $l = 0$. Also, u'_λ is increasing on (d, ∞) (because $u''_\lambda(x) = (V(x) - \lambda)u_\lambda(x) > 0$) and negative, so it also has a limit at infinity. Since u_λ converges (to zero) at infinity, we must have $\lim_{x\to\infty} u'_\lambda(x) = 0$. Now we may apply l'Hôspital's rule to get

$$\lim_{x\to\infty} \frac{(u'_\lambda(x))^2}{u^2_\lambda(x)} = \lim_{x\to\infty} \frac{u''_\lambda(x)}{u_\lambda(x)} = \lim_{x\to\infty} (V(x) - \lambda) = -\lambda.$$

Thus $\frac{u'_\lambda(x)}{u_\lambda(x)} \longrightarrow -\sqrt{-\lambda}$ as $x \longrightarrow \infty$ because u_λ and u'_λ have opposite sign at infinity. Given $\epsilon > 0$, there exists $M > d$ such that

$$-\sqrt{-\lambda+\epsilon} < \frac{u'_\lambda(x)}{u_\lambda(x)} < -\sqrt{-\lambda-\epsilon} \qquad \text{on } [M,\infty). \tag{2.6}$$

Integrating (2.6) on $[M,x]$ we obtain, as in case 1°,

$$u_\lambda(M)e^{-\sqrt{-\lambda+\epsilon}(x-M)} < u_\lambda(x) < u_\lambda(M)e^{-\sqrt{-\lambda-\epsilon}(x-M)} \qquad \text{for any } x > M. \tag{2.7}$$

Finally, (2.2) and (2.3) follow from (2.5), respectivey (2.7) and the fact that $\lim_{x\to\infty}\frac{u''_\lambda(x)}{u_\lambda(x)} = -\lambda$, $\lim_{x\to\infty}\frac{u'_\lambda(x)}{u_\lambda(x)} = \pm\sqrt{-\lambda}$. It is obvious that λ is an eigenvalue of L if and only if $u_\lambda \in \mathbf{H}$, i.e. if and only if we are in case 2°. Therefore assertion iv) is proved.

Note also that u_λ has only a finite number of zeroes. Indeed, it follows from the above arguments that u_λ has at most one zero in (d,∞) and we know that any zero is isolated, so there are only finitely many zeroes in $[0,d]$.

The proofs of v) and vi) are rather classical and are similar to the proofs of Theorems XIII.8 and XIII.9 p. 90-94 in [13]. The bound on the number of eigenvalues given by vi) is due to Bargmann (see [13] and references therein). □

Corollary 2.2 *The linear operator $A = -\frac{d^2}{dx^2} + q^2(1+r_{2c\varepsilon})^2$ (considered on $\mathbf{L}$ with domain $D(A) = \mathbf{H}$) is self-adjoint and has the following properties :*

i) $A \geq 2c^2\varepsilon^2q^2$ and $\sigma_{ess}(A) = [q^2,\infty)$.

ii) A has at least one eigenvalue in $[2c^2\varepsilon^2q^2, q^2)$.

iii) Any eigenvalue of A is simple. If $\mu < q^2$ is an eigenvalue and u_μ is a corresponding eigenvector, then for any $\epsilon > 0$, there exist $C_1, C_2, M > 0$ such that

$$C_1e^{-\sqrt{q^2-\mu+\epsilon}|x|} \leq |u_\mu^{(m)}(x)| \leq C_2e^{-\sqrt{q^2-\mu-\epsilon}|x|} \qquad \text{if } |x| \geq M, \quad m = 0,1,2. \tag{2.8}$$

iv) Let N_q be the number of eigenvalues of A in $[2c^2\varepsilon^2q^2, q^2)$. We have $N_q < 1+(2\ln 2)q^2$. In particular, if $q \leq \frac{1}{\sqrt{2\ln 2}}$, then A has exactly one eigenvalue less than q^2.

v) We have $N_q \longrightarrow \infty$ as $q \longrightarrow \infty$.

It can be proved that there exist $c_1, c_2, q_0 > 0$ such that $c_1q \leq N_q \leq c_2q$ for any $q \geq q_0$, but we will not make use of this result in what follows.

Proof. Recall that $r_{2c\varepsilon}$ is given by (1.12). We have $A = -\frac{d^2}{dx^2} + q^2V(x) + q^2$, where the function V given by $V(x) = (1+r_{2c\varepsilon}(x))^2 - 1 = (1-2c^2\varepsilon^2)\Big(-1+\tanh^2(\sqrt{\frac{1-2c^2\varepsilon^2}{2}}x)\Big)$ is even, negative, tends exponentially to zero as $x \longrightarrow \pm\infty$ and $\inf_{x\in\mathbf{R}} V(x) = 2c^2\varepsilon^2 - 1$. Obviously, μ is an eigenvalue of A if and only if $\mu - q^2$ is an eigenvalue of $-\frac{d^2}{dx^2} + q^2V(x)$, so i), ii) and iii) follow at once from Proposition 2.1. An easy computation gives

$$\int_0^\infty x|V(x)|dx = (1-2c^2\varepsilon^2)\int_0^\infty x\Big(1-\tanh^2(\sqrt{\frac{1-2c^2\varepsilon^2}{2}}x)\Big)dx$$
$$= 2\int_0^\infty y(1-\tanh^2 y)dy = 2\int_0^\infty y(\tanh y - 1)'dy = 2\ln 2.$$

Now iv) is a direct consequence of Proposition 2.1, vi).

v) Fix $n \in \mathbf{N}$, $n \geq 1$ and take n symmetric functions $\varphi_1, \ldots, \varphi_n \in C_0^\infty(\mathbf{R})$, $\varphi_i \not\equiv 0$, such that $supp(\varphi_i) \cap supp(\varphi_j) = \emptyset$ if $i \neq j$. Clearly,

$$\langle A\varphi_i, \varphi_i\rangle_{\mathbf{L}} - q^2\langle\varphi_i,\varphi_i\rangle_{\mathbf{L}} = \int_{\mathbf{R}} |\nabla\varphi_i|^2dx + q^2\int_{\mathbf{R}} V(x)|\varphi_i(x)|^2dx \longrightarrow -\infty \qquad \text{as } q \longrightarrow \infty$$

hence there exists $q_0 > 0$ such that for any $q \geq q_0$ and any $i = 1, \ldots, n$ we have $\langle A\varphi_i, \varphi_i \rangle_{\mathbf{L}} - q^2 \langle \varphi_i, \varphi_i \rangle_{\mathbf{L}} < 0$. Since the φ_i's have disjoint supports we get

$$\begin{aligned}&\langle A\Big(\sum_{i=1}^{n} \alpha_i \varphi_i\Big), \sum_{i=1}^{n} \alpha_i \varphi_i \rangle_{\mathbf{L}} - q^2 || \sum_{i=1}^{n} \alpha_i \varphi_i ||_{\mathbf{L}}^2 \\ &= \sum_{i=1}^{n} |\alpha_i|^2 \Big(\int_{\mathbf{R}} |\nabla \varphi_i|^2 dx + q^2 \int_{\mathbf{R}} V(x) |\varphi_i(x)|^2 dx \Big) < 0\end{aligned}$$

Therefore we have found an n-dimensional subspace of $\mathbf{H}$, $V_n = Span\{\varphi_1, \ldots, \varphi_n\}$ such that $\langle Au, u \rangle_{\mathbf{L}} - q^2 ||u||_{\mathbf{L}} < 0$ for any $u \in V_n$ and any $q \geq q_0$. By the Min-Max Principle (see, e.g., [13], Theorem XIII.1 p.76) it follows that for $q \geq q_0$, A has at least n eigenvalues less than q^2, that is $N_q \geq n$ if $q \geq q_0$. This proves v). $\square$

We have the following result concerning the existence of non-trivial solitary waves:

Theorem 2.3 *Let $\lambda_* < q^2$ be an eigenvalue of A and let u_* be a corresponding eigenvector. There exists $\eta > 0$ and C^∞ functions*

$$s \longmapsto (\lambda(s), r(s), u(s)) \in \mathbf{R} \times \mathbf{H} \times (u_*^\perp \cap \mathbf{H})$$

defined on $(-\eta, \eta)$ such that $\lambda(0) = \lambda_*$, $r(0) = 0$, $u(0) = 0$ and

$$S(r_{2c\varepsilon} + sr(s), s(u_* + u(s))) = 0, \quad T(\lambda(s), r_{2c\varepsilon} + sr(s), s(u_* + u(s))) = 0.$$

Moreover, there exists a neighbourhood U of $(\lambda_*, r_{2c\varepsilon}, 0)$ in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$ such that any solution of $S(r, u) = 0$, $T(\lambda, r, u) = 0$ in U is either of the form $(\lambda(s), r_{2c\varepsilon} + sr(s), s(u_* + u(s)))$ or of the form $(\lambda, r_{2c\varepsilon}, 0)$.

That is, $r = r_{2c\varepsilon} + sr(s)$, $u = s(u_* + u(s))$ are nontrivial solutions of (1.9)-(1.10) for $\lambda = \lambda(s)$.

Let $g_{2c\varepsilon} : (-1, \infty) \longrightarrow \mathbf{R}$, $g_{2c\varepsilon}(x) = (1+x)^3 - (1+x) - c^2\varepsilon^2\Big(1 + x - \frac{1}{(1+x)^3}\Big)$. Then $S(r, u)$ can be written as $S(r, u) = -r'' + g_{2c\varepsilon}(r) + (1 + r)u^2$. It is easily seen that $d_r S(r_{2c\varepsilon}, 0) = -\frac{d^2}{dx^2} + g'_{2c\varepsilon}(r_{2c\varepsilon})$.

For the proof of Theorem 2.3, we need the following lemmas :

Lemma 2.4 *The linear operator $J := -\frac{d^2}{dx^2} + g'_{2c\varepsilon}(r_{2c\varepsilon}) : \mathbf{H} \longrightarrow \mathbf{L}$ has the following properties :*

i) J is self-adjoint, invertible and has the essential spectrum $\sigma_{ess}(J) = [2 - 4c^2\varepsilon^2, \infty)$.

ii) J has exactly one negative eigenvalue and any eigenvalue of J is simple.

Proof. i) The linear operator $B = -\frac{d^2}{dx^2} + g'_{2c\varepsilon}(r_{2c\varepsilon})$ with domain $D(B) = H^2(\mathbf{R})$ is self-adjoint in $L^2(\mathbf{R})$. We claim that $Ker(B) = Span\{\frac{d}{dx} r_{2c\varepsilon}\}$. Indeed, we have

$$(2.9) \qquad \frac{d^2}{dx^2} r_{2c\varepsilon} = g_{2c\varepsilon}(r_{2c\varepsilon}).$$

Thus $r''_{2c\varepsilon} \in C^1(\mathbf{R})$. Differentiating (2.9) with respect to x we get $\frac{d}{dx} r_{2c\varepsilon} \in Ker(B)$.

Conversely, let $h \in Ker(B)$. Then $h'' = g'_{2c\varepsilon}(r_{2c\varepsilon})h$, so that

$$(h' r'_{2c\varepsilon})' = h'' r'_{2c\varepsilon} + h' r''_{2c\varepsilon} = h g'_{2c\varepsilon}(r_{2c\varepsilon}) r'_{2c\varepsilon} + h' g_{2c\varepsilon}(r_{2c\varepsilon}) = (h g_{2c\varepsilon}(r_{2c\varepsilon}))'.$$

Hence $h' r'_{2c\varepsilon} = h g_{2c\varepsilon}(r_{2c\varepsilon}) + C$ on $\mathbf{R}$. Taking the limits as $|x| \longrightarrow \infty$, we get $C = 0$, so $h' r'_{2c\varepsilon} = h g_{2c\varepsilon}(r_{2c\varepsilon}) = h r''_{2c\varepsilon}$. Since $r'_{2c\varepsilon} \neq 0$ on $(-\infty, 0)$ and on $(0, \infty)$, on each of these intervals we have $\left(\frac{h}{r'_{2c\varepsilon}}\right)' = \frac{h'r'_{2c\varepsilon}-hr''_{2c\varepsilon}}{(r'_{2c\varepsilon})^2} = 0$. Thus there exist constants C_1, C_2 such that $h(x) = C_1 r'_{2c\varepsilon}(x)$ on $(-\infty, 0)$ and $h(x) = C_2 r'_{2c\varepsilon}(x)$ on $(0, \infty)$. Consequently, $h'(x) = C_1 r''_{2c\varepsilon}(x) = C_1 g(r_{2c\varepsilon}(x))$ on $(-\infty, 0)$ and $h'(x) = C_2 r''_{2c\varepsilon}(x) = C_2 g_{2c\varepsilon}(r_{2c\varepsilon}(x))$ on $(0, \infty)$. But h' is continuous because $h \in H^2(\mathbf{R})$ and therefore $C_1 = C_2$, which proves our claim.

