Existence of nonstationary bubbles in higher dimensions

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Abstract

We are interested in the existence of travelling-waves for the nonlinear Schrödinger equation in $\mathbf{R}^N$ with "$\psi^3 - \psi^5$"-type nonlinearity. First, we prove an abstract result in critical point theory (a local variant of the classical saddle-point theorem). Using this result, we get the existence of travelling-waves moving with sufficiently small velocity in space dimension $N \geq 4$.

Résumé

Nous nous intéressons à l'existence des ondes progressives pour l'équation de Schrödinger non-linéaire dans $\mathbf{R}^N$ avec une non-linéarité de type "$\psi^3 - \psi^5$". D'abord on montre un résultat abstrait en théorie des points critiques (une variante locale du théorème du point selle). À l'aide de ce résultat on prouve l'existence des ondes progressives de petite vitesse en dimension $N \geq 4$.

1 Introduction

The aim of this work is to prove the existence of travelling "bubbles" for the nonlinear Schrödinger equation

$$(1.1) \qquad i\frac{\partial \varphi}{\partial t} + \Delta\varphi + F(|\varphi|^2)\varphi = 0 \quad \text{in } \mathbf{R}^N,$$

where the function φ is complex-valued and satisfies the "boundary condition" $|\varphi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$, and r_0 is a positive real constant such that $F(r_0^2) = 0$. The case of the "$\psi^3 - \psi^5$" nonlinear Schrödinger equation

$$(1.1') \qquad i\frac{\partial \psi}{\partial t} + \Delta\psi - \alpha_1\psi + \alpha_3|\psi|^2\psi - \alpha_5|\psi|^4\psi = 0$$

with $\alpha_1, \alpha_3, \alpha_5 > 0$ and $\frac{3}{16} < \frac{\alpha_1\alpha_5}{\alpha_3^2} < \frac{1}{4}$ fits in this framework.

Equation (1.1) (and in particular (1.1')) appears in a large variety of physical problems, see [1]. For example, (1.1') describes the boson gas with 2-body attractive and 3-body repulsive δ-function interaction. These equations have applications to superfluidity, where the "$\psi^3 - \psi^5$" NLS equation arises on the level of the Ginzburg-Landau two-liquid theory. They also occur in the description of defectons, in the theory of one-dimensional ferromagnetic and molecular chains and in other similar problems in condensed matter. Equation (1.1') with $N = 3$ models the evolution of a monochromatic wave complex envelope in a medium with weakly saturating nonlinearity.

There is a special kind of solutions of (1.1), the "stationary bubbles". These are solutions of the form $e^{i\omega t}\psi(x)$. It was proved in [4] under general conditions on the nonlinearity F that the stationary bubbles exist and are unstable.

It was also proved (see [2]) that in space dimension one there exist some localized solutions travelling with velocity c, having the form $\varphi(t,x) = \Phi(x - ct)$ and corresponding to "nonstationary bubbles". The boundary condition is then $\lim_{x \to \pm\infty} \Phi(x) = r_0 e^{\mp i\mu}$, where μ is a real number depending on c and $\mu = 0$ when $c = 0$.

The travelling waves (or nonstationary bubbles) of (1.1) are solutions of the form $\varphi(t, x_1, \dots, x_N) = \Phi(x_1 - ct, x_2, \dots, x_N)$. In view of the boundary condition, we will seek for solutions Φ of the form $\Phi(x) = r_0 - u(x)$ with $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. The function u must satisfy

$$icu_{x_1} - \Delta u + F(|r_0 - u|^2)(r_0 - u) = 0. \tag{1.2}$$

Now let us describe the assumptions that we make on the nonlinearity F (which are essentially the same as in [4] or [6]). We assume throughout that $F \in C^1(\mathbf{R}_+, \mathbf{R})$ and

$$F(r_0^2) = 0, \quad F'(r_0^2) < 0. \tag{\textbf{H1}}$$

We will need a little bit more regularity on F only in a neighbourhood of r_0^2. We suppose that there exists $\alpha > 0$ such that $|F'(r_0^2 + s) - F'(r_0^2)| \le C|s|^\alpha$ for s small. Set

$$V(s) = \int_s^{r_0^2} F(\tau)d\tau. \tag{1.3}$$

In particular, condition (H1) implies $|F(r_0^2 + s)| \le C|s|$ and $V(r_0^2 + s) \le C's^2$ for some $C, C' > 0$ and s small.

We also have to impose some restrictions on the behaviour of F at infinity. We suppose that there exists $C > 0$ such that

$$|F'(s)| \le C|s|^{\frac{\sigma}{2}-1} \quad \text{for } s \ge 1, \text{ where } \sigma = \frac{4}{N-2}. \tag{\textbf{H2}}$$

(Note that $2 + \sigma$ is the critical exponent for the embedding of $H^1(\mathbf{R}^N)$ in some $L^p(\mathbf{R}^N)$.) Of course this implies

$$|F(s)| \le C's^{\frac{\sigma}{2}} \quad \text{if } s \ge 1 \quad \text{and} \tag{1.4}$$

$$|V(s)| \le C'' s^{\frac{\sigma}{2}+1} \tag{1.5}$$

for some positive constants C', C''.

We will always make the assumption

(H3) there exists $\rho_1 \in [0, r_0^2)$ such that $V(\rho_1) < 0$.

Note that assumptions (H1), (H2), (H3) are "almost" needed for the existence of stationary bubbles (see [3] and [4]). In addition, for technical reasons we impose the following condition:

(H4) there exists $M > 0$ such that $F(s) \le 0$ for $s \ge M$.

We need (H4) only in Section 5, to prove the regularity of the nonstationary bubbles.

Let $a_0 = \sup\{a > 0 \mid F(|r_0 - u|^2)(r_0 - u) > 0,\ \forall u \in (0, a)\}$. In view of (H1) and (H3), it is clear that $0 < a_0 < r_0$.

We define $J(\lambda, u) = [2u - (\lambda+2)r_0]F(|r_0 - u|^2) - 2\lambda u(r_0 - u)^2 F'(|r_0 - u|^2)$ and we suppose that the following condition is satisfied: for any $U \in (a_0, r_0)$ there exists $\lambda(U) > 0$ continuously depending on U such that

(H5) $$\begin{array}{ll} J(\lambda(U), u) \le 0, & \forall u \in [0, U] \quad \text{and} \\ J(\lambda(U), u) \ge 0, & \forall u \in [U, r_0]. \end{array}$$

Note that assumption (H5) is the analogous of conditions (5)-(6) in [8] and we need it only to prove an uniqueness result in section 2 (Theorem 2.6).