Since $r'_{2c\varepsilon} \notin \mathbf{H}$, it is clear that the restriction of B to $\mathbf{H}$ is one-to-one from $\mathbf{H}$ into $\mathbf{L}$. It remains to prove that $B\mathbf{H} = \mathbf{L}$. It is well-known that $Im(B) = Ker(B)^{\perp} = (r'_{2c\varepsilon})^{\perp}$ since B is self-adjoint. We have $\mathbf{L} \subset Im(B)$ because $r'_{2c\varepsilon}$ is an odd function. Let $f \in \mathbf{L}$. Clearly there exists $r \in H^2(\mathbf{R})$ such that $Br = f$. Let $\tilde{r}(x) = r(-x)$. It is easy to see that $B\tilde{r} = f$, hence there exists C such that $r - \tilde{r} = Cr'_{2c\varepsilon}$. Then $r - \frac{1}{2}Cr'_{2c\varepsilon} = \frac{1}{2}(r + \tilde{r}) \in \mathbf{H}$ and $B(r - \frac{1}{2}Cr'_{2c\varepsilon}) = f$.

Now it is clear that J, which is the restriction of B to $\mathbf{H}$, is self-adjoint in $\mathbf{L}$ and invertible. The function $g'_{2c\varepsilon}(r_{2c\varepsilon})$ tends (exponentially) to $g'_{2c\varepsilon}(0) = 2 - 4c^2\varepsilon^2$ as $x \longrightarrow \infty$. It follows from Weyl's theorem that $\sigma_{ess}(J) = \sigma_{ess}(B) = [2 - 4c^2\varepsilon^2, \infty)$. This completes the proof of i).

ii) It follows from Proposition 2.1 iii) and v) that any eigenvalue of J is simple and the number of negative eigenvalues of J is exactly the number of zeroes of u in $(0, \infty)$, where u is the solution of the Cauchy problem

$$\begin{cases} -u'' + g'_{2c\varepsilon}(r_{2c\varepsilon})u = 0 & \text{in } [0, \infty), \\ u(0) = 1, \quad u'(0) = 0. \end{cases} \tag{2.10}$$

We use the following simplified version of the well-known Sturm oscillation lemma (this is also a paticular case of Lemma 5 in [8]) :

Sturm oscillation lemma. *Let Y and Z be nontrivial solutions of the differential equation*

$$-\varphi'' + h(x)\varphi = 0$$

on some interval (μ, ν), where h is continuous on (μ, ν). If Y and Z are linearly independent and $Y(\mu) = Y(\nu) = 0$, then Z has at least one zero in (μ, ν).

From this lemma it follows at once that J has at most one negative eigenvalue. Indeed, suppose that J has at least two negative eigenvalues. Then the solution u of (2.10) has at least two zeroes in $(0, \infty)$, say, $x_1 < x_2$. But the function $r'_{2c\varepsilon}$ also satisfies the differential equation in (2.10) and obviously u and $r'_{2c\varepsilon}$ are linearly independent (because $r'_{2c\varepsilon}(0) = 0$). Using Sturm's oscillation lemma, we infer that $r'_{2c\varepsilon}$ must have a zero on (x_1, x_2), which is absurd because $r'_{2c\varepsilon}(x) > 0$ on $(0, \infty)$.

Now let us prove that J has (at least) one negative eigenvalue. We argue again by contradiction and we suppose that J has no negative eigenvalues. Then the solution u of (2.10) has no zeroes in $[0, \infty)$, consequently $u(x) > 0$ for any $x \in [0, \infty)$. Since $g'_{2c\varepsilon}(r_{2c\varepsilon}(x)) \longrightarrow 2 - 4c^2\varepsilon^2 > 0$ as $x \longrightarrow \infty$, repeating the argument used in the proof of Proposition 2.1 iv) we infer that either $u(x) \longrightarrow \infty$ or $u(x) \longrightarrow 0$ as $x \longrightarrow \infty$. In the latter case we have also

$$|u^{(m)}(x)| \le Ce^{-\sqrt{2-4c^2\varepsilon^2-\delta}|x|}, \qquad m = 0, 1, 2$$

for some constant $C > 0$, $\delta \in (0, 2 - 4c^2\varepsilon^2)$ and x sufficiently large. Consequently, $u \in \mathbf{H}$ and 0 is an eigenvalue of J. But this is excluded by i). Therefore we must have $u(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Since $u(0) = 1$, we have $u > 0$ in a neighbourhood of 0. Note that $g'_{2c\varepsilon}(r_{2c\varepsilon}(0)) = (5 + \frac{3}{2c^2\varepsilon^2})(c^2\varepsilon^2 - \frac{1}{2}) < 0$, hence $g'_{2c\varepsilon}(r_{2c\varepsilon}) < 0$ near 0. From (2.10) we get $u''(x) < 0$ for

$x > 0$ and x close to 0. We have $u'(0) = 0$, so there exists $\delta > 0$ such that $u'(x) < 0$ on $(0, \delta]$. We may choose δ so small that $u(\delta) > 0$ and $r''_{2c\varepsilon}(\delta) > 0$ (note that $r''_{2c\varepsilon}(0) = g_{2c\varepsilon}(r_{2c\varepsilon}(0)) = \frac{(1-2c^2\varepsilon^2)^2}{2\sqrt{2}c\varepsilon} > 0$). Let $\beta = \frac{u(\delta)}{r'_{2c\varepsilon}(\delta)} > 0$ and let $h(x) = \beta r'_{2c\varepsilon}(x) - u(x)$. Clearly, h is a solution of the differential equation in (2.10) and $h(\delta) = 0$, $h'(\delta) = \beta r''_{2c\varepsilon}(\delta) - u'(\delta) > 0$. Hence $h(x) > 0$ for $x > \delta$ and x close to δ. On the other hand, we have $\lim_{x \longrightarrow \infty} h(x) = -\infty$, so there exists $\eta > \delta$ such that $h(\eta) = 0$. Since both $r'_{2c\varepsilon}$ and h satisfy the differential equation in (2.10), by the Sturm oscillation lemma we infer that $r'_{2c\varepsilon}$ must have a zero in (δ, η), which is absurd. This finishes the proof of Lemma 2.4. □

Lemma 2.5 *We have:*
i) $Ker(T(\lambda_*, r_{2c\varepsilon}, \cdot)) = Span(u_*)$;
ii) $Im(T(\lambda_*, r_{2c\varepsilon}, \cdot)) = u_*^\perp \cap \mathbf{L}$.

The proof is obvious.

Proof of Theorem 2.3. Let $\tilde{V} = \{r \in \mathbf{H} \mid \sup_{x \in \mathbf{R}} |r(x)| < 1\}$ and $I = (-\sqrt{2}c\varepsilon, \sqrt{2}c\varepsilon)$. Clearly $\tilde{V}$ is open in $\mathbf{H}$. We define $F : I \times \mathbf{R} \times \tilde{V} \times (\mathbf{H} \cap u_*^\perp) \longrightarrow \mathbf{L} \times \mathbf{L}$ by

$$F(s, \lambda, r, u) = \begin{cases} \begin{pmatrix} \frac{1}{s} S(r_{2c\varepsilon} + sr, s(u_* + u)) \\ \\ \frac{1}{s} T(\lambda, r_{2c\varepsilon} + sr, s(u_* + u)) \end{pmatrix} & \text{if } s \neq 0, \\ \\ \begin{pmatrix} (d_r S(r_{2c\varepsilon}, 0).r \\ T(\lambda, r_{2c\varepsilon}, u_* + u) \end{pmatrix} & \text{if } s = 0. \end{cases}$$

It is easily seen that F is C^∞ because

$$\begin{aligned} F_1(s, \lambda, r, u) &= \tfrac{1}{s} \left(S(r_{2c\varepsilon} + sr, s(u_* + u)) - S(r_{2c\varepsilon}, 0) \right) \\ &= \tfrac{1}{s} \int_0^1 \frac{d}{dt} S(r_{2c\varepsilon} + tsr, ts(u_* + u)) dt \\ &= \tfrac{1}{s} \int_0^1 d_r S(r_{2c\varepsilon} + tsr, ts(u_* + u)).sr + d_u S(r_{2c\varepsilon} + tsr, ts(u_* + u)).s(u_* + u) dt \\ &= \int_0^1 d_r S(r_{2c\varepsilon} + tsr, ts(u_* + u)).r + d_u S(r_{2c\varepsilon} + tsr, ts(u_* + u)).(u_* + u) dt \end{aligned}$$

and $F_2(s, \lambda, r, u) = T(\lambda, r_{2c\varepsilon} + sr, u_* + u)$.

It is also clear that $F(0, \lambda_*, 0, 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and

$$d_{(\lambda, r, u)} F(0, \lambda_*, 0, 0)(\tilde{\lambda}, \tilde{r}, \tilde{u}) = \begin{pmatrix} 0 \\ -\tilde{\lambda} u_* \end{pmatrix} + \begin{pmatrix} d_r S(r_{2c\varepsilon}, 0).\tilde{r} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ T(\lambda_*, r_{2c\varepsilon}, \tilde{u}) \end{pmatrix}$$

In view of Lemmas 2.4 and 2.5, $d_{(\lambda, r, u)} F(0, \lambda_*, 0, 0)$ is invertible. By the Implicit Function Theorem, there exist $\eta > 0$ and C^∞ functions defined on $(-\eta, \eta)$,

$$s \longmapsto (\lambda(s), r(s), u(s)) \in \mathbf{R} \times \mathbf{H} \times (\mathbf{H} \cap u_*^\perp)$$

such that $\lambda(0) = \lambda_*$, $r(0) = 0$, $u(0) = 0$ and $F(s, \lambda(s), u(s), r(s)) = (0, 0)$. It is obvious that for $s \neq 0$, $(\lambda(s), (r_{2c\varepsilon} + sr(s), s(u_0 + u(s))))$ satisfy the system (1.9)-(1.10). Finally, the uniqueness part in Theorem 2.3 is proved exactly in the same way as in the Bifurcation from a Simple Eigenvalue Theorem. □

Remark 2.6 Let $\lambda(s)$, $r(s)$, $u(s)$ be given by Theorem 2.3. We have $\dot{\lambda}(0) = 0$, $\dot{u}(0) = 0$ and

$$\ddot{\lambda}(0) = -\frac{4q^2}{||u_*||^2_{\mathbf{L}}}\langle(1 + r_{2c\varepsilon})u_*^2, J^{-1}((1 + r_{2c\varepsilon})u_*^2)\rangle_{\mathbf{L}}, \tag{2.11}$$

where the dots denote derivatives with respect to s and J is the operator in Lemma 2.4.

To see this, we differentiate with respect to s the equation $T(\lambda(s), r_{2c\varepsilon} + sr(s), u_* + u(s)) = 0$ and then we take $s = 0$ to obtain

$$-\frac{d^2}{dx^2}\dot{u}(0) + [q^2(1 + r_{2c\varepsilon})^2 - \lambda_*]\dot{u}(0) - \dot{\lambda}(0)u_* = 0, \tag{2.12}$$

that is $(A - \lambda_*)\dot{u}(0) - \dot{\lambda}(0)u_* = 0$. But $Im(A - \lambda_*)$ and $Ker(A - \lambda_*) = Span\{u_*\}$ are orthogonal (because A is self-adjoint), so (2.12) implies that $\dot{\lambda}(0) = 0$ and $\dot{u}(0) = 0$.

We differentiate twice with respect to s the equation $T(\lambda(s), r_{2c\varepsilon} + sr(s), u_* + u(s)) = 0$, then we take $s = 0$ to get

$$(A - \lambda_*)\ddot{u}(0) + 4q^2(1 + r_{2c\varepsilon})\dot{r}(0)u_* - \ddot{\lambda}(0)u_* = 0. \tag{2.13}$$

Substracting the equation $-r''_{2c\varepsilon} + g_{2c\varepsilon}(r_{2c\varepsilon}) = 0$ from the equation $S(r_{2c\varepsilon} + sr(s), s(u_* + u(s))) = 0$ and then dividing by s we get

$$-\frac{d^2}{dx^2}r(s) + \int_0^1 g'_{2c\varepsilon}(r_{2c\varepsilon} + tsr(s))\, dt \cdot r(s) + s(1 + r_{2c\varepsilon} + sr(s))(u_* + u(s))^2 = 0. \tag{2.14}$$

We differentiate (2.14) with respect to s, then we take $s = 0$ to obtain

$$-\frac{d^2}{dx^2}\dot{r}(0) + g'_{2c\varepsilon}(r_{2c\varepsilon})\dot{r}(0) + (1 + r_{2c\varepsilon})u_*^2 = 0,$$

that is $J\dot{r}(0) + (1 + r_{2c\varepsilon})u_*^2 = 0$, which can still be written as

$$\dot{r}(0) = -J^{-1}((1 + r_{2c\varepsilon})u_*^2). \tag{2.15}$$

Taking the scalar product of (2.13) with u_* we find $\ddot{\lambda}(0)||u_*||^2_{\mathbf{L}} = 4q^2\langle 1 + r_{2c\varepsilon})u_*^2, \dot{r}(0)\rangle_{\mathbf{L}}$. We replace $\dot{r}(0)$ from (2.15) in the last equality to obtain (2.11).