A complex-valued function $u = u_1 + iu_2$ is a solution of equation (1.2) if and only if its real and imaginary parts satisfy the system

$$-cu_{2_{x_1}} - \Delta u_1 + F((r_0 - u_1)^2 + u_2^2)(r_0 - u_1) = 0, \tag{1.6}$$

$$cu_{1_{x_1}} - \Delta u_2 - F((r_0 - u_1)^2 + u_2^2)u_2 = 0. \tag{1.7}$$

In what follows, $H^1(\mathbf{R}^N)$ always denotes the space $H^1(\mathbf{R}^N, \mathbf{R})$ and $\mathcal{D}^{1,2}(\mathbf{R}^N) = \mathcal{D}^{1,2}(\mathbf{R}^N, \mathbf{R}) = \{v \in L^{2+\sigma}(\mathbf{R}^N) \mid \nabla v \in L^2(\mathbf{R}^N)\}$, with norm $||v||^2_{\mathcal{D}^{1,2}} = \int_{\mathbf{R}^N} |\nabla v|^2 dx$. We shall identify a function $u = u_1 + iu_2$ with the pair (u_1, u_2) and we seek for solutions with $u_1 \in H^1(\mathbf{R}^N)$, $u_2 \in \mathcal{D}^{1,2}(\mathbf{R}^N)$. Let $\mathbf{H} = H^1(\mathbf{R}^N) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. On $\mathbf{H}$ we consider the norm $||(u_1, u_2)||^2 = ||u_1||^2_{H^1} + ||u_2||^2_{\mathcal{D}^{1,2}}$. We identify $H^1(\mathbf{R}^N) \times \{0\}$ with $H^1(\mathbf{R}^N)$ and $\{0\} \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ with $\mathcal{D}^{1,2}(\mathbf{R}^N)$. We introduce the following functionals:

$$\begin{aligned}
T(u) &= T(u_1, u_2) = \int_{\mathbf{R}^N} |\nabla u|^2 dx = \int_{\mathbf{R}^N} |\nabla u_1|^2 dx + \int_{\mathbf{R}^N} |\nabla u_2|^2 dx, \\
I(u) &= I(u_1, u_2) = \int_{\mathbf{R}^N} V(|r_0 - u|^2) dx = \int_{\mathbf{R}^N} V((r_0 - u_1)^2 + u_2^2) dx, \\
Q(u) &= Q(u_1, u_2) = -2\int_{\mathbf{R}^N} u_1 u_{2_{x_1}} dx, \\
E(u) &= E(u_1, u_2) = T(u) + I(u), \\
E_c(u) &= E_c(u_1, u_2) = T(u) + I(u) + cQ(u) = E(u) + cQ(u).
\end{aligned}$$

Obviously T and Q are of class C^∞ on $\mathbf{H}$. It is easy to check that under assumptions (H1) and (H2), I is of class C^2 on $H^1(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$. It will be verified at the beginning of Section 4 that I is well-defined and of class C^2 on $\mathbf{H}$ if $N \geq 4$.

Therefore E and E_c are of class C^2 on $\mathbf{H}$ if $N \geq 4$ and the $\mathbf{H}$-solutions of (1.2) are exactly the critical points of E_c, while the critical points u of E satisfy the equation

$$-\Delta u + F(|r_0 - u|^2)(r_0 - u) = 0. \tag{1.8}$$

The following theorem gives the existence of a special solution of (1.8):

Theorem 1.1.([4]) *There exists a real-valued function $u_0 \in H^1(\mathbf{R}^N)$ which satisfies equation (1.8) and has the following properties:*

i) u_0 is radially symmetric, i.e $u_0(x) = u_0(|x|) = u_0(r)$;

ii) $0 < u_0(r) < r_0, \ \ \forall r \in [0, \infty)$, $u_{0r}(0) = 0$ and $u_{0r}(r) < 0, \ \ \forall r > 0$ (i.e. u_0 is strictly decreasing in r);

iii) $u_0 \in C^2(\mathbf{R}^N)$ and there exist constants $C, \delta > 0$ such that $|\partial_x^\alpha u_0(x)| \leq Ce^{-\delta|x|}, \ \ \forall x \in \mathbf{R}^N, \ \forall \alpha \in \mathbf{N}^N$ with $|\alpha| \leq 2$.

iv) u_0 is a solution of the minimization problem:

"minimize $T(u)$ under the constraint $I(u) = I(u_0)$";

v) equivalently, u_0 is a solution of the maximization problem:

"maximize $I(u)$ under the constraint $T(u) = T(u_0)$".

Theorem 1.1 was proved in [4] by using a general result of H. Berestycki and P.-L. Lions (see [3]). A solution having the properties listed in Theorem 1.1 will be called a *ground state* for equation (1.8).

Note that $\lim\limits_{s \to r_0^2} \frac{V(s)}{(s-r_0)^2} = -\frac{1}{2}F'(r_0^2) > 0$, so $V(s)$ is positive on an interval $((r_0 - \eta)^2, (r_0 + \eta)^2)$. Suppose that $V \geq 0$ on $[r_0^2, \infty)$ (remark that this is the case for the "$\psi^3 - \psi^5$" nonlinearity). Then $V(|r_0 - z|^2) < 0$ implies that z belongs to the ball (in $\mathbf{C}$) of center r_0 and radius $r_0 - \eta$. Let $N = \{z \in \mathbf{C} \mid V(|r_0 - z|^2) < 0\} \subset B_{\mathbf{C}}(r_0, r_0 - \eta)$. If $u \in \mathbf{H}$ and $E(u) < 0$, we have

$$\begin{aligned} E(u) &\geq \int_{\mathbf{R}^N} V(|r_0 - u|^2)dx \geq \int_{\{x|u(x)\in N\}} V(|r_0 - u|^2)dx \\ &\geq \inf_{[0,r_0^2]} V \cdot meas(\{x \mid u(x) \in N\}), \end{aligned}$$

so that $meas(\{x \mid u(x) \in N\}) \geq \frac{\int V(|r_0-u|^2)dx}{\inf_{[0,r_0^2]} V} \geq \frac{E(u)}{\inf_{[0,r_0^2]} V}$. On the other hand, by the Sobolev embedding and the fact that $dist(N, 0) \geq \eta$ we have

$$\int_{\mathbf{R}^N} |\nabla u|^2 dx \geq C_S ||u||_{L^{2^*}}^2 \geq C_S \eta^{\frac{2}{2^*}} (meas(\{x \mid u(x) \in N\}))^{\frac{2}{2^*}},$$

so that

$$E(u) \geq C_1 (meas(\{x \mid u(x) \in N\}))^{\frac{2}{2^*}} - C_2 meas(\{x \mid u(x) \in N\})$$

for some positive constants C_1, C_2. Clearly, $meas(\{x \mid u(x) \in N\})$ *does not* depend continuously on u. However, using the simple observations made above, it is possible to find a radial function $v_0 \in H^1(\mathbf{R}^N)$ such that $E(v_0) < 0$ and

$\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} E(\gamma(t)) > 0$, where $\Gamma = \{\gamma \in C([0,1], \mathbf{H}) \mid \gamma(0) = 0, \gamma(1) = v_0\}$. Therefore the functional E admits a Palais-Smale sequence (nevertheless, it is not obvious at this stage that this sequence converges in $\mathbf{H}$).