3 Global branches of solutions

Our purpose is to obtain information about the global structure of the set of nontrivial solutions of (1.9)-(1.10). We give a nonexistence result first.

Proposition 3.1 *i) The system (1.9)-(1.10) does not admit solutions* $(\lambda, r, u) \in \mathbf{R} \times V \times H^1(\mathbf{R})$ *with* $(r, u) \neq (0, 0)$ *if* $c \geq \frac{1}{\varepsilon\sqrt{2}}$.

ii) Suppose that $c < \frac{1}{\varepsilon\sqrt{2}}$ *and let* $(\lambda, r, u) \in \mathbf{R} \times V \times H^1(\mathbf{R})$ *be a nontrivial solution of the system (1.9)-(1.10). Then* $2c^2\varepsilon^2q^2 < \lambda \leq q^2$ *and* $-1 + \sqrt{2}c\varepsilon < r(x) \leq 0$ *for any* $x \in \mathbf{R}$.

Proof. Let $(\lambda, r, u) \in \mathbf{R} \times V \times H^1(\mathbf{R})$ be a solution of (1.9)-(1.10). Since $H^1(\mathbf{R}) \subset C_b(\mathbf{R})$, the equations (1.9)-(1.10) imply that r'' and u'' are continuous, hence $r, u \in C^2(\mathbf{R})$.

If $u \equiv 0$ and $c \geq \frac{1}{\varepsilon\sqrt{2}}$, the only solution of (1.9) which tends to zero at $\pm\infty$ is $r \equiv 0$ (this was proved in [7], but can be easily deduced from the arguments below). From now on we suppose that $u \not\equiv 0$. Multiplying (1.10) by u and integrating we find

$$\int_{\mathbf{R}} |u'|^2 dx + q^2 \int_{\mathbf{R}} (1+r)^2 |u|^2 dx = \lambda \int_{\mathbf{R}} |u|^2 dx. \tag{3.1}$$

Since $u \not\equiv 0$, we have necessarily $\lambda > 0$. Let
$G_{2c\varepsilon}(s) = \displaystyle\int_0^s g_{2c\varepsilon}(\tau) d\tau = \frac{1}{4}((1+s)^2 - 1)^2 \Big(1 - \frac{2c^2\varepsilon^2}{(1+s)^2}\Big)$. Multiplying (1.9) by r' gives

$$-\frac{1}{2}[(r')^2]' + [G_{2c\varepsilon}(r)]' + \frac{1}{2}[(1+r)^2]' u^2 = 0, \tag{3.2}$$

and multiplying (1.10) by u' leads to

$$-\frac{1}{2}[(u')^2]' + \frac{1}{2} q^2 (1+r)^2 (u^2)' - \frac{\lambda}{2}(u^2)' = 0. \tag{3.3}$$

From (3.2) and (3.3) we get

$$-\frac{1}{2}[(r')^2]' - \frac{1}{2q^2}[(u')^2]' + [G_{2c\varepsilon}(r)]' + \frac{1}{2}[(1+r)^2 u^2]' - \frac{\lambda}{2q^2}(u^2)' = 0. \tag{3.4}$$

Integrating (3.4) from $-\infty$ to x and taking into account that $r(x) \longrightarrow 0$, $r'(x) \longrightarrow 0$, $u(x) \longrightarrow 0$ and $u'(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$ we obtain

$$|r'|^2(x) + \frac{1}{q^2}|u'|^2(x) + \Big(\frac{\lambda}{q^2} - (1+r(x))^2\Big) u^2(x) = 2G_{2c\varepsilon}(r(x)) \qquad \text{for any } x \in \mathbf{R}. \tag{3.5}$$

Suppose that there exists $x_0 \in \mathbf{R}$ such that $r(x_0) < \min(-1 + \frac{\sqrt{\lambda}}{q}, -1 + \sqrt{2}c\varepsilon)$. Then $\frac{\lambda}{q^2} - (1 + r(x_0))^2 > 0$ and the left hand side of (3.5) is positive at x_0 (because $u(x_0) = u'(x_0) = 0$ and (1.10) would imply $u \equiv 0$) while $G_{2c\varepsilon}(r(x_0)) < 0$, a contradiction. Thus $r(x) \geq \min(-1 + \frac{\sqrt{\lambda}}{q}, -1 + \sqrt{2}c\varepsilon)$ for any $x \in \mathbf{R}$.

Suppose that $\lambda \leq 2c^2\varepsilon^2 q^2$ (that is, $\frac{\sqrt{\lambda}}{q} \leq \sqrt{2}c\varepsilon$). Then we have $(1 + r(x))^2 \geq \frac{\lambda}{q^2}$ for any $x \in \mathbf{R}$ and (3.1) gives

$$\int_{\mathbf{R}} |u'|^2 dx + q^2 \int_{\mathbf{R}} \Big((1+r)^2 - \frac{\lambda}{q^2}\Big) u^2 dx = 0,$$

which implies $u \equiv 0$, again a contradiction. Therefore we have $\lambda > 2c^2\varepsilon^2 q^2$ and $r(x) \geq -1 + \sqrt{2}c\varepsilon$ for any $x \in \mathbf{R}$. This is impossible if $\sqrt{2}c\varepsilon > 1$ because $r(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$.

Hence we cannot have other solutions than $(\lambda, 0, 0)$ if $\sqrt{2}c\varepsilon > 1$. From now on we suppose that $\sqrt{2}c\varepsilon \leq 1$. In this case we have $r \leq 0$ on $\mathbf{R}$ by the Maximum Principle. Indeed, the function $g_{2c\varepsilon}$ is strictly increasing and positive on $(0, \infty)$. Suppose that r achieves a positive maximum at x_0. Then $r''(x_0) \leq 0$. On the other hand, from (1.9) we infer that $r''(x_0) = g_{2c\varepsilon}(r(x_0)) + (1 + r(x_0))u^2(x_0) > 0$, which is absurd.

If $\sqrt{2}c\varepsilon = 1$ we have seen that $0 \geq r(x) \geq -1 + \sqrt{2}c\varepsilon = 0$, hence $r \equiv 0$. Then (1.10) becomes $u'' = (q^2 - \lambda)u$; together with the boundary condition $u(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$, this gives $u \equiv 0$. Thus i) is proved.

From now on we suppose throughout that $2c^2\varepsilon^2 < 1$. Clearly, if $r(x_0) = -1 + \sqrt{2}c\varepsilon$ for some $x_0 \in \mathbf{R}$, then (3.5) would imply $u(x_0) = u'(x_0) = 0$ (because $\lambda > 2c^2\varepsilon^2 q^2$), hence $u \equiv 0$ by (1.10), which is impossible. Hence $0 \geq r(x) > -1 + \sqrt{2}c\varepsilon$ for any $x \in \mathbf{R}$.

It only remains to show that we cannot have nontrivial solutions with $\lambda > q^2$. Suppose that (λ, r, u) is such a solution. First, observe that r cannot vanish because (3.5) would give a contradiction. We prove that r decays sufficiently fast at infinity. Take $0 < \epsilon < \frac{\lambda}{q^2} - 1$.There exists $M_\epsilon > 0$ such that $(1 + r(x))^2 \leq 1 + \epsilon$ on $[M_\epsilon, \infty)$ (because $r(x) \longrightarrow 0$ as $x \longrightarrow \infty$). Using (3.5), we have on $[M_\epsilon, \infty)$

$$0 \leq \Big(\frac{\lambda}{q^2} - 1 - \epsilon\Big) u^2(x) \leq 2G_{2c\varepsilon}(r(x)),$$

hence $0 \leq \left(\frac{\lambda}{q^2} - 1 - \epsilon\right) \frac{u^2(x)}{|r(x)|} \leq 2\frac{|G_{2c\varepsilon}(r(x))|}{|r(x)|}$. Passing to the limit as $x \longrightarrow \infty$ we obtain $\lim\limits_{x\to\infty} \frac{u^2(x)}{r(x)} = 0$. Dividing (1.9) by r we get

$$\text{(3.6)} \qquad \frac{r''(x)}{r(x)} = \frac{g_{2c\varepsilon}(r(x))}{r(x)} + (1 + r(x))\frac{u^2(x)}{r(x)} \longrightarrow g'_{2c\varepsilon}(0) > 0 \qquad \text{as } x \longrightarrow \infty.$$

Since r'' must have at least one zero between two zeroes of r', (3.6) shows that r' has no zeroes in some neighbourhood of infinity. In that neighbourhood we have

$$\frac{(|r'(x)|^2)'}{(r^2(x))'} = \frac{r''(x)}{r(x)} \longrightarrow g'_{2c\varepsilon}(0) > 0 \qquad \text{as } x \longrightarrow \infty.$$

Since $r(x) \longrightarrow 0$ and $r'(x) \longrightarrow 0$ at infinity, we may apply l'Hôspital's rule to get $\lim\limits_{x\to\infty} \left(\frac{r'(x)}{r(x)}\right)^2 = g'_{2c\varepsilon}(0)$. We know that r and r' have constant sign in a neighbourhood of infinity and they cannot have the same sign because r tends to 0 at infinity, so necessarily $\lim\limits_{x\to\infty} \frac{r'(x)}{r(x)} = -\sqrt{g'_{2c\varepsilon}(0)}$. The argument already used in the proof of Proposition 2.1 shows that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|r(x)| \leq C_\epsilon e^{-\sqrt{g'_{2c\varepsilon}(0) - \epsilon}\, x} \qquad \text{for any } x \in [0, \infty).$$

Of course that a similar estimate is valid on $(-\infty, 0]$. In particular, $r^2 + 2r$ is a continuous, bounded function on $\mathbf{R}$ and $\lim\limits_{x\to\pm\infty} |x|(r^2(x) + 2r(x)) = 0$. Moreover, multiplication by $r^2 + 2r$ is a bounded aperator on $L^2(\mathbf{R})$, hence it is also bounded with respect to $-\frac{d^2}{dx^2}$ with relative bound zero. Consequently, by the Kato-Agmon-Simon Theorem (see, e.g., [13], Theorem XIII.58 p. 226), the operator $-\frac{d^2}{dx^2} + q^2(r^2 + 2r)$ (with domain $H^2(\mathbf{R})$ and range $L^2(\mathbf{R})$) cannot have eigenvalues embedded in the continuous spectrum $(0, \infty)$. This means exactly that the operator $-\frac{d^2}{dx^2} + q^2(1 + r)^2$ has no eigenvalues in (q^2, ∞) and contradicts the existence of a non-tivial solution (λ, r, u) with $\lambda > q^2$. □

We will use the following variant of the Global Bifurcation Theorem of Rabinowitz :

Proposition 3.2 *Let E be a real Banach space and $\Omega \subset \mathbf{R} \times E$ an open set. Suppose that $G : \Omega \longrightarrow E$ is compact on closed, bounded subsets $\omega \subset \Omega$ such that $dist(\omega, \partial\Omega) > 0$ and is of the form $G(a, u) = L(a, u) + H(a, u)$, where L and H satisfy the following assumptions :*

a) $L(a, \cdot)$ is linear, compact for any fixed a and $(a, u) \longmapsto L(a, u)$ is continuous and compact on closed, bounded subsets $\omega \subset \Omega$ such that $dist(\omega, \partial\Omega) > 0$.

b) For any closed, bounded subset $\omega \subset \Omega$ such that $dist(\omega, \partial\Omega) > 0$, there exists a function h_ω such that $h_\omega(s) \longrightarrow 0$ as $s \longrightarrow 0$ and

$$||H(a,u)|| \leq ||u||h_\omega(||u||) \qquad \text{for any } (a,u) \in \omega.$$

c) There exists a_0 and $\epsilon > 0$ such that
- $(a_0, 0) \in \Omega$,
- *for any $a \in [a_0 - \epsilon, a_0 + \epsilon] \setminus \{a_0\}$ we have $Ker(Id - L(a,\cdot)) = \{0\}$,*
- *if $a_1 \in [a_0 - \epsilon, a_0)$ and $a_2 \in (a_0, a_0 + \epsilon]$, then*

$$ind(Id - L(a_1,\cdot), 0) \neq ind(Id - L(a_2,\cdot), 0).$$

Let

$$\mathcal{S} = \{(a,u) \in \Omega \mid u \neq 0 \text{ and } u = G(a,u)\}$$

be the set of nontrivial solutions of the equation $u = G(a,u)$. Then $\mathcal{S} \cup \{(a_0, 0)\}$ possesses a maximal subcontinuum (i.e. a maximal closed connected subset) $\mathcal{C}_{a_0}$ which contains $(a_0, 0)$ and has at least one of the following properties :

i) $\mathcal{C}_{a_0}$ is unbounded ;

ii) $dist(\mathcal{C}_{a_0}, \partial\Omega) = 0$;

iii) $\mathcal{C}_{a_0}$ meets $(a_1, 0)$, where $a_1 \neq a_0$ and $Ker(Id - L(a_1,\cdot)) \neq \{0\}$.

From the first assertion in c) it follows that the index $ind(Id - L(a,\cdot), 0) = deg(Id - L(a,\cdot), B(0,\rho), 0)$ is well defined for any $a \in [a_0 - \epsilon, a_0 + \epsilon] \setminus \{a_0\}$. By a) and the homotopy invariance of the Leray-Schauder degree, it is a continuous function of a. So we have necessarily $Ker(Id - L(a_0,\cdot)) \neq \{0\}$ (since otherwise $ind(Id - L(a_0,\cdot), 0)$ would be defined and $ind(Id - L(a,\cdot)\ 0)$ would be constant for $a \in [a_0 - \epsilon, a_0 + \epsilon]$, contradicting the last assertion in c)).

The proof of Proposition 3.2 is similar to that of Theorem 1.3, p. 490 in [12] (see also Corollary 1.12 in [12]).

Next, we give a reformulation of problem (1.9)-(1.10) suitable for the use of Proposition 3.2.

Equation (1.9) can be written as $-r'' + g_{2c\varepsilon}(r) + (1+r)u^2 = 0$, where $g_{2c\varepsilon}(x) = (1+x)^3 - (1+x) - c^2\varepsilon^2\left(1 + x - \frac{1}{(1+x)^3}\right)$. We will seek for solutions of the form $r(x) = r_{2c\varepsilon}(x) + w(x)$. Taking into account that $r_{2c\varepsilon}$ satisfies $-r''_{2c\varepsilon} + g_{2c\varepsilon}(r_{2c\varepsilon}) = 0$, equation (1.9) becomes

$$-w'' + g_{2c\varepsilon}(r_{2c\varepsilon} + w) - g_{2c\varepsilon}(r_{2c\varepsilon}) + (1 + r_{2c\varepsilon} + w)u^2 = 0. \tag{3.7}$$

Note that $g'_{2c\varepsilon}(0) = 2 - 4c^2\varepsilon^2 > 0$, thus the linear operator $-\frac{d^2}{dx^2} + g'_{2c\varepsilon}(0)$ (with domain $\mathbf{H}$ and range $\mathbf{L}$) is invertible, so equation (3.7) is equivalent to

(3.8)

$$w = -\Big(-\frac{d^2}{dx^2} + g'_{2c\varepsilon}(0)\Big)^{-1}[g_{2c\varepsilon}(r_{2c\varepsilon} + w) - g_{2c\varepsilon}(r_{2c\varepsilon}) - g'_{2c\varepsilon}(r_{2c\varepsilon})w + (1 + r_{2c\varepsilon} + w)u^2]$$

$$-\Big(-\frac{d^2}{dx^2} + g'_{2c\varepsilon}(0)\Big)^{-1}[(g'_{2c\varepsilon}(r_{2c\varepsilon}) - g'_{2c\varepsilon}(0))w].$$

In the same way, equation (1.10) can be written as

$$-u'' + (q^2 - \lambda)u = q^2(1 - (1 + r_{2c\varepsilon} + w)^2)u.$$

For $\lambda < q^2$, the linear operator $-\frac{d^2}{dx^2} + q^2 - \lambda$ is invertible and (1.10) becomes

$$u = -q^2\Big(-\frac{d^2}{dx^2} + q^2 - \lambda\Big)^{-1}[(r^2_{2c\varepsilon} + 2r_{2c\varepsilon})u] - q^2\Big(-\frac{d^2}{dx^2} + q^2 - \lambda\Big)^{-1}[(w^2 + 2wr_{2c\varepsilon} + 2w)u]. \tag{3.9}$$

We denote

$$H_1(w,u) = \Big(-\tfrac{d^2}{dx^2} + g'_{2c\varepsilon}(0)\Big)^{-1}[g_{2c\varepsilon}(r_{2c\varepsilon}+w) - g_{2c\varepsilon}(r_{2c\varepsilon}) - g'_{2c\varepsilon}(r_{2c\varepsilon})w + (1 + r_{2c\varepsilon}+w)u^2],$$

$$H_2(\lambda,w,u) = q^2\Big(-\tfrac{d^2}{dx^2} + q^2 - \lambda\Big)^{-1}[(w^2 + 2wr_{2c\varepsilon} + 2w)u],$$

$$A_\lambda(u) = A(\lambda,u) = q^2\Big(-\tfrac{d^2}{dx^2} + q^2 - \lambda\Big)^{-1}[(r_{2c\varepsilon}^2 + 2r_{2c\varepsilon})u],$$

$$B(w) = \Big(-\tfrac{d^2}{dx^2} + g'_{2c\varepsilon}(0)\Big)^{-1}[(g'_{2c\varepsilon}(r_{2c\varepsilon}) - g'_{2c\varepsilon}(0))w].$$

It is easy to see that $A_\lambda,\ B : \mathbf{L} \longrightarrow \mathbf{H}$ are linear and continuous. Denote $V_{2c\varepsilon} = \{r \in \mathbf{H} \mid r + r_{2c\varepsilon} \in V\}$. It is obvious that $V_{2c\varepsilon}$ is open in $\mathbf{H}$. Since $\mathbf{H} \subset C_b^1(\mathbf{R})$ and $\mathbf{H}$ is an algebra, H_1 and H_2 are well-defined and continuous from $V_{2c\varepsilon} \times \mathbf{H}$ and $(-\infty, q^2) \times \mathbf{H} \times \mathbf{H}$, respectively, to $\mathbf{H}$.

If $\lambda < q^2$, then (λ, r, u) satisfies the system (1.9)-(1.10) if and only if (λ, w, u) (where $w = r - r_{2c\varepsilon}$) satisfies the system (3.8)-(3.9) which is equivalent to

$$\begin{pmatrix} w \\ u \end{pmatrix} = -\begin{pmatrix} B & 0 \\ 0 & A_\lambda \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} - \begin{pmatrix} H_1(w,u) \\ H_2(\lambda,w,u) \end{pmatrix}. \tag{3.10}$$

We have already shown in Introduction that we cannot expect to have properness for problem (1.9)-(1.10). The counterexample that we have seen is essentially due to the invariance by translations of the system and to the fact that we have localized solutions. Of course that passing from (1.9)-(1.10) to (3.10) should not prevent the same problems to appear. To overcome this difficulty, we shall work on some weighted Sobolev space. As a "weight", we take a function $W : \mathbf{R} \longrightarrow \mathbf{R}$ which satisfies the following properties :

(W1) W is continuous and even, i.e. $W(x) = W(-x)$;

(W2) $W \geq 1$ and $\lim\limits_{x\to\infty} W(x) = \infty$;

(W3) There exists $C_W > 0$ such that $W(a+b) \leq C_W(W(a) + W(b))$.

It follows easily from (W1) and (W3) that there exist $K, s > 0$ such that $W(x) \leq K|x|^s$ for $|x| \geq 1$. Indeed, from (W3) we infer that $\forall a \in \mathbf{R}$, $W(2^n a) \leq (2C_W)^n W(a)$. If $x \in [2^{n-1}, 2^n]$ and $M = \max\limits_{x\in[0,1]} W(x)$, then

$$W(x) \leq (2C_W)^n W(\frac{x}{2^n}) \leq 2C_W M (2C_W)^{n-1} = 2C_W M 2^{(n-1)(1+\log_2 C_W)} \leq 2C_W M x^{1+\log_2 C_W}.$$

In particular, we get

(W4) $\forall a > 0, \quad e^{-a|\cdot|}W(\cdot) \in L^1 \cap L^\infty(\mathbf{R}).$

For a function W satisfying (W1)-(W3) we consider the spaces

$$\mathbf{L}_W = \{\varphi \in \mathbf{L} \mid W\varphi \in \mathbf{L}\},$$

$$\mathbf{H}_W = \{\varphi \in \mathbf{H} \mid W\varphi, W\varphi', W\varphi'' \in \mathbf{L}\},$$

endowed with the norms $||\varphi||_{\mathbf{L}_W} = ||W\varphi||_{L^2}$, respectively $||\varphi||^2_{\mathbf{H}_W} = ||W\varphi||^2_{L^2} + ||W\varphi'||^2_{L^2} + ||W\varphi''||^2_{L^2}$. Equiped with these norms, $\mathbf{L}_W$ and $\mathbf{H}_W$ are Hilbert spaces. It is clear that $||\varphi||_{L^2} \leq ||\varphi||_{\mathbf{L}_W}$, $||\varphi||_{H^2} \leq ||\varphi||_{\mathbf{H}_W}$ and $\mathbf{L}_W$ (respectively $\mathbf{H}_W$) is a dense subspace of $\mathbf{L}$ (respectively of $\mathbf{H}$).

Lemma 3.3 *The embedding $\mathbf{H}_W \subset C^1_b(\mathbf{R})$ is compact.*

Proof. It is clear that the embeddings $\mathbf{H}_W \subset H^2(\mathbf{R}) \subset C^1_b(\mathbf{R})$ are continuous. To prove compactness, consider an arbitrary sequence $u_n \rightharpoonup 0$ in $\mathbf{H}_W$ and let us show that $u_n \longrightarrow 0$ in $C^1_b(\mathbf{R})$. Fix $\epsilon > 0$. Let $K = \sup_n ||u_n||_{\mathbf{H}_W}$. There exists $M > 0$ such that $W(x) \geq \frac{K}{\epsilon}$ if $|x| \geq M$. It follows that $||u_n||_{H^2((-\infty,M)\cup(M,\infty))} \leq \epsilon$. By the Sobolev embedding theorem, we have $||u_n||_{L^\infty((-\infty,M]\cup[M,\infty))} + ||u_n'||_{L^\infty((-\infty,M]\cup[M,\infty))} \leq C_S\epsilon$. On the other hand $u_{n|[-M,M]} \rightharpoonup 0$ in $H^2(-M,M)$. Since the embedding $H^2(-M,M) \subset C^1([-M,M])$ is compact, it follows that $u_n \longrightarrow 0$ in $C^1([-M,M])$, so $||u_n||_{L^\infty([-M,M])} + ||u_n'||_{L^\infty([-M,M])} \leq \epsilon$ if n is sufficiently big. Thus $||u_n||_{L^\infty(\mathbf{R})} + ||u_n'||_{L^\infty(\mathbf{R})} \leq (C_S+1)\epsilon$ for n sufficiently big. As ϵ was arbitrary, we infer that $u_n \longrightarrow 0$ in $C^1_b(\mathbf{R})$ and the lemma is proved. □

Lemma 3.4 *Let W satisfy (W1)-(W3). For any $a > 0$, the operator $-\frac{d^2}{dx^2}+a : \mathbf{H}_W \longrightarrow \mathbf{L}_W$ is bounded and invertible. Moreover, the norm of $(-\frac{d^2}{dx^2}+a)^{-1}$ is uniformly bounded in $\mathcal{L}(\mathbf{L}_W, \mathbf{H}_W)$ when a remains in a compact subinterval of $(0,\infty)$.*

Proof. It is clear that

$$||(-\frac{d^2}{dx^2}+a)v||_{\mathbf{L}_W} = ||-v''+av||_{\mathbf{L}_W} \leq C||v||_{\mathbf{H}_W},$$

so the operator is bounded. Since $-\frac{d^2}{dx^2}+a : \mathbf{H} \longrightarrow \mathbf{L}$ is bounded and invertible, it is clear that the restriction of $-\frac{d^2}{dx^2}+a$ to $\mathbf{H}_W$ is one to one and for any $f \in \mathbf{L}_W \subset \mathbf{L}$ there exists an unique $v \in \mathbf{H}$ such that $(-\frac{d^2}{dx^2}+a)v = f$. It remains only to prove that $v \in \mathbf{H}_W$ and $||v||_{\mathbf{H}_W} \leq ||f||_{\mathbf{L}_W}$. Using the Fourier transform we get $(\xi^2+a)\widehat{v}(\xi) = \widehat{f}(\xi)$ or equivalently $\widehat{v}(\xi) = \frac{1}{\xi^2+a}\widehat{f}(\xi)$. Since $\mathcal{F}(e^{-\sqrt{a}|\cdot|})(\xi) = \frac{2\sqrt{a}}{\xi^2+a}$, we infer that

$$v = \frac{1}{2\sqrt{a}}(e^{-\sqrt{a}|\cdot|}) * f. \tag{3.11}$$

From (3.11) we get

$$\begin{aligned}
&|v(x)W(x)| = \tfrac{1}{2\sqrt{a}}W(x)\Big|\int_{\mathbf{R}} e^{-\sqrt{a}|x-y|}f(y)dy\Big| \\
&\leq \tfrac{C_W}{2\sqrt{a}}\int_{\mathbf{R}} W(x-y)e^{-\sqrt{a}|x-y|}|f(y)| + e^{-\sqrt{a}|x-y|}W(y)|f(y)|dy \\
&\leq C_1(a)[((We^{-\sqrt{a}|\cdot|}) * |f|)(x) + (e^{-\sqrt{a}|\cdot|}) * (|f|W)(x)],
\end{aligned}$$

that is $|vW| \leq C_1(a)[(We^{-\sqrt{a}|\cdot|}) * |f| + e^{-\sqrt{a}|\cdot|} * (|f|W)]$. But

$$||(We^{-\sqrt{a}|\cdot|}) * |f|\,||_{L^2} \leq ||We^{-\sqrt{a}|\cdot|}||_{L^1}||f||_{L^2} \leq ||We^{-\sqrt{a}|\cdot|}||_{L^1}||f||_{\mathbf{L}_W}$$

and

$$||e^{-\sqrt{a}|\cdot|} * (|f|W)||_{L^2} \leq ||e^{-\sqrt{a}|\cdot|}||_{L^1}||Wf||_{L^2}$$

so we obtain from (3.11) that

$$||v||_{\mathbf{L}_W} \leq C_2(a)||f||_{\mathbf{L}_W}, \tag{3.12}$$

where $C_2(a)$ remains bounded if $a \in [d,e]$, $0 < d < e < \infty$.