Since $E_c(u) \longrightarrow E(u)$ as $c \longrightarrow 0$ uniformly on bounded sets of $\mathbf{H}$, one should expect that $\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} E_c(\gamma(t)) > 0$, at least for small values of c. However, the observations made above fail when E is replaced by E_c: it is not possible to bound $E_c(u)$ from below in terms of $meas(\{x \mid u(x) \in N\})$. There exist continuous paths connecting v_0 to functions of arbitrarily low "energy" E_c such that E_c decreases and $meas(\{x \mid u(x) \in N\})$ is constant along these paths. To be more precise, for any $c \neq 0$ one can find a continuous path $\tilde{\gamma}_c : [0,\infty) \longrightarrow \mathbf{H}$ such that $\tilde{\gamma}_c(0) = v_0$, $\tilde{\gamma}_c(\tau)$ is of the form $r_0 - (r_0 - v_0)e^{i\varphi_\tau}$ (hence $|r_0 - \tilde{\gamma}_c(\tau)(x)| = |r_0 - v_0(x)|$) and $E_c(\tilde{\gamma}_c(\cdot))$ is strictly decreasing on $[0,\infty)$ with $\lim_{\tau\longrightarrow\infty} E_c(\tilde{\gamma}_c(\tau)) = -\infty$. We do not know whether it is possible or not to connect some $\tilde{\gamma}_c(\tau)$ for large τ (thus for $E_c(\tilde{\gamma}_c(\tau))$ very small) to zero by a continuous path in $\mathbf{H}$ such that E_c remains negative along this path. (Of course, if such a path existed, we would be able to connect zero to v_0 in the set $\{u \in \mathbf{H} \mid E_c(u) \leq 0\}$, which is not possible in the set $\{u \in \mathbf{H} \mid E(u) \leq 0\}$. Anyway, the preceeding arguments suggest that it should be extremely difficult to find Palais-Smale sequences for E_c by using a Mountain-Pass Theorem on the entire $\mathbf{H}$. Even if such a sequence is found, it should be still more difficult to prove that it converges (in some sense) to a non-trivial solution of (1.2).

We want to prove that (1.2) admits non-trivial solutions by showing that E_c possesses non-trivial critical points. But instead of searching for a change of topology of the level sets of E_c on the entire $\mathbf{H}$, we analyze what happens locally on a small neighbourhood of u_0, where u_0 is a ground state of equation (1.8) as given by Theorem 1.1.

Remark that the system (1.6)-(1.7) is of the form $\Phi_1(c, u_1, u_2) = 0$, $\Phi_2(c, u_1, u_2) = 0$ with

$$\begin{pmatrix} \dfrac{\partial \Phi_1}{\partial u_1} & \dfrac{\partial \Phi_1}{\partial u_2} \\ \\ \dfrac{\partial \Phi_2}{\partial u_1} & \dfrac{\partial \Phi_2}{\partial u_2} \end{pmatrix} (c, u_0, 0) = \begin{pmatrix} A & -c\dfrac{\partial}{\partial x_1} \\ \\ c\dfrac{\partial}{\partial x_1} & B \end{pmatrix}.$$

where A and B are linear operators in $L^2(\mathbf{R}^N)$ defined by $D(A) = D(B) = H^2(\mathbf{R}^N)$ and

$$\begin{aligned} Au &= -\Delta u - [2F'((r_0-u_0)^2)(r_0-u_0)^2 + F((r_0-u_0)^2)]u, \\ Bu &= -\Delta u - F((r_0-u_0)^2)u, \end{aligned} \tag{1.9}$$

u_0 being the ground state. It is easy to see that A and B are self-adjoint. It follows from a classical theorem of Weyl that the essential spectrum of A is $\sigma_{ess}(A) = [-2F'(r_0^2)r_0^2, \infty)$ and the essential spectrum of B is $\sigma_{ess}(B) = [0,\infty)$. Note that $-2F'(r_0^2)r_0^2 > 0$ by (H1) and it is not hard to see that for $c < -2F'(r_0^2)r_0^2$, the essential spectrum of $(\Phi_1, \Phi_2)'(c, u_0, 0)$ is $[0, \infty)$. So even if restricted to the space orthogonal to its kernel, the linear operator $(\Phi_1, \Phi_2)'(0, u_0, 0)$ is not invertible. Therefore we cannot solve the equation $(\Phi_1, \Phi_2)(c, u_1, u_2) = (0, 0)$ for c near zero and (u_1, u_2) near $(u_0, 0)$ by an argument based on the Implicit Function Theorem (such as, for example, the Lyapunov-Schmidt method).

Our strategy is as follows: we consider the spectral decomposition

$$L^2(\mathbf{R}^N) = X \oplus Ker(A) \oplus \tilde{Y},$$

where X, $\tilde{Y}$ are the subspaces corresponding to the negative part of $\sigma(A)$, respectively to the positive part of $\sigma(A)$. It will be seen in the next section that X is one-dimensional and $X \subset H^1(\mathbf{R}^N)$. Let $Y = \tilde{Y} \cap H^1(\mathbf{R}^N)$. We consider the restrictions of the functionals E and E_c to $(X \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. We prove in Section 4 that $E(u_0 + u_1, u_2) > E(u_0, 0)$ for $u_1 \in Y$, $u_2 \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, $(u_1, u_2) \neq (0,0)$, $||(u_1, u_2)||_{\mathbf{H}}$ small and $E(u_0 + v_1, 0) < E(u_0, 0)$ for $v_1 \in X$, $v_1 \neq 0$, $||v_1||_{H^1}$ small.

Therefore u_0 is a saddle-point for E restricted to $(X \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. We shall prove that for c sufficiently small, there exists an open neighbourhood Ω_c of $(0,0)$ in $Y \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ such that for all $(u_1, u_2) \in \Omega_c$ and (u_1, u_2) "close" to $\partial\Omega_c$ we have $E_c(u_0 + u_1, u_2) > E_c(u_0, 0)$ and $E_c(u_0 + v, 0) = E(u_0 + v, 0) < E_c(u_0, 0)$ for $v \in X$, $v \neq 0$, $||v||_{H^1}$ small. By a local Mountain-Pass type argument we infer that for c sufficiently small, there exists a critical point $(u_0 + u_1^c, u_2^c)$ of E_c restricted to $(X \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ and $||(u_1^c, u_2^c)||_{\mathbf{H}} \longrightarrow 0$ as $c \longrightarrow 0$.

It remains only to prove that $E_c'(u_0 + u_1^c, u_2^c).u = 0$ for all $u \in Ker(A)$. It is obvious that $\frac{\partial u_0}{\partial x_i} \in Ker(A)$, $i = 1, \dots, N$. It will be proved in section 2 that $Ker(A)$ is spanned by $\frac{\partial u_0}{\partial x_i}$, $i = 1, \dots, N$ and we shall get the desired conclusion thanks to the invariance of equation (1.2) by translations in $\mathbf{R}^N$. Our main result is:

Theorem. *Let $N \geq 4$. There exists $c_0 > 0$ such that for any $c \in [-c_0, c_0]$ there exists a critical point $u_c \in \mathbf{H}$ of E_c. Moreover, $u_c \longrightarrow u_0$ in $\mathbf{H}$ as $c \longrightarrow 0$ and u_c can be chosen radially symmetric in the transverse variables $(x_2, \dots, x_N)$.*

Similar results were obtained in space dimension $N = 2, 3$ by Zhiwu Lin in [6]. He used the hydrodynamical formulation of the nonlinear Schrödinger equation, searching for solitary waves of (1.1) of the form $\sqrt{\rho}e^{i\varphi}$ and he applied the Lyapunov-Schmidt method of finite-dimensional reduction to the equations in ρ and φ. He used implicitly the fact that $Ker(A) = Span\{\frac{\partial u_0}{\partial x_1}, \dots, \frac{\partial u_0}{\partial x_N}\}$.