In the same way, we have $\widehat{v'}(\xi) = i\xi\widehat{v}(\xi) = \frac{i\xi}{\xi^2+a}\widehat{f}(\xi)$, hence $v'(x) = -\frac{1}{2}\zeta_a * f(x)$, where $\zeta_a(x) = \text{sgn}(x)e^{-\sqrt{a}|x|}$. Repeating the above argument we find

$$||v'W||_{L^2} \leq C_3(a)||f||_{\mathbf{L}_W}, \tag{3.13}$$

where $C_3(a)$ remains bounded if a is in a compact interval of $(0,\infty)$.

Finally, using the equation satisfied by v we get $v'' = -f + av$, hence

$$||v''W||_{L^2} \leq ||f||_{\mathbf{L}_W} + a||v||_{\mathbf{L}_W} \leq (1 + aC_2(a))||f||_{\mathbf{L}_W}. \tag{3.14}$$

Lemma 3.4 follows from (3.12), (3.13) and (3.14). □

Note that the operator $-\frac{d^2}{dx^2}+a : \mathbf{H}_W \longrightarrow \mathbf{L}_W$ is not invertible if the weight W increases too fast at infinity. Indeed, if $f \in C_0^\infty(\mathbf{R})$ and $f \geq 0$, it is easily seen (e.g., from (3.11)) that the solution v of $-v'' + av = f$ behaves like $e^{-\sqrt{a}|\cdot|}$ at $\pm\infty$. If we take $W(x) = e^{b|x|}$ and $a < b^2$, then v does not belong to $\mathbf{H}_W$, so $-\frac{d^2}{dx^2} + a : \mathbf{H}_W \longrightarrow \mathbf{L}_W$ is not surjective.

The next lemma shows that we do not loose solutions if we work in $\mathbf{H}_W$ instead of $\mathbf{H}$.

Lemma 3.5 *Let (λ, r, u) be a solution of (1.9)-(1.10) with $r \in \mathbf{H}$, $u \in \mathbf{H}$ and $\lambda < q^2$. Then r and u belong to $\mathbf{H}_W$.*

Proof. We have already seen in Proposition 3.1 that $-1 + \sqrt{2}c\varepsilon < r \leq 0$. Applying Proposition 2.1 iv) (see also Corollary 2.2, iii)) for $V(x) = q^2(r^2(x) + 2r(x))$, we infer that for any $\epsilon > 0$, u, u' and u'' decay at $\pm\infty$ faster than $e^{-\sqrt{q^2-\lambda-\epsilon}|x|}$, hence $u \in \mathbf{H}_W$.

Since $g'_{2c\varepsilon}(0) > 0$ and $r(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$, there exists $M > 0$ such that $r(x)g_{2c\varepsilon}(r(x)) \geq \frac{1}{2}g'_{2c\varepsilon}(0)r^2(x)$ if $|x| > M$.

Consider a symmetric function $\chi \in C_0^\infty(\mathbf{R})$ such that $\chi \equiv 1$ on $[-1,1]$, χ is non-increasing on $[0,\infty)$ and $supp(\chi) \subset [-2,2]$. We multiply (1.9) by $xr(x)\chi(\frac{x}{n})$ and integrate on $[0,\infty)$. Integrating by parts, we get :

$$\begin{aligned}&\int_0^\infty |r'|^2(x)x\chi(\tfrac{x}{n})dx - \tfrac{1}{2}r^2(0) - \tfrac{1}{2}\int_0^\infty r^2(x)\Big(\tfrac{2}{n}\chi'(\tfrac{x}{n}) + \tfrac{x}{n^2}\chi''(\tfrac{x}{n})\Big)dx \\ &+\int_0^M g_{2c\varepsilon}(r(x))r(x)x\chi(\tfrac{x}{n})dx + \int_M^\infty g_{2c\varepsilon}(r(x))r(x)x\chi(\tfrac{x}{n})dx \\ &+\int_0^\infty (1+r(x))u^2(x)r(x)x\chi(\tfrac{x}{n})dx = 0.\end{aligned} \tag{3.15}$$

By the Monotone Convergence Theorem, the first integral in (3.15) tends to $\int_0^\infty |r'(x)|^2 x dx$ as $n \longrightarrow \infty$, while the fourth integral tends to $\int_M^\infty g_{2c\varepsilon}(r(x))r(x)xdx$. The other three integrals converge as $n \longrightarrow \infty$ by Lebesgue's theorem on dominated convergence. Letting $n \longrightarrow \infty$ in (3.15) we obtain :

$$\begin{aligned}&\int_0^\infty |r'|^2(x)xdx - \tfrac{1}{2}r^2(0) + \int_0^M g_{2c\varepsilon}(r(x))r(x)xdx \\ &+\int_M^\infty g_{2c\varepsilon}(r(x))r(x)xdx + \int_0^\infty r(x)(1+r(x))xu^2(x)dx = 0.\end{aligned} \tag{3.16}$$

Since the second and the last integral in (3.16) are finite (because u decays exponentially at $\pm\infty$), we infer that $\int_0^\infty |r'|^2(x)xdx < \infty$ and $\int_M^\infty g_{2c\varepsilon}(r(x))r(x)xdx < \infty$. Consequently, $|x|^{\frac{1}{2}}r'(x)$ and $|x|^{\frac{1}{2}}r(x)$ belong to $L^2(\mathbf{R})$.

We have $g_{2c\varepsilon}(s) = g'_{2c\varepsilon}(0)s + h(s)s^2$, where h is continuous on $(-1, \infty)$, hence $h(r(x))$ is bounded. Equation (1.9) can be written as

$$-r'' + g'_{2c\varepsilon}(0)r = -(1+r)u^2 - h(r)r^2, \tag{3.17}$$

which gives, as in the proof of Lemma 3.4,

$$r = -\frac{1}{2\sqrt{g'_{2c\varepsilon}(0)}} e^{-\sqrt{g'_{2c\varepsilon}(0)}|\cdot|} * ((1+r)u^2 + h(r)r^2). \tag{3.18}$$

Suppose that $|x|^\alpha r(x) \in L^2(\mathbf{R})$ for some $\alpha > 0$. Since $|x|^\beta u(x) \in L^p(\mathbf{R})$ for any $\beta > 0$ and $1 \le p \le \infty$, we have :

$$\begin{aligned} |x|^{2\alpha}|r(x)| \ \le \ & C[(|\cdot|^{2\alpha} e^{-\sqrt{g'_{2c\varepsilon}(0)}|\cdot|}) * ((1+r)u^2 + h(r)r^2)(x) \\ & + e^{-\sqrt{g'_{2c\varepsilon}(0)}|\cdot|} * ((1+r)u^2|\cdot|^{2\alpha} + h(r)(|\cdot|^\alpha r)^2)](x) \end{aligned} \tag{3.19}$$

and we infer that $|\cdot|^{2\alpha} r \in L^p(\mathbf{R})$ for $1 \le p \le \infty$.

We have already proved that $|x|^{\frac{1}{2}} r(x) \in L^2(\mathbf{R})$, so it follows easily by induction that $|x|^\sigma r(x) \in L^p(\mathbf{R})$ for any $\sigma > 0$ and $1 \le p \le \infty$. Since $W(x) \le K|x|^s$ for some $K, s > 0$, we infer that $(1+r)u^2 + h(r)r^2 \in \mathbf{L}_W$. Now it follows form (3.17) and Lemma 3.4 that $r \in \mathbf{H}_W$ and Lemma 3.5 is proved. □

Now we turn our attention to the operators A, B, H_1 and H_2 appearing in (3.10).

Lemma 3.6 *We have :*

i) For any $\lambda \in (-\infty, q^2)$, $A_\lambda : \mathbf{H}_W \longrightarrow \mathbf{H}_W$ is linear, compact and the mapping $(\lambda, u) \longmapsto A_\lambda(u)$ is continuous from $(-\infty, q^2) \times \mathbf{H}_W$ to $\mathbf{H}_W$ and compact on closed bounded subsets of $[d, e] \times \mathbf{H}_W$ for $-\infty < d < e < q^2$.

ii) The linear operator $B : \mathbf{H}_W \longrightarrow \mathbf{H}_W$ is compact.

iii) $H_1 : ((V - r_{2c\varepsilon}) \cap \mathbf{H}_W) \times \mathbf{H}_W \longrightarrow \mathbf{H}_W$ is continuous, compact on closed bounded subsets ω_1 of $((V - r_{2c\varepsilon}) \cap \mathbf{H}_W) \times \mathbf{H}_W$ such that $dist(\omega_1, (\mathbf{H}_W \setminus (V - r_{2c\varepsilon})) \times \mathbf{H}_W) > 0$ and

$$||H_1(w,u)||_{\mathbf{H}_W} \le C_{\omega_1}(||w||^2_{\mathbf{H}_W} + ||u||^2_{\mathbf{H}_W}). \tag{3.20}$$

iv) $H_2 : (-\infty, q^2) \times \mathbf{H}_W \times \mathbf{H}_W \longrightarrow \mathbf{H}_W$ is continuous, compact on closed bounded subsets of $[d, e] \times \mathbf{H}_W \times \mathbf{H}_W$ for $-\infty < d < e < q^2$ and

$$||H_2(\lambda, w, u)||_{\mathbf{H}_W} \le C_{d,e}(||w||^2_{\mathbf{H}_W} + ||w||^4_{\mathbf{H}_W} + ||u||^2_{\mathbf{H}_W}) \qquad \text{for any } \lambda \in [d, e]. \tag{3.21}$$

Proof. It is easy to see that $u_n \rightharpoonup u_*$ in $\mathbf{H}_W$ and $v_n \rightharpoonup v_*$ in $\mathbf{H}_W$ imply that $u_n v_n \longrightarrow u_* v_*$ in $\mathbf{L}_W$. Indeed, (u_n) and (v_n) are bounded in $\mathbf{H}_W$ and by Lemma 3.3 we have

$$||u_n v_n - u_* v_*||_{\mathbf{L}_W} \le ||v_n - v_*||_{L^\infty} ||u_n||_{\mathbf{L}_W} + ||u_n - u_*||_{L^\infty} ||v_*||_{\mathbf{L}_W} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.22}$$

i) It is now clear that $u \longmapsto (r^2_{2c\varepsilon} + 2r^2_{2c\varepsilon})u$ is a linear compact mapping from $\mathbf{H}_W$ to $\mathbf{L}_W$ and we get i) by using Lemma 3.4 and the resolvent formula

$$\Big(-\frac{d^2}{dx^2} + q^2 - \lambda_1\Big)^{-1} - \Big(-\frac{d^2}{dx^2} + q^2 - \lambda_2\Big)^{-1} = (\lambda_1 - \lambda_2)\Big(-\frac{d^2}{dx^2} + q^2 - \lambda_1\Big)^{-1}\Big(-\frac{d^2}{dx^2} + q^2 - \lambda_2\Big)^{-1}.$$

ii) is obvious.

iii) Let ω_1 be as in Lemma 3.6. We claim that there exists $\eta > 0$ such that for any $(w, u) \in \omega_1$ we have $\inf_{x\in\mathbf{R}}(w(x) + r_{2c\varepsilon}(x)) \geq -1 + \eta$. We argue by contradiction and suppose that there exists a sequence $(w_n, u_n) \in \omega_1$ such that $a_n := \inf_{x\in\mathbf{R}}(w_n(x) + r_{2c\varepsilon}(x)) = (w_n + r_{2c\varepsilon})(x_n)$ tends to -1. The sequence (w_n) is bounded in $\mathbf{H}_W$, hence we may assume (passing to a subsequence if necessary) that $w_n \rightharpoonup w_*$ in $\mathbf{H}_W$. By Lemma 3.3, $w_n + r_{2c\varepsilon} \longrightarrow w_* + r_{2c\varepsilon}$ in $C_b^1(\mathbf{R})$. Since $w_*(x) + r_{2c\varepsilon}(x) \longrightarrow 0$ as $x \longrightarrow \infty$, the sequence (x_n) is bounded, say, $x_n \in [-M, M]$. Take $\chi \in C_0^\infty(\mathbf{R})$ such that $supp(\chi) \subset [-M-1, M+1]$ and $\chi \equiv 1$ on $[-M, M]$. Then $\inf_{x\in\mathbf{R}}(w_n(x)+r_{2c\varepsilon}(x)-(a_n+1)\chi(x)) = w_n(x_n)+r_{2c\varepsilon}(x_n)-(a_n+1)\chi(x_n) = -1$, so that $w_n + r_{2c\varepsilon} - (a_n+1)\chi \notin V$ and

$$dist(w_n, \mathbf{H}_W \setminus (V - r_{2c\varepsilon})) \leq dist(w_n, w_n - (a_n+1)\chi) = |1 + a_n| \, ||\chi||_{\mathbf{H}_W} \longrightarrow 0$$

as $n \longrightarrow \infty$, contradicting the fact that $(w_n, u_n) \in \omega_1$. This proves the claim.