This paper is organized as follows: the next section is devoted to the study of the operator A introduced in (1.9). Its properties are essential for our proof of existence of nonstationary bubbles. It will be shown that A has a first negative eigenvalue, 0 is its second eigenvalue and $Ker(A) = Span\{\frac{\partial u_0}{\partial x_1}, \dots, \frac{\partial u_0}{\partial x_N}\}$. In Section 3 we prove an abstract result in critical point theory (a local Saddle-Point Theorem). This result will be applied in Section 4 to find critical points of the functional E_c. Finally, Section 5 is devoted to the regularity of nonstationary bubbles.

2 Properties of the operator A

We have already defined the operator A in $L^2(\mathbf{R}^N)$ by formula (1.9). In this section we study its properties and we are particularly interested in the structure of its kernel. It turns out that the results obtained here still hold in a slightly more general framework. Therefore, consider $g \in C^1([0, \infty))$ with $g(0) = 0$, $g'(0) > 0$ and $|g'(s) - g'(0)| \leq C|s|^\alpha$ for small s and some C, $\alpha > 0$. Let $G(t) = \displaystyle\int_0^t g(s)ds$

and suppose that there exists $\zeta > 0$ with $G(\zeta) < 0$ (this corresponds to assumption (H3) on F). Suppose that the problem

$$-\Delta u + g(u) = 0 \tag{2.1}$$

admits a positive radial solution having the properties listed in Theorem 1.1, where I is replaced by $I(u) = \int_{\mathbf{R}^N} G(u)dx$. If $N \geq 3$ and $\limsup_{x\to\infty} \frac{g(s)}{s^{1+\sigma}} \leq 0$, it follows from a classical result of H. Berestycki and P.-L. Lions that such a solution always exists (see Theorem 1 in [3]); it is called a *ground state* for (2.1). In this section, we denote by u_0 a ground state for (2.1) and we define the operator L on $L^2(\mathbf{R}^N)$ by $D(L) = H^2(\mathbf{R}^N)$ and $Lu = -\Delta u + g'(u_0)u$. Note that in the particular case $g(s) = F((r_0 - s)^2)(r_0 - s)$, (2.1) becomes (1.8) and L equals A.

Remark that L is bounded from below. Since $g'(u_0(x))$ tends exponentially to $g'(0)$ as $|x| \longrightarrow \infty$ (at this point we use the fact that $|g'(s)-g'(0)| \leq C|s|^\alpha$ for small s) it follows from a theorem of Weyl that the essential spectrum of L is the same as the essential spectrum of $-\Delta+g'(0)$, that is $\sigma_{ess}(L) = \sigma_{ess}(-\Delta+g'(0)) = [g'(0),\infty)$. Hence $\sigma(L)$ consists precisely in $\sigma_{ess}(L)$ and a finite number of discrete eigenvalues below $g'(0)$.

Lemma 2.1. *The first eigenvalue of L exists and is negative.*

Proof. It suffices to show that $\inf_{u\in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\langle Lu, u\rangle}{||u||^2_{L^2}} < 0$. We will find a function $v \in H^1(\mathbf{R}^N)$ such that $\langle Lu, u\rangle < 0$.

Because $u_0(x) = u_0(|x|) = u_0(r)$ is a solution of (2.1), u_0 (as a function of the real variable r) must satisfy

$$-u_0'' - \frac{N-1}{r}u_0' + g(u_0) = 0 \qquad \text{on } (0,\infty). \tag{2.2}$$

This implies that $u_0 \in C^3(0,\infty)$; differentiating (2.2) we get

$$-u_0''' - \frac{N-1}{r}u_0'' + g'(u_0)u_0' + \frac{N-1}{r^2}u_0' = 0. \tag{2.3}$$

Let $v(x) = u_0(|x|)$. In view of Theorem 1.1 iii), $v \in H^1(\mathbf{R}^N)$ and from (2.3) we see that v satisfies $Lv + \frac{N-1}{r^2}v = 0$. Therefore $\langle Lv, v\rangle = -(N-1)\int_{\mathbf{R}^N} \frac{|v(x)|^2}{|x|^2}dx < 0$. This proves Lemma 2.1. $\square$

We denote by $-\lambda_1$ the first eigenvalue of L. It is known that $-\lambda_1$ is simple and the corresponding eigenvector is radially symmetric, has constant sign and tends exponentially to zero at infinity. Denote by e_1 an eigenvector corresponding to $-\lambda_1$ with $||e_1||_{L^2} = 1$.

Differentiating equation (2.1) with respect to x_i, we get $\frac{\partial u_0}{\partial x_i} \in Ker(L)$. Therefore 0 is an eigenvalue of L. Using the fact that u_0 minimizes $T(u) = \int_{\mathbf{R}^N} |\nabla u|^2 dx$ subject to the constraint $I(u) = I(u_0)$, where $I(u) = \int_{\mathbf{R}^N} G(u)dx$, we obtain:

Lemma 2.2. *0 is the second eigenvalue of L.*

Proof. Since $-\lambda_1 < 0$ and 0 is an eigenvalue, it is clear that the second eigenvalue of L exists and is ≤ 0. In order to show that the second eigenvalue of L is ≥ 0, we will find a function $f_0 \in H^1(\mathbf{R}^N)$ such that L is positive on $f_0^\perp \cap H^1(\mathbf{R}^N)$ and we use the Min-Max Principle. We claim that for any $v \in H^1(\mathbf{R}^N)$ such that

$$I'(u_0).v = \int_{\mathbf{R}^N} g(u_0) v dx = 0 \tag{2.4}$$

we have $\langle Lv, v\rangle \geq 0$. Indeed, fix $v \in H^1(\mathbf{R}^N)$ such that $I'(u_0).v = 0$. Since $I'(u_0) \neq 0$, there exists $w \in H^1(\mathbf{R}^N)$ such that

$$I''(u_0).(v,v) + I'(u_0).w = 0. \tag{2.5}$$

Using the Implicit Function Theorem, it is not hard to see that there exists $\delta > 0$ and a C^2-curve $\psi : (-\delta, \delta) \longrightarrow H^1(\mathbf{R}^N)$ such that

$$\psi(0) = u_0, \quad \psi'(0) = v, \quad \psi''(0) = w \quad \text{ and } I(\psi(t)) = I(u_0). \tag{2.6}$$

Recall that we have assumed that u_0 satisfies the conditions of Theorem 1.1, in particular u_0 minimizes $T(u)$ under the constraint $I(u) = I(u_0)$. The Euler-Lagrange equation of u_0 is exactly equation (2.1), that is $\frac{1}{2}T'(u_0) + I'(u_0) = 0$. Moreover, the real function $t \longrightarrow T(\psi(t))$ achieves a local minimum at $t = 0$, therefore $\frac{d}{dt}T(\psi(t)) \mid_{t=0} = 0$ and $\frac{d^2}{dt^2}T(\psi(t)) \mid_{t=0} \geq 0$. This gives $T'(u_0).v = 0$ and

$$T''(u_0).(v,v) + T'(u_0).w \geq 0.$$

Using the Euler-Lagrange equation and (2.5) we get

$$\frac{1}{2}T''(u_0).(v,v) \geq -\frac{1}{2}T'(u_0).w = I'(u_0).w = -I''(u_0).(v,v),$$

i.e. $\frac{1}{2}T''(u_0)(v,v) + I''(u_0)(v,v) \geq 0$, which is exactly $\langle Lv, v\rangle \geq 0$. Our claim is thus proved.

It is clear that $g(u_0) \in H^1(\mathbf{R}^N)$. By the Min-Max Principle (see, for example, [11], vol. IV, Theorem XIII.1 p. 76 and Theorem XIII.2 p. 78) the second eigenvalue of L is exactly

$$\inf_{u \in e_1^\perp \setminus \{0\}} \frac{\langle Lu, u\rangle}{||u||^2_{L^2}} = \sup_{f \in H^1(\mathbf{R}^N)} \inf_{u \in f^\perp \setminus \{0\}} \frac{\langle Lu, u\rangle}{||u||^2_{L^2}} \geq 0. \tag{2.7}$$

Therefore 0 is the second eigenvalue of L. □

Corollary 2.3. *i) For any $v \in H^1(\mathbf{R}^N) \cap e_1^\perp$ we have $\langle Lv, v\rangle \geq 0$.*

ii) For any $v \in H^1(\mathbf{R}^N) \cap g'(u_0)^\perp$ we have $\langle Lv, v\rangle \geq 0$.

Corollary 2.3 follows directly from the proof of Lemma 2.2.

Because $\sigma_{ess}(L) = [g'(0), \infty)$ and 0 is a discrete eigenvalue, we have $\beta = \inf(\sigma(L) \cap (0, \infty)) > 0$. Consider the functional calculus associated to the self-adjoint operator L. Let $L_+ = \chi_{(0,\infty)}(L)$ and $\tilde{Y} = Im(L_+)$. Then we have the orthogonal decomposition $L^2(\mathbf{R}^N) = \mathbf{R}e_1 \oplus Ker(L) \oplus \tilde{Y}$. Let $Y = \tilde{Y} \cap H^1(\mathbf{R}^N)$. We have

$$\langle Lu, u\rangle \geq \beta ||u||^2_{L^2}, \quad \forall u \in Y.$$

Lemma 2.4. *There exists $\alpha > 0$ such that*

$$\langle Lu, u\rangle \geq \alpha ||u||^2_{H^1}, \quad \forall u \in Y. \tag{2.8}$$

Proof. For any $u \in Y$ we have

$$\langle Lu, u\rangle = \int_{\mathbf{R}^N} |\nabla u|^2 + g'(u_0)|u|^2 dx \geq \beta ||u||_{L^2}^2 \geq -\beta\delta \int_{\mathbf{R}^N} g'(u_0)|u|^2 dx,$$

where $\delta = \frac{1}{||g'(u_0)||_{L^\infty}}$. It follows that $\int_{\mathbf{R}^N} |\nabla u|^2 dx + (1+\beta\delta)\int_{\mathbf{R}^N} g'(u_0)|u|^2 dx \geq 0$ (or equivalently $\frac{1}{1+\beta\delta}\int_{\mathbf{R}^N} |\nabla u|^2 dx + \int_{\mathbf{R}^N} g'(u_0)|u|^2 dx \geq 0$), which gives $\langle Lu, u\rangle \geq \frac{\beta\delta}{1+\beta\delta}\int_{\mathbf{R}^N} |\nabla u|^2 dx$. □

Now we focus our attention on the kernel of L. First we have to introduce some notation. Let $\mathcal{H}_k$ be the space of spherical harmonics of degree k with $dim\ \mathcal{H}_k = a_k = C_{N+k-1}^k - C_{N+k-3}^{k-2}$. For each k let $\{Y_1^{(k)}, \ldots, Y_{a_k}^{(k)}\}$ be an orthonormal basis of $\mathcal{H}_k$. Let $\mathcal{P}_k$ be the space of linear combinations of the form $\sum_{i=1}^{a_k} f_i(|x|)Y_i^{(k)}\left(\frac{x}{|x|}\right)$ with $f_i \in L^2((0,\infty), r^{N-1}dr)$. Then $\mathcal{P}_k \subset L^2(\mathbf{R}^N)$, the spaces $\mathcal{P}_k$ are mutually orthogonal and invariant under the Fourier transform. More precisely, if $Y \in \mathcal{H}_k$, $f \in L^2((0,\infty), r^{N-1}dr)$ then $\mathcal{F}\left(f(|x|)Y\left(\frac{x}{|x|}\right)\right)(\xi) = g(|\xi|)Y\left(\frac{\xi}{|\xi|}\right)$ for some $g \in L^2((0,\infty), r^{N-1}dr)$. Moreover, $\sum_{k=0}^{\infty} \mathcal{P}_k = L^2(\mathbf{R}^N)$, that is any function $u \in L^2(\mathbf{R}^N)$ has an unique expansion

$$u = \sum_{k=0}^{\infty}\sum_{i=1}^{a_k} c_{k,i}(|x|)Y_i^{(k)}\left(\frac{x}{|x|}\right), \tag{2.9}$$

where $c_{k,i}(|x|) = \int_{S^{N-1}} u(|x|\theta)\overline{Y_i^{(k)}}(\theta)d\theta$. Let $p_{k,i}$ be the projection $p_{k,i}(u) = c_{k,i}(|x|)Y_i^{(k)}\left(\frac{x}{|x|}\right)$. Then $p_{k,i}$ is bounded (has norm 1) as an operator from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$, $s \geq 0$ and commutes with Δ.

After this preparation, we may prove

Theorem 2.5. *$Ker(L)$ is spanned by $\{\frac{\partial u_0}{\partial x_1}, \ldots, \frac{\partial u_0}{\partial x_N}\} \cup (Ker(L) \cap H_{rad}^2(\mathbf{R}^N))$, where $H_{rad}^2(\mathbf{R}^N) = \{u \in H^2(\mathbf{R}^N) \mid u$ is radially symmetric $\}$.*

Proof. The proof was inspired by an idea of M. Weinstein (see the proof of Proposition 2.8 b), p. 483 in [12]). Let $u \in Ker(L)$ and consider its decomposition given by (2.9). Since $u \in H^2(\mathbf{R}^N)$, we have $p_{k,i}(u) \in H^2(\mathbf{R}^N)$. Because $g'(u_0)$ is a radial function, it is clear that $p_{k,i}(g'(u_0)u) = g'(u_0)p_{k,i}(u)$. Therefore we have $L(p_{k,i}(u)) = p_{k,i}(Lu) = 0$. This implies that $c_{k,i}(r)$ satisfies