For a given $w \in V - r_{2c\varepsilon}$, we have

$$\begin{aligned}(g_{2c\varepsilon}(r_{2c\varepsilon} + w) - g_{2c\varepsilon}(r_{2c\varepsilon}) - g'_{2c\varepsilon}(r_{2c\varepsilon})w)(x) &= \int_0^1 g'_{2c\varepsilon}(r_{2c\varepsilon} + tw)w(x)dt - g'_{2c\varepsilon}(r_{2c\varepsilon})w(x) \\ = w^2(x)\int_0^1\int_0^1 g''_{2c\varepsilon}(r_{2c\varepsilon} + tsw)(x)ds\, t\, dt &= w^2(x)h_1(w)(x),\end{aligned}$$

where $h_1(w)(x) = \int_0^1\int_0^1 g''_{2c\varepsilon}(r_{2c\varepsilon} + tsw)(x)ds\, t\, dt$.

To prove iii) it suffices to show that for any sequence $(w_n, u_n) \in \omega_1$ such that $w_n \rightharpoonup w_*$ and $u_n \rightharpoonup u_*$ in $\mathbf{H}_W$, we have $H_1(w_n, u_n) \longrightarrow H_1(w_*, u_*)$ in $\mathbf{H}_W$. In view of Lemma 3.4, it suffices to show that

$$h_1(w_n)w_n^2 + (1 + r_{2c\varepsilon} + w_n)u_n^2 \longrightarrow h_1(w_*)w_*^2 + (1 + r_{2c\varepsilon} + w_*)u_*^2 \qquad \text{in } \mathbf{L}_W. \tag{3.23}$$

The sequence (w_n) being bounded in $\mathbf{H}_W$, there exists $K > 0$ such that $-1+\min(\eta, \sqrt{2}c\varepsilon) \leq r_{2c\varepsilon}(x) + stw_n(x) \leq K$ for any $x \in \mathbf{R}$, $n \in \mathbf{N}$ and $s, t \in [0, 1]$. Since $g''_{2c\varepsilon}$ is uniformly continuous on $[-1 + \min(\eta, \sqrt{2}c\varepsilon), K]$, it is standard to prove that $h_1(w_n) \longrightarrow h_1(w_*)$ in $L^\infty(\mathbf{R})$ and then (3.23) follows from (3.22). Finally, using Lemma 3.4 we have for any $(w, u) \in \omega_1$

$$||H_1(w, u)||_{\mathbf{H}_W} \leq C||h_1(w)w^2 + (1 + r_{2c\varepsilon} + w)u^2||_{\mathbf{L}_W} \leq C_{\omega_1}(||w||^2_{\mathbf{H}_W} + ||u||^2_{\mathbf{H}_W}).$$

iv) From the preceeding arguments it is easy to see that the mapping $(w, u) \longmapsto (w^2 + 2wr_{2c\varepsilon} + 2w)u$ is continuous from $\mathbf{H}_W \times \mathbf{H}_W$ to $\mathbf{L}_W$ and the image of any bounded set in $\mathbf{H}_W \times \mathbf{H}_W$ is precompact in $\mathbf{L}_W$, so iv) follows from Lemma 3.4 and the resolvent formula above. The estimate (3.21) is straightforward. □

Lemma 3.7 *For any $\lambda < q^2$ we have :*

i) $Ker(Id_{\mathbf{H}_W} + A_\lambda) \neq \{0\}$ if and only if λ is an eigenvalue of the operator $A = -\frac{d^2}{dx^2} + q^2(1 + r_{2c\varepsilon})^2$. In this case we have $Ker(Id_{\mathbf{H}_W} + A_\lambda)^n = Span\{u_\lambda\}$ for any $n \in \mathbf{N}^*$.

ii) If λ is not an eigenvalue of A, then $ind(Id_{\mathbf{H}_W} + A_\lambda, 0) = (-1)^{n(\lambda)}$ (where $n(\lambda)$ is the number of eigenvalues of A less than λ).

Proof. i) It is easy to see that $u \in \mathbf{L}$ and $u + A_\lambda u = 0$ is equivalent to $u \in \mathbf{H}$ and $Au = \lambda u$. Recall that if $\lambda < q^2$ is an eigenvalue of A in $\mathbf{L}$, then the corresponding eigenvector u_λ is in $\mathbf{H}_W$ by Corollary 2.2 iii). Consequently, we have $Ker(Id_{\mathbf{H}_W} + A_\lambda) = Ker(Id_{\mathbf{L}} + A_\lambda) = Ker(\lambda Id_{\mathbf{H}} - A) = Span\{u_\lambda\}$.

To prove i), it suffices to show that $u_\lambda \notin Im(Id_{\mathbf{L}} + A_\lambda)$. Suppose by contradiction that there exists $v \in \mathbf{L}$ such that $v + A_\lambda v = u_\lambda$. This is equivalent to $v \in \mathbf{H}$ and $Av - \lambda v = -u_\lambda'' + (q^2 - \lambda)u_\lambda$, that is $-u_\lambda'' + (q^2-\lambda)u_\lambda \in Im(A-\lambda)$. Since $A - \lambda$ is self-adjoint on $\mathbf{L}$, $-u_\lambda'' + (q^2-\lambda)u_\lambda$ must be orthogonal (in $\mathbf{L}$) to $Ker(A-\lambda) = Span\{u_\lambda\}$, which gives $\int_{\mathbf{R}} |u_\lambda'|^2 dx + (q^2-\lambda)\int_{\mathbf{R}} |u_\lambda|^2 dx = 0$, a contradiction.

ii) A well-known result of Leray and Schauder asserts that if K is a compact operator on a real Banach space X and 1 is not an eigenvalue of K, then

$$ind(Id - K, 0) = (-1)^\beta,$$

where β is the sum of all the (algebraic) multiplicities of eigenvalues of K greater than 1. (see, e.g., [6], Theorem 4.6 p. 133).

Thus, for a given λ which is not an eigenvalue of A, we are interested by the eigenvalues $\mu > 1$ of $-A_\lambda$. Clearly, $-A_\lambda u = \mu u$ is equivalent to

$$q^2\Big(-\frac{d^2}{dx^2} + q^2 - \lambda\Big)^{-1}((r_{2c\varepsilon}^2 + 2r_{2c\varepsilon})u) + \mu u = 0,$$

that is

$$-u'' + q^2(1+r_{2c\varepsilon})^2 u + q^2\Big(1-\frac{1}{\mu}\Big)[1-(1+r_{2c\varepsilon})^2]u = \lambda u.$$

In other words, $\mu > 1$ is an eigenvalue of $-A_\lambda$ if and only if λ is an eigenvalue of the operator

$$M_\mu = -\frac{d^2}{dx^2} + q^2(1+r_{2c\varepsilon})^2 + q^2\Big(1-\frac{1}{\mu}\Big)[1-(1+r_{2c\varepsilon})^2] = A + q^2\Big(1-\frac{1}{\mu}\Big)[1-(1+r_{2c\varepsilon})^2].$$

Remark that $M_\mu \geq A$ for any $\mu \geq 1$ and $\sigma_{ess}(M_\mu) = [q^2, \infty)$ by Weyl's theorem. By Proposition 2.1 iv), $\lambda \in (-\infty, q^2)$ is an eigenvalue of M_μ considered as an operator on $\mathbf{L}_W$ if and only if λ is an eigenvalue of M_μ considered as an operator on $\mathbf{L}$. We will work on $\mathbf{L}$ because on this space M_μ is self-adjoint.

Given $\lambda < q^2$ not an eigenvalue of A, we will prove that there are exactly $n(\lambda)$ values $\mu \in (1, \infty)$ such that λ is an eigenvalue of M_μ.

For $\mu \in [1, \infty)$, we define

$$\alpha_n(\mu) = \sup_{\varphi_1,\dots,\varphi_{n-1}\in\mathbf{H}} \; \inf_{\psi\in\{\varphi_1,\dots,\varphi_{n-1}\}^\perp} \frac{\langle M_\mu\psi,\psi\rangle_{\mathbf{L}}}{||\psi||_{\mathbf{L}}^2}. \tag{3.24}$$

By the Min-Max Principle ([13], Theorem XIII.1 p. 76), either $\alpha_n(\mu)$ is the n^{th} eigenvalue of M_μ (counted with multiplicity) or $\alpha_n(\mu) = q^2$. By Proposition 2.1 iii), the eigenvalues of M_μ are simple, thus we have $\alpha_p(\mu) < \alpha_n(\mu)$ if $p < n$ and $\alpha_p(\mu) < q^2$.

It is obvious that the functions $\mu \longmapsto \alpha_n(\mu)$ are increasing on $[1,\infty)$ because $M_{\mu_1} \leq M_{\mu_2}$ if $1 \leq \mu_1 < \mu_2$. In fact, α_n is strictly increasing on $\{\mu \in [1,\infty) \mid \alpha_n(\mu) < q^2\}$. To see this, consider $\mu_1 < \mu_2$ such that $\alpha_n(\mu_2) < q^2$. Then $\alpha_1(\mu_2), \dots, \alpha_n(\mu_2)$ are eigenvalues of M_{μ_2}. Let $u_1, \dots, u_n \in \mathbf{H}$ be corresponding eigenvectors with $||u_i||_{\mathbf{L}} = 1$. Clearly, $u_1, \dots, u_n$ are mutually orthogonal in $\mathbf{L}$ and it is easily seen from the definition of M_μ that $\langle M_{\mu_1}u_i, u_i\rangle_{\mathbf{L}} < \langle M_{\mu_2}u_i, u_i\rangle_{\mathbf{L}} = \alpha_i(\mu_2)$, $i = 1, \dots, n$. Remark that the quantity $N(u) =$

$\left(\int_{\mathbf{R}}[1-(1+r_{2c\varepsilon})^2]|u|^2dx\right)^{\frac{1}{2}}$ is a norm on $\mathbf{L}$. Since $Span\{u_1,\dots,u_n\}$ is finite-dimensional, there exists $N_1>0$ such that $N(u)\geq N_1||u||_{\mathbf{L}}$ for any $u\in Span\{u_1,\dots,u_n\}$. Therefore

$$\begin{aligned}
&\langle M_{\mu_1}\Big(\sum_{i=1}^n a_iu_i\Big),\Big(\sum_{i=1}^n a_iu_i\Big)\rangle_{\mathbf{L}}\\
&=\langle M_{\mu_2}\Big(\sum_{i=1}^n a_iu_i\Big),\Big(\sum_{i=1}^n a_iu_i\Big)\rangle_{\mathbf{L}}-\langle (M_{\mu_2}-M_{\mu_1})\Big(\sum_{i=1}^n a_iu_i\Big),\Big(\sum_{i=1}^n a_iu_i\Big)\rangle_{\mathbf{L}}\\
&=\sum_{i=1}^n\alpha_i(\mu_2)|a_i|^2-q^2(\frac{1}{\mu_1}-\frac{1}{\mu_2})\int_{\mathbf{R}}[1-(1+r_{2c\varepsilon})^2]\Big|\sum_{i=1}^n a_iu_i\Big|^2dx\\
&\leq\alpha_n(\mu_2)||\sum_{i=1}^n a_iu_i||_{\mathbf{L}}^2-q^2(\frac{1}{\mu_1}-\frac{1}{\mu_2})N_1^2||\sum_{i=1}^n a_iu_i||_{\mathbf{L}}^2.
\end{aligned}\tag{3.25}$$

Thus for any u in the n-dimensional subspace $Span\{u_1,\dots,u_n\}$ we have

$$\langle M_{\mu_1}u,u\rangle_{\mathbf{L}}\leq\Big(\alpha_n(\mu_2)-q^2\Big(\frac{1}{\mu_1}-\frac{1}{\mu_2}\Big)N_1^2\Big)||u||_{\mathbf{L}}^2.$$

By the Min-Max Principle it follows that $\alpha_n(\mu_1)\leq\alpha_n(\mu_2)-q^2\Big(\frac{1}{\mu_1}-\frac{1}{\mu_2}\Big)N_1^2$.