$$A_k c_{k,i} = 0 \text{ on } (0,\infty),$$

where $A_k = -\frac{d^2}{dr^2} - \frac{N-1}{r}\frac{d}{dr} + g'(u_0(r)) + \frac{k(k+N-2)}{r^2}$. Putting $u_{k,i}(x) = c_{k,i}(|x|)$ we obtain $L_k u_{k,i} = 0$, in particular $\langle L_k u_{k,i}, u_{k,i}\rangle = 0$, where $L_k = -\Delta + g'(u_0) + \frac{k(k+N-2)}{|x|^2}$ on $\mathbf{R}^N$. Taking $v(x) = u_0'(|x|)$ (as in the proof of Lemma 2.1), we see that $L_1 v = 0$, that is v is an eigenvector of L_1 corresponding to the eigenvalue 0. Moreover, v is radially symmetric and has constant sign. But it is known that L_1 possesses a first eigenvalue and the corresponding eigenvector (i.e. the ground state of L_1) is radial, does not change sign and any other eigenvector of L_1 changes sign (because it is orthogonal to the ground state). We infer that v must be the ground state of L_1, 0 its first eigenvalue and therefore $L_1 \geq 0$. Since $L_1 u_{1,i} = 0$, we have necessarily $u_{1,i} = c_i v$ for some constants c_i, so that $c_{1,i}(|x|)Y_i^{(1)}\left(\frac{x}{|x|}\right) = c_i u_0'(|x|)\frac{x_i}{|x|} = c_i \frac{\partial u_0}{\partial x_i}$. For $k \geq 2$ we have $L_k = L_1 + \frac{(k-1)(k-1+N)}{|x|^2}$, so that $\langle L_k u_{k,i}, u_{k,i}\rangle = 0$ implies $u_{k,i} = 0$, that is $c_{k,i} = 0$. Thus $u = p_{0,1}(u) + \sum_{i=1}^{N} p_{1,i}(u) = p_{0,1}(u) + \sum_{i=1}^{N} c_i \frac{\partial u_0}{\partial x_i}$ and $p_{0,1}(u) \in H^2_{rad}(\mathbf{R}^N) \cap Ker(L)$. □

Let $a_0 = \sup\{a > 0 \mid g(s) > 0,\ \forall s \in (0,a)\}$. It is clear that $G > 0$ on $(0,a]$ and (2.1) implies that u_0 satisfies the Pohozaev's identity $\int_{\mathbf{R}^N} G(u_0(x))dx = -\frac{N-2}{N}\int_{\mathbf{R}^N} |\nabla u_0|^2 dx < 0$, thus necessarily $u_0(0) > a_0$. We define

$$I(u,\lambda) = \lambda u g'(u) - (\lambda+2)g(u).$$

In the remainder of this section we will make the following assumption: there exists a continuous function $\lambda : (a_0, u_0(0)] \longrightarrow (0,\infty)$ such that for any $U \in (a_0, u_0(0)]$ we have

$$(\mathbf{H5'}) \qquad \begin{array}{ll} I(u,\lambda(U)) \leq 0, & \forall u \in [0,U] \text{ and} \\ I(u,\lambda(U)) \geq 0, & \forall u \in [U, u_0(0)]. \end{array}$$

Note that in the particular case $g(u) = F((r_0-u)^2)(r_0-u)$, we have $I(u,\lambda) = J(u,\lambda)$ and the condition (H5') is in fact assumption (H5).

Theorem 2.6. *Under assumption (H5'), we have $Ker(L) \cap H^2_{rad}(\mathbf{R}^N) = \{0\}$.* Consequently, $Ker(L) = Span\{\ \frac{\partial u_0}{\partial x_1}, \ldots, \frac{\partial u_0}{\partial x_N}\ \}$.

Proof. An easy boot-strap argument shows that any $u \in Ker(L)$ belongs to $W^{2,p}(\mathbf{R}^N)$, $\forall p \in [2,\infty)$, so that $u \in C^{1,\alpha}(\mathbf{R}^N)$ $\forall \alpha \in [0,1)$ and u as well as $\frac{\partial u}{\partial x_i}$, $i = 1,\ldots,N$ are bounded and tend to zero at infinity. Let $u(x) = \tilde{\delta}(|x|) = \tilde{\delta}(r) \in Ker(L) \cap H^2_{rad}(\mathbf{R}^N)$. Because u is C^1, necessarily $\tilde{\delta}'(0) = 0$ so $\tilde{\delta}$ must satisfy

$$-\delta'' - \frac{N-1}{r}\delta' + g'(u_0)\delta = 0 \quad \text{on } (0,\infty) \tag{2.10}$$

together with the boundary conditions

$$\tilde{\delta}'(0) = 0, \qquad \lim_{r\longrightarrow\infty} \tilde{\delta}(r) = 0. \tag{2.11}$$

Since $\tilde{\delta} \in C^1([0,\infty))$, (2.10) implies that in fact $\tilde{\delta} \in C^3(0,\infty)$.

It is clear that the linear equation (2.10) with the condition $\delta'(0) = 0$ admits a global solution δ defined on $[0,\infty]$ and any other such solution is a multiple of δ. We may suppose without loss of generality that $\delta(0) = 1$. In order to prove Theorem 2.6, it suffices to show that the function $u_1(x) = \delta(|x|)$ does not belong to $H^2(\mathbf{R}^N)$.

Suppose by contradiction that $u_1 \in H^2_{rad}(\mathbf{R}^N)$. This implies that δ and δ' tend to zero as $r \longrightarrow \infty$. First, we prove that δ has exactly one zero in $(0,\infty)$. Since $u_1 \in L^2(\mathbf{R}^N)$, necessarily $\delta \in L^2((0,\infty), r^{N-1}dr)$. Let $w_1(r) = r^{\frac{N-1}{2}}\delta(r)$. Then $w_1 \in L^2(0,\infty)$ and satisfies

$$Mw_1 = 0, \tag{2.12}$$

where $M = -\frac{d^2}{dr^2} + g'(u_0) + \frac{(N-1)(N+1)}{4r^2}$. Remark that $Mw = \lambda w$ if and only if $u(x) = |x|^{-\frac{N-1}{2}} w(|x|)$ satisfies $Lu = \lambda u$. Using Lemmas 2.1 and 2.2 we infer that 0 is the second eigenvalue of M, the first being $-\lambda_1$ (with corresponding eigenvector $r^{\frac{N-1}{2}} e_1(r)$). Since w_1 satisfies (2.12), a well-known result (see, for example, Theorem XIII.8, p. 90 in [11], vol. IV) implies that the number of zeroes of w_1 in $(0,\infty)$ is exactly the number of eigenvalues of M below 0, that is one. It is obvious that $\delta(r) = 0$ for $r \in (0,\infty)$ if and only if $w_1(r) = 0$, thus δ has exactly one zero, say, r_1. Because δ and δ' cannot vanish simultaneously, δ must change sign at r_1. Therefore $\delta > 0$ on $[0, r_1)$, $\delta < 0$ on (r_1, ∞) and necessarily $\delta'(r_1) < 0$.

The rest of the proof was inspired by the ideas developed by K. McLeod in [8].

We show that $u_0(r_1) > a_0$. Suppose that $u_0(r_1) \le a_0$. Then $u_0(r) < a_0$ and $g(u_0(r)) > 0$ on (r_1, ∞). Remark that equations (2.2) and (2.10) can be written as

$$(r^{N-1}u_0'(r))' = r^{N-1}g(u_0(r)), \tag{2.13}$$

respectively

$$(r^{N-1}\delta'(r))' = r^{N-1}g'(u_0(r))\delta(r). \tag{2.14}$$

We obtain from (2.13) and (2.14)

$$[(r^{N-1}u_0'(r))(r^{N-1}\delta'(r))]' = (r^{N-1}u_0'(r))' r^{N-1}\delta'(r) + r^{N-1}u_0'(r)(r^{N-1}\delta'(r))'$$

$$= r^{2N-2}[g(u_0(r))\delta'(r) + g'(u_0(r))u_0'(r)\delta(r)] = r^{2N-2}[g(u_0(r))\delta(r)]'.$$

Integrating this equality from r_1 to ∞ and then integrating by parts we get, taking into account that u_0, u_0' and $g'(u_0)$ tend exponentially to zero and δ, δ' tend to zero as $r \longrightarrow \infty$,

$$\begin{aligned}
&-r_1^{2N-2}u_0'(r_1)\delta'(r_1) = \int_{r_1}^{\infty} r^{2N-2}[g(u_0(r))\delta(r)]' dr \\
&= r^{2N-2}g(u_0(r))\delta(r)\Big|_{r_1}^{\infty} - (2N-2)\int_{r_1}^{\infty} r^{2N-3}g(u_0(r))\delta(r)dr \\
&= -(2N-2)\int_{r_1}^{\infty} r^{2N-3}g(u_0(r))\delta(r)dr.
\end{aligned}$$

But $r_1^{2N-2}u_0'(r_1)\delta'(r_1) > 0$ and $\int_{r_1}^{\infty} r^{2N-3}g(u_0(r))\delta(r)dr < 0$ because $g(u_0) > 0$ and $\delta < 0$ on that interval, so we obtain a contradiction which proves that $u_0(r_1) > a_0$.

We need the following oscillation result which appears as Lemma 5 in [8] and is a special case of the Sturm comparison theorem:

Lemma 2.7.([8]) *Let Y and Z be nontrivial solutions of*

$$-Y'' - \frac{N-1}{r}Y' + H(r)Y = 0, \qquad \textit{respectively} \tag{2.15}$$

$$-Z'' - \frac{N-1}{r}Z' + h(r)Z = 0 \tag{2.16}$$

on some interval $(\mu, \nu) \subset (0, \infty)$, where H and h are continuous on (μ, ν), $H \ge h$ on (μ, ν) and $H \not\equiv h$. If either *a)* $\mu > 0$ *and* $Y(\mu) = Y(\nu) = 0$, *or*

b) $\mu = 0$, Y *and* Z *are continuous at 0,* $Y'(0) = Z'(0) = 0$ *and* $Y(\nu) = 0$, *then* Z *has at least one zero on* (μ, ν). *The same conclusion holds if* $H \equiv h$ *on* (μ, ν), *provided* Y *and* Z *are linearly independent.*

Suppose that (2.15) has at least one solution which does not vanish in some neighbourhood of ∞. We define

$$\rho = \inf\{r \in (0, \infty) \mid \text{there exists a solution of (2.15) with no zeroes in } (0, \infty)\}.$$

The interval (ρ, ∞) is called the disconjugacy interval of (2.15). It is not hard to see that any solution of (2.15) has at least one zero in $[\rho, \infty)$; in fact, it has exactly one by Lemma 2.7,a). The following result holds (for the proof, the reader may consult [8]) :

Lemma 2.8.([8]) *Assume that* H *is continuous on* $(0, \infty)$ *and* $H(r) > 0$ *for large* r. *Let the disconjugacy inteval of (2.15) be* (ρ, ∞) *with* $\rho > 0$ *and suppose that (2.15) has a solution* Y_0 *with* $\lim_{r \to \infty} Y_0(r) = 0$. *Then:*

a) $Y_0(\rho) = 0$ *and if* Y *is a nontrivial solution of (2.15) such that* $Y(\rho) = 0$, *there exists* c *such that* $Y = cY_0$.

b) If Y *is a nontrivial solution of (2.15) with a zero in* (ρ, ∞), *then* $Y(r) \longrightarrow \pm\infty$ *as* $r \longrightarrow \infty$.

We will also make use of the following well-known result about the ground state u_0 (for a proof, see [10]) :

Lemma 2.9.([10]) *We have* $\lim_{r \to \infty} \dfrac{u_0'(r)}{u_0(r)} = -\sqrt{g'(0)} < 0$.

Now let us show how assumption (H5') implies the conclusion of Theorem 2.6. For $\lambda > 0$, define

$$v_\lambda(r) = r u_0'(r) + \lambda u_0(r). \tag{2.17}$$

A simple calculation using (2.2) shows that v_λ satisfies

$$-v_\lambda'' - \frac{N-1}{r} v_\lambda' + g'(u_0) v_\lambda = \lambda g'(u_0(r)) u_0(r) - (\lambda + 2) g(u_0(r)) = I(u_0(r), \lambda). \tag{2.18}$$

Equivalently, v_λ is a solution of

$$-v_\lambda'' - \frac{N-1}{r} v_\lambda' + \left(g'(u_0) - \frac{I(u_0(r), \lambda)}{v_\lambda} \right) v_\lambda = 0 \tag{2.19}$$

on any interval which does not contain any zero of v_λ.

Let $\lambda_1 = \lambda(u(r_1))$ and $\lambda_2 = \lambda(u(0))$, where $\lambda(U)$ is given by assumption (H5'). Then $I(u(r), \lambda_1) \geq 0$ on $[0, r_1]$ and $I(u(r), \lambda_1) \leq 0$ on $[r_1, \infty)$, while $I(u(r), \lambda_2) \leq 0$ for all $r \in [0, \infty)$. By (2.10), (2.19) and Lemma 2.7, v_{λ_1} oscillates faster than δ on any subinterval of $[0, r_1]$ on which $v_{\lambda_1} > 0$. Since $v_{\lambda_1}(0) = \lambda_1 u_0(0) > 0$ and $(\delta(r_1)) = 0$, it follows that the first zero of v_{λ_1} occurs in $(0, r_1]$. Similarly, v_{λ_2} oscillates slower than δ as long as $v_{\lambda_2} > 0$, hence the first zero of v_{λ_2} occurs in $[r_1, \infty)$.

Lemma 2.10. *Assume that for a certain $\lambda > 0$ we have $I(u_0(r), \lambda) \le 0$ on $[r_1, \infty)$ and there exists $r_2 \ge r_1$ such that $v_\lambda(r_2) < 0$. Then $v_\lambda < 0$ on $[r_2, \infty)$.*

Proof. Suppose by contradiction that there exists $r > r_2$ such that $v_\lambda(r) = 0$. Let $r_3 = \inf\{r > r_2 \mid v_\lambda(r) = 0\}$. Obviously $v_\lambda(r_3) = 0$ and $v'_\lambda(r_3) \ge 0$.

We claim that $v'_\lambda(r_3) > 0$. Indeed, if $v'_\lambda(r_3) = 0$, (2.18) would imply $v''_\lambda(r_3) = -I(u_0(r_3), \lambda) \ge 0$. Since $v_\lambda < 0$ on (r_2, r_3), r_3 cannot be a local minimum of v_λ, so necessarily $v''_\lambda(r_3) = 0$ and $I(u_0(r_3), \lambda) = 0$. From the equalities $v_\lambda(r_3) = v'_\lambda(r_3) = v''_\lambda(r_3) = 0$, $I(u_0(r_3), \lambda) = 0$ it can be easily deduced that $u'_0(r_3) = 0$, a contradiction. Thus $v'_\lambda(r_3) > 0$.