A standard argument shows that each α_n is continuous. Indeed, suppose by contradiction that $\mu_*\in(1,\infty)$ is a discontinuity point. Then necessarily $l_1:=\sup\limits_{\mu<\mu_*}\alpha_n(\mu)<\inf\limits_{\mu>\mu_*}\alpha_n(\mu):=l_2$. Take $0<\epsilon<\frac{l_2-l_1}{4}$ and $\mu_1<\mu_*$, $\mu_2>\mu_*$ such that $q^2\Big(\frac{1}{\mu_1}-\frac{1}{\mu_2}\Big)<\epsilon$. Since $\alpha_n(\mu_2)>l_2-\epsilon$, there exist $\varphi_1,\dots,\varphi_{n-1}\in\mathbf{H}$ such that $\langle M_{\mu_2}\psi,\psi\rangle_{\mathbf{L}}>l_2-\epsilon$ for any $\psi\in\{\varphi_1,\dots,\varphi_{n-1}\}^{\perp}$ with $||\psi||_{\mathbf{L}}=1$. We have

$$\begin{aligned}
&\langle M_{\mu_2}\psi,\psi\rangle_{\mathbf{L}}-\langle M_{\mu_1}\psi,\psi\rangle_{\mathbf{L}}\\
&=q^2(\tfrac{1}{\mu_1}-\tfrac{1}{\mu_2})\int_{\mathbf{R}}[1-(1+r_{2c\varepsilon})^2]|\psi|^2dx\leq q^2(\frac{1}{\mu_1}-\frac{1}{\mu_2})||\psi||_{\mathbf{L}}^2<\epsilon,
\end{aligned}$$

thus $\langle M_{\mu_1}\psi,\psi\rangle_{\mathbf{L}}>l_2-2\epsilon$ for any $\psi\in\{\varphi_1,\dots,\varphi_{n-1}\}^{\perp}$ with $||\psi||_{\mathbf{L}}=1$. Therefore $\alpha_n(\mu_1)>l_2-2\epsilon$, which is a contradiction.

We have also for any $u\in\mathbf{H}$,

$$\langle M_\mu u,u\rangle_{\mathbf{L}}=||u'||_{L^2}^2+q^2||u||_{\mathbf{L}}^2-\frac{q^2}{\mu}\int_{\mathbf{R}}[1-(1+r_{2c\varepsilon})^2]|u|^2dx\geq q^2||u||_{\mathbf{L}}^2-\frac{C}{\mu}||u||_{\mathbf{L}}^2,$$

hence $\alpha_1(\mu)\geq q^2-\frac{C}{\mu}\longrightarrow q^2$ as $\mu\longrightarrow\infty$. Consequently, $\alpha_n(\mu)\longrightarrow q^2$ as $\mu\longrightarrow\infty$ for any $n\geq 1$.

Note that $\lambda<q^2$ is an eigenvalue of M_μ if and only if $\lambda=\alpha_n(\mu)$ for some $n\in\mathbf{N}^*$. We know that there are exactly $n(\lambda)$ eigenvalues of A less than λ, say, $\lambda_1<\lambda_2<\ldots<\lambda_{n(\lambda)}<\lambda$. We have $\alpha_i(1)=\lambda_i$ because $M_1=A$, the functions α_i are strictly increasing (until they reach the value q^2, if this happens), continuous and tend to q^2 at infinity. We infer that for each $i\in\{1,\dots,n(\lambda)\}$, there exists exactly one value μ_i such that $\alpha_i(\mu_i)=\lambda$. Moreover, $\mu_1>\mu_2>\ldots>\mu_{n(\lambda)}>1$. For any $n>n(\lambda)$, we have $\alpha_n(1)>\lambda$, hence $\alpha_n(\mu)>\lambda$ for $\mu\in(0,\infty)$ because α_n is increasing.

Thus we have shown that the operator $-A_\lambda$ has exactly $n(\lambda)$ eigenvalues greater than 1, $\mu_1>\mu_2>\ldots>\mu_{n(\lambda)}$. Moreover, $Ker(\mu_i+A_\lambda)=Ker(M_{\mu_i}-\lambda)$. We know by Proposition 2.1 iii) that $Ker(M_{\mu_i}-\lambda)$ is one dimensional. If this kernel is spanned by a function v_i, then $v_i\notin Im(\mu_i+A_\lambda)$. Indeed, $\mu_iu+A_\lambda u=v_i$ would imply $(M_{\mu_i}-\lambda)u=\frac{1}{\mu_i}(-v_i''+(q^2-\lambda)v_i)$.

Since M is self-adjoint, $-v_i'' + (q^2 - \lambda)v_i$ would be orthogonal to $Ker(M_{\mu_i} - \lambda) = Span\{v_i\}$, which gives a contradiction. Consequently, we have $Ker(\mu_i + A_\lambda)^n = Span\{v_i\}$ for any $n \in \mathbf{N}^*$, that is μ_i is a simple eigenvalue of $-A_\lambda$.

As a consequence, we have $ind(Id_{\mathbf{H}_W} + A_\lambda, 0) = (-1)^{n(\lambda)}$ and Lemma 3.7 is proved. □

We are now in position to state the main result of this paper.

Theorem 3.8 *Let $\mathcal{S}$ be the set of nontrivial solutions of the system (1.9)-(1.10) in $\mathbf{R} \times (\mathbf{H} \cap V) \times \mathbf{H}$. For any eigenvalue $\lambda_m < q^2$ of $A = -\frac{d^2}{dx^2} + (1 + r_{2c\varepsilon})^2$, the set $\mathcal{S} \cup \{(\lambda_m, r_{2c\varepsilon}, 0)\}$ contains a maximal closed connected subset $\mathcal{C}_m$ in $(-\infty, q^2) \times \mathbf{H}_W \times \mathbf{H}_W$ such that $\mathcal{C}_m \cap \mathcal{C}_p = \emptyset$ if $m \neq p$ and $\mathcal{C}_m$ satisfies at least one of the two properties :*

i) $\mathcal{C}_m$ is unbounded in $\mathbf{R} \times \mathbf{H}_W \times \mathbf{H}_W$ or

ii) there exists a sequence $(\lambda_n, r_n, u_n) \in \mathcal{C}_m$ such that $\lambda_n \longrightarrow q^2$ as $n \longrightarrow \infty$.

Proof.

We have already seen that $(\lambda, r, u) \in (-\infty, q^2) \times (\mathbf{H} \cap V) \times \mathbf{H}$ is a nontrivial solution of (1.9)-(1.10) if and only if $(\lambda, r - r_{2c\varepsilon}, u)$ belongs to $(-\infty, q^2) \times (\mathbf{H}_W \cap (V - r_{2c\varepsilon})) \times \mathbf{H}_W$ and satisfies the system (3.8)-(3.9) (or, equivalently, (3.10)).

Let $E = \mathbf{H}_W \times \mathbf{H}_W$, $\Omega = (-\infty, q^2) \times (\mathbf{H}_W \cap (V - r_{2c\varepsilon})) \times \mathbf{H}_W$, $L_\lambda = \begin{pmatrix} -B & 0 \\ 0 & -A_\lambda \end{pmatrix}$ and $H(\lambda, w, u) = \begin{pmatrix} -H_1(w, u) \\ -H_2(\lambda, w, u) \end{pmatrix}$. Let $G(\lambda, w, u) = L_\lambda(w, u) + H(\lambda, w, u)$. It is obvious that on Ω, (3.10) is equivalent to the equation $(w, u) = G(\lambda, w, u)$. It follows easily from Lemma 3.6 that L and H satisfy the assumptions a) and b) in Proposition 3.2.

We claim that $Id_{\mathbf{H}_W} + B : \mathbf{H}_W \longrightarrow \mathbf{H}_W$ is invertible. Indeed, $(Id_{\mathbf{H}_W} + B)u = v$ is equivalent to $-u'' + g'_{2c\varepsilon}(r_{2c\varepsilon})u = \left(-\frac{d^2}{dx^2} + g'_{2c\varepsilon}(0)\right)v$. By Lemma 2.4, there exists an unique $u \in \mathbf{H}$ satisfying this equation. We have

$$-u'' + g'_{2c\varepsilon}(0)u = \left(-\frac{d^2}{dx^2} + g'_{2c\varepsilon}(0)\right)v + (g'_{2c\varepsilon}(0) - g'_{2c\varepsilon}(r_{2c\varepsilon}))u \in \mathbf{L}_W$$

(recall that $v \in \mathbf{H}_W$ and $g'_{2c\varepsilon}(0) - g'_{2c\varepsilon}(r_{2c\varepsilon})$ decays exponentially at infinity). Using Lemma 3.4, we infer that $u \in \mathbf{H}_W$.

For $\lambda < q^2$, is is clear that $Id_{\mathbf{H}_W \times \mathbf{H}_W} - L_\lambda$ is not invertible if and only if $Id_{\mathbf{H}_W} + A_\lambda$ is not invertible, i.e. if and only if λ is an eigenvalue of A. Let $\lambda_1 < \lambda_2 < \ldots < \lambda_{N_q} < q^2$ be the eigenvalues of A below q^2. If λ is not an eigenvalue of A, we infer using Lemma 3.7 that $i(\lambda) := ind(Id_{\mathbf{H}_W \times \mathbf{H}_W} - L_\lambda, 0) = ind(Id_{\mathbf{H}_W} + A_\lambda, 0) \cdot ind(Id_{\mathbf{H}_W} + B, 0) = (-1)^{n(\lambda)} ind(Id_{\mathbf{H}_W} + B, 0)$ is constant on each of the intervals $(-\infty, \lambda_1), (\lambda_i, \lambda_{i+1}), (\lambda_{N_q}, q^2)$ and changes sign at each λ_i. Consequently, L_λ also satisfies assumption c) in Proposition 3.2 at any point $(\lambda_i, 0, 0)$. Let $\tilde{\mathcal{S}}_0 = \{(\lambda, w, u) \in \Omega \mid (w, u) \neq (0, 0)$ and (λ, w, u) satisfies (3.10)$\}$ and let $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_0 \setminus \{(\lambda, -r_{2c\varepsilon}, 0) \mid \lambda \in (-\infty, q^2)\}$. Note that the solutions $(\lambda, -r_{2c\varepsilon}, 0)$ of (3.10) correspond to the solutions $(\lambda, 0, 0)$ of (1.9)-(1.10) and $\mathcal{S} \cap ((-\infty, q^2) \times (V \cap \mathbf{H}_W) \times \mathbf{H}_W) = \tilde{\mathcal{S}} + (0, r_{2c\varepsilon}, 0)$. We may apply Proposition 3.2 to infer that for any $1 \leq m \leq N_q$, there exists a maximal closed connected subset $\mathcal{D}_m$ (in Ω) of $\tilde{\mathcal{S}}_0 \cup \{(\lambda_m, 0, 0)\}$ which contains $(\lambda_m, 0, 0)$ and satisfies at least one of the following properties :

1°. $\mathcal{D}_m$ is unbounded.

2°. There exists a sequence $(\lambda_n, w_n, u_n) \in \mathcal{D}_m$ such that $\lambda_n \longrightarrow q^2$ as $n \longrightarrow \infty$.

3°. There exists a sequence $(\lambda_n, w_n, u_n) \in \mathcal{D}_m$ such that $dist(w_n, \partial((V - r_{2c\varepsilon}) \cap \mathbf{H}_W)) \longrightarrow 0$, that is $\inf_{x \in \mathbf{R}} (w_n(x) + r_{2c\varepsilon}(x)) \longrightarrow -1$ as $n \longrightarrow \infty$.

4°. The closure in Ω of $\mathcal{D}_m$ contains a point $(\lambda_i, 0, 0)$ with $i \neq m$.

Let us show first that $\mathcal{D}_m$ cannot meet $\{(\lambda, -r_{2c\varepsilon}, 0) \mid \lambda \in (-\infty, q^2)\}$. A straightforward computation gives $d_{(w,u)}(Id_E - G)(\lambda, -r_{2c\varepsilon}, 0) = Id_E$ for any $\lambda < q^2$. By the Implicit Functions Theorem, there exists a neighbourhood N_λ of $(\lambda, -r_{2c\varepsilon}, 0)$ in $\mathbf{R} \times E$ such that the only solutions of the equation $(w, u) = G(\lambda, w, u)$ in N_λ are $(\mu, -r_{2c\varepsilon}, 0)$. Hence $\cup_\lambda N_\lambda$ is a neighbourhood of $\{(\lambda, -r_{2c\varepsilon}, 0) \mid \lambda < q^2\}$ in Ω which contains no other solutions of (3.10). Consequently, we have $\mathcal{D}_m \subset \tilde{\mathcal{S}}$.