It follows that $v_\lambda > 0$ on an interval $(r_3, r_3 + \eta)$. On the other hand, it follows from (2.17) and Lemma 2.9 that $v_\lambda(r)$ is negative for large r, therefore v_λ must vanish after r_3. Let $r_4 = \inf\{r > r_3 \mid v_\lambda(r) = 0\}$. Then $v_\lambda > 0$ on (r_3, r_4) and comparing (2.10) and (2.19) we infer that δ oscillates faster than v_λ on (r_3, r_4), thus δ must vanish on $[r_3, r_4]$, contradicting the fact that r_1 is the unique zero of δ. This proves the lemma. □

Coming back to the proof of Theorem 2.6, we show that the first zero of v_{λ_1} occurs in $(0, r_1)$. Suppose by contradiction that it occurs exactly at r_1. Then we have $v_{\lambda_1}(r_1) = \delta(r_1) = 0$, $v_{\lambda_1} \longrightarrow 0$ exponentially and $\delta, \delta' \longrightarrow 0$ as $r \longrightarrow \infty$. Using (2.14) and (2.19) and integrating by parts we get

$$\begin{aligned} \int_{r_1}^{\infty} r^{N-1} g'(u_0)\delta v_{\lambda_1} dr &= \int_{r_1}^{\infty} (r^{N-1}\delta')' v_{\lambda_1} dr = -\int_{r_1}^{\infty} r^{N-1}\delta' v'_{\lambda_1} dr \\ = \int_{r_1}^{\infty} (r^{N-1} v'_{\lambda_1})'\delta(r) dr &= \int_{r_1}^{\infty} r^{N-1}[g'(u_0)v_{\lambda_1} - I(u_0(r), \lambda_1)]\delta(r) dr. \end{aligned}$$

Thus $\int_{r_1}^{\infty} r^{N-1} I(u_0(r), \lambda_1)\delta(r) dr = 0$. But $I(u_0(r), \lambda_1) \le 0$ and $\delta < 0$ on (r_1, ∞), so necessarily $I(u_0(r), \lambda_1) \equiv 0$ on $[r_1, \infty)$, that is $\lambda_1 u g'(u) - (\lambda_1 + 2)g(u) = 0$ for $u \in (0, u_0(r_1)]$, which implies $g(u) = Au^{\frac{\lambda_1+2}{\lambda_1}}$ on $(0, u_0(r_1)]$ for some constant A, contradicting the fact that $g'(0) > 0$. Hence the first zero of v_{λ_1} occurs in $(0, r_1)$.

It is clear that $v_{\lambda_1} - v_{\lambda_2} = (\lambda_1 - \lambda_2)u_0$ has the same sign as $\lambda_1 - \lambda_2$ on $[0, \infty)$. Since the first zero of v_{λ_1} occurs before the first zero of v_{λ_2}, we must have $\lambda_1 < \lambda_2$.

We infer that there exists $\lambda'_0 \in (\lambda_1, \lambda_2]$ such that the first zero of $v_{\lambda'_0}$ occurs exactly at r_1. Choose $\lambda_0 \in (\lambda_1, \lambda'_0)$ such that the first zero of v_{λ_0} occurs before r_1 and $v_{\lambda_0}(r_1) < 0$. Let $r_0^\star$ be the last zero of v_{λ_0} before r_1. Since $\lambda_1 = \lambda(u_0(r_1))$, $\lambda_2 = \lambda(u_0(0))$ and $r \longmapsto \lambda(u(r))$ is continuous, there exists $\tilde{r}_0 \in (0, r_1)$ such that $\lambda_0 = \lambda(u_0(\tilde{r}_0))$. Let $r_0 = \max(r_0^\star, \tilde{r}_0)) < r_1$. Then $I(u_0(r), \lambda_0) \le 0$, $\forall r \in [r_0, \infty)$ and $v_{\lambda_0}(r_1) < 0$. By Lemma 2.10 we have $v_{\lambda_0} < 0$ on $[r_1, \infty)$, hence $v_{\lambda_0} < 0$ on (r_0, ∞).

Consider the solution δ_0 of (2.10) with $\delta_0(r_0) = 0$, $\delta'_0(r_0) = 1$. Then δ_0 cannot have any zero in (r_0, ∞) since if $\delta_0(r_4) = 0$ for some $r_4 \in (r_0, \infty)$ we would infer from (2.10), (2.19) and Lemma 2.7 that v_{λ_0} has a zero in (r_0, r_4), which is absurd. Consequently (r_0, ∞) is contained in the disconjugacy interval of (2.10). But δ is a solution of (2.10) which vanishes at r_1 and r_1 is an interior point of the disconjugacy interval of (2.10). Using lemma 2.8b) we infer that $\delta(r) \longrightarrow -\infty$ as $r \longrightarrow \infty$, which contradicts the assumption $u_1(x) = \delta(|x|) \in H^2(\mathbf{R}^N)$. This finishes the proof of Theorem 2.6. □

3 A local variant of the Saddle-Point Theorem

In this section we present a general abstract result in critical point theory which generalizes the classical Saddle-Point Theorem. The proof is based on a sharp deformation result (the Quantitative Deformation Lemma) due to M. Willem.

Theorem 3.1. *Let E be a Banach space and $\varphi : E \longrightarrow \mathbf{R}$ a C^1-functional. Let F be a finite-dimensional subspace and G a closed subspace of E such that $F+G=E$ and $F \cap G = \{0\}$. Suppose that there exist $r>0$ and an open set $\Omega \subset G$ containing 0 with the following properties:*

i) $\varphi(x) \leq 0$ if $x \in B_F(0,r)$;

ii) $\varphi(x+y) \leq \mu_0 < 0$ if $x \in F$, $r_1 \leq ||x|| \leq r$ for some $r_1 < r$ and $y \in \Omega$;

iii) $\varphi(y) \geq \mu_1 > \mu_0$ if $y \in \Omega$;

iv) there exists $0 < \delta_0 < dist(0, \partial\Omega)$ and a continuous function $h : \Omega(\delta_0) = \{y \in \Omega \mid dist(y, \partial\Omega) \leq \delta_0\} \longrightarrow [0,r)$ such that for all $x \in F$ with $||x|| = r$ and for all $y \in \Omega(\delta_0)$, the function $t \longmapsto \varphi(tx+y)$ is not increasing on $[\frac{h(y)}{r}, 1]$;

v) $\varphi(x+y) \geq 0$ if $y \in \Omega(\delta_0)$ and $||x|| \leq h(y)$.

Then there exists $c \in [\mu_1, 0]$ and a sequence $z_n \in B_F(0,r) + \Omega$ such that:

a) $\varphi(z_n) \longrightarrow c$ and

b) $\varphi'(z_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Remark 3.2. A sequence satisfying a) and b) is called a Palais-Smale sequence for φ. The functional φ is said to have the Palais-Smale property if any Palais-Smale sequence contains a convergent subsequence. Thus if φ satisfies the assumptions of Theorem 3.1 and has the Palais-Smale property, it has a critical point in $B_F(0,r)+\Omega$.

Remark 3.3. If φ' is bounded on bounded sets of E, we may replace assumption ii) by $\varphi(x+y) \leq \mu_0 < 0$ if $x \in F$, $||x|| = r$ and $y \in \Omega$.

Proof of Theorem 3.1 We denote