By Proposition 3.1, for any $(\lambda, w, u) \in \tilde{\mathcal{S}}_0$ we have $\inf\limits_{x \in \mathbf{R}} (w(x) + r_{2c\varepsilon}(x)) > -1 + \sqrt{2}c\varepsilon$, hence $\mathcal{D}_m$ cannot satisfy property 3° above.

We will also eliminate the alternative 4°. Observe that if $(\lambda, r, u) \in (-\infty, q^2) \times \mathbf{H} \times \mathbf{H}$ is a nontrivial solution of (1.9)-(1.10), then, in particular, u is an eigenvector of the linear operator $-\frac{d^2}{dx^2} + q^2(1+r)^2$ corresponding to the eigenvalue λ. It is easily checked that this operator is a compact perturbation of $-\frac{d^2}{dx^2} + q^2$, so it has the essential spectrum $[q^2, \infty)$. Since $\lambda < q^2$, the operator $-\frac{d^2}{dx^2} + q^2(1+r)^2$ has only a finite number of eigenvalues less than λ, say, p. We define $z(\lambda, r, u) = p$. By Proposition 2.1 v), we know that u has exactly p zeroes in $(0, \infty)$. We also define $z(\lambda_i, r_{2c\varepsilon}, 0) = i - 1$. We have :

Lemma 3.9 *The function z is continuous on* $(\mathcal{S} \cup \{(\lambda_i, r_{2c\varepsilon}, 0) \mid i = 1, \ldots, N_q\}) \cap ((-\infty, q^2) \times \mathbf{H} \times \mathbf{H})$.

Assume for the moment that Lemma 3.9 holds. Obviously, the function z is also continuous for the $\mathbf{R} \times E$ topology. Since z takes values in $\mathbf{N}$, it must be constant on each connected component of $(\mathcal{S} \cup \{(\lambda_i, r_{2c\varepsilon}, 0) \mid i = 1, \ldots, N_q\}) \cap ((-\infty, q^2) \times \mathbf{H} \times \mathbf{H}) = (\tilde{\mathcal{S}} + (0, r_{2c\varepsilon}, 0)) \cup \{(\lambda_i, r_{2c\varepsilon}, 0) \mid i = 1, \ldots, N_q\}$. In particular, it is constant on $\mathcal{D}_m + (0, r_{2c\varepsilon}, 0)$ and we find $z(\mathcal{D}_m + (0, r_{2c\varepsilon}, 0)) = z(\lambda_m, r_{2c\varepsilon}, 0) = m - 1$. We have also $z(\mathcal{D}_i + (0, r_{2c\varepsilon}, 0)) = i - 1$, hence $\mathcal{D}_m$ and $\mathcal{D}_i$ are disjoint if $i \neq m$ (in fact, we see that the closures of $\mathcal{D}_m$ and $\mathcal{D}_i$ in $(-\infty, q^2) \times \mathbf{H} \times \mathbf{H}$ are disjoint if $i \neq m$). Thus $\mathcal{D}_m$ cannot satisfy the alternative 4° above, hence it necessarily satisfies one of the alternatives 1° or 2°. Let $\mathcal{C}_m = \mathcal{D}_m + (0, r_{2c\varepsilon}, 0)$. It is now clear that $\mathcal{C}_m$ satisfies i) or ii) in Theorem 3.8. □

Proof of Lemma 3.9. Let $(\lambda, r, u), (\nu_n, r_n, u_n) \in (\mathcal{S} \cup \{(\lambda_i, r_{2c\varepsilon}, 0) \mid i = 1, \ldots, N_q\}) \cap ((-\infty, q^2) \times \mathbf{H} \times \mathbf{H})$ be such that $z(\lambda, r, u) = p$ and $(\nu_n, r_n, u_n) \longrightarrow (\lambda, r, u)$ as $n \longrightarrow \infty$. Let $\mu_1 < \mu_2 < \ldots < \mu_{p+1} = \lambda$ be the eigenvalues of the operator $B = -\frac{d^2}{dx^2} + q^2(1+r)^2$ in $\mathbf{L}$ and let $u_1^*, \ldots, u_{p+1}^* = u$ be corresponding eigenvectors. Denote $B_n = -\frac{d^2}{dx^2} + q^2(1+r_n)^2$.

We prove that $z(\nu_n, r_n, u_n) \geq p$ if n is sufficiently big. There is nothing to do if $p = 0$. Suppose that $p \geq 1$. Take $0 < \epsilon < \frac{\mu_{p+1} - \mu_p}{4}$ and let n_0 be suficiently big, so that $||(r_n - r)(2 + r_n + r)||_{L^\infty} < \frac{\epsilon}{q^2}$ and $\lambda - \epsilon < \nu_n < \lambda + \epsilon$ for any $n \geq n_0$. For $n \geq n_0$ and $v \in Span\{u_1^*, \ldots u_p^*\}$ we have

$$\begin{aligned} &\langle B_n v, v \rangle_{\mathbf{L}} = \langle B v, v \rangle_{\mathbf{L}} + \langle (B_n - B) v, v \rangle_{\mathbf{L}} \\ &\leq \mu_p ||v||_{\mathbf{L}}^2 + q^2 \int_{\mathbf{R}} (r_n - r)(2 + r_n + r)|v|^2 dx \leq (\mu_p + \epsilon)||v||_{\mathbf{L}}^2 < (\nu_n - \epsilon)||v||_{\mathbf{L}}^2. \end{aligned}$$

By the Min-Max Principle, B_n has at least p eigenvalues less than or equal to $\nu_n - \epsilon$, so $z(\nu_n, r_n, u_n) \geq p$.

Let $\mu_{p+2} = \sup\limits_{\varphi_1, \ldots, \varphi_{p+1} \in \mathbf{H}} \inf\limits_{\psi \in \{\varphi_1, \ldots, \varphi_{p+1}\}^\perp} \dfrac{\langle B\psi, \psi \rangle_{\mathbf{L}}}{||\psi||_{\mathbf{L}}^2}$. Since $\lambda = \mu_{p+1} < q^2$ and λ is a simple eigenvalue of B by Proposition 2.1 iii), we know by the Min-Max Principle that either $\mu_{p+2} = q^2$ or μ_{p+2} is an eigenvalue of B and $\mu_{p+2} > \mu_{p+1}$. Let $\epsilon \in (0, \frac{\mu_{p+2} - \mu_{p+1}}{4})$. Take n_0 as above and $\varphi_1, \ldots, \varphi_{p+1} \in \mathbf{H}$ such that $\inf\limits_{\psi \in \{\varphi_1, \ldots, \varphi_{p+1}\}^\perp} \dfrac{\langle B\psi, \psi \rangle_{\mathbf{L}}}{||\psi||_{\mathbf{L}}^2} \geq \mu_{p+2} - \epsilon$. For any

$\psi \in \{\varphi_1, \dots, \varphi_{p+1}\}^{\perp}$, $\psi \neq 0$ we have :

$$\langle B_n\psi, \psi\rangle_{\mathbf{L}} = \langle B\psi, \psi\rangle_{\mathbf{L}} + \langle (B_n - B)\psi, \psi\rangle_{\mathbf{L}} \geq (\mu_{p+2} - \epsilon)||\psi||^2_{\mathbf{L}} - \epsilon||\psi||^2_{\mathbf{L}} \geq (\nu_n + \epsilon)||\psi||^2_{\mathbf{L}}.$$

It follows from the Min-Max Principle that for $n \geq n_0$, either B_n has at most $p+1$ eigenvalues, or the $(p+2)^{th}$ eigenvalue is greater than $\nu_n + \epsilon$. Since ν_n is an eigenvalue of B_n, there are at most p eigenvalues of B_n less than ν_n, hence $z(\nu_n, r_n, u_n) \leq p$ for any $n \geq n_0$. This finishes the proof of Lemma 3.9 and that of Theorem 3.8. □

We were not able to eliminate one or another of the alternatives in Theorem 3.8.

Up to now, we have proved the existence of branches of nontrivial *symmetric* solutions (λ, r, u) to the system (1.9)-(1.10). For any such solution, $(\tilde{\psi}, \tilde{\varphi})$ is a travelling wave of (1.1) for $\varepsilon^2(c^2\delta^2 + k^2) = \lambda$ and satisfies the boundary condition (1.2), where $\tilde{\varphi}(x) = \frac{1}{\varepsilon}u(\frac{x}{\varepsilon})e^{ic\delta x}$ and $\tilde{\psi}(x) = (1 + r(\frac{x}{\varepsilon}))e^{i\psi_0(x)}$ (with $\psi_0(x) = c\int_0^x \left[1 - \frac{1}{(1+r(\frac{s}{\varepsilon}))^2}\right]ds = c\varepsilon\int_0^{\frac{x}{\varepsilon}} \frac{2r(\tau)+r^2(\tau)}{(1+r(\tau))^2}d\tau$). Note also that $\tilde{\psi}(-x) = \overline{\tilde{\psi}(x)}$, $\tilde{\varphi}(-x) = \overline{\tilde{\varphi}(x)}$, $|\tilde{\psi}| > \sqrt{2}c\varepsilon$ by Proposition 2.1 and the phase ψ_0 of $\tilde{\psi}$ remains bounded because r decays at infinity faster than $|x|^{\beta}$ for any $\beta > 0$ (see the end of the proof of Lemma 3.5). Since $2c^2\varepsilon^2q^2 < \lambda \leq q^2$, we have bounds on the single-particle impurity energy : $c^2(2q^2 - \delta^2) < k^2 \leq \frac{q^2}{\varepsilon^2} - c^2\delta^2$.

Remark 3.10 It follows from Corollary 2.2 iv)-v) that there is exactly one branch of travelling-waves bifurcating from the trivial solutions if $q \leq \frac{1}{\sqrt{2\ln 2}}$. The number of these branches is the same as the number of eigenvalues of A, so it tends to infinity as $q \longrightarrow \infty$.

It is natural to ask how the branches $\mathcal{C}_m$ given by Theorem 3.8 behave in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$. The topology of $\mathbf{H}_W$ being stronger than that of $\mathbf{H}$, any of the sets $\mathcal{C}_m$ is also connected in $\mathbf{R}\times\mathbf{H}\times\mathbf{H}$. Roughly speaking, either $\mathcal{C}_m$ approaches $\{q^2\}\times(\mathbf{H}\cap V)\times\mathbf{H}$, or $\mathcal{C}_m$ is unbounded in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$ or it remains bounded in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$ but the norm in $\mathbf{R} \times \mathbf{H}_W \times \mathbf{H}_W$ tends to infinity along $\mathcal{C}_m$, i.e. "there is some mass moving to infinity".

Remark 3.11 The importance of Theorem 2.3 is that it gives a precise description of $\mathcal{C}_m$ in a neighbourhood of $(\lambda_m, r_{2c\varepsilon}, 0)$ in $\mathbf{R}\times\mathbf{H}\times\mathbf{H}$. Let $\mathcal{C}_m^+$ (respectively $\mathcal{C}_m^-$) be the maximal subcontinuum in $\mathbf{R} \times \mathbf{H}_W \times \mathbf{H}_W$ of $\mathcal{C}_m \setminus \{(\lambda(s), r_{2c\varepsilon} + sr(s), s(u_m + u(s))) \mid s \in (-\eta, 0)\}$, (respectively of $\mathcal{C}_m \setminus \{(\lambda(s), r_{2c\varepsilon} + sr(s), s(u_m + u(s))) \mid s \in (0, \eta)\}$), where the curve $s \longmapsto (\lambda(s), r(s), u(s))$ is given by Theorem 2.3. It can be proved by using a variant of a classical result of Rabinowitz (Theorem 1.40 p. 500 in [12]) that each of $\mathcal{C}_m^+$ and $\mathcal{C}_m^-$ satisfies i) or ii) in Theorem 3.8.

Remark 3.12 It is not hard to prove that in dimension $N = 1$, 2 or 3 the Cauchy problem for the system (1.1) is globally well-posed in $(1 + H^1(\mathbf{R}^N)) \times H^1(\mathbf{R}^N)$. However, the dynamics associated to (1.1) and the asymptotic behavior of solutions are not yet understood.

Remark 3.13 The existence of solitary waves for (1.1) in dimension greater than 1 is an open problem. Even the existence of "trivial" solitary waves (i.e., solutions of the form $(\psi(x_1 - ct, x_2, \dots, x_N), 0)$ is a difficult problem. Note that if $\varphi \equiv 0$, the system (1.1) reduces to the Gross-Pitaevskii equation

$$2i\frac{\partial\psi}{\partial t} = -\Delta\psi + (|\psi|^2 - 1)\psi, \quad |\psi| \longrightarrow 1 \text{ as } |x| \longrightarrow \infty$$

The existence of travelling-waves moving with small speed for this equation was proved, for instance, in [2] (in dimension $N = 2$) and [1], [3] (in dimension $N \geq 3$).

Acknowledgement. I am indebted to Professor Jean-Claude Saut, who brought the subject to my attention, for interesting and stimulating discussions. I am also very grateful to Professor Louis Jeanjean for constant help and encouragement and to Professor Patrick J. Rabier for useful remarks.

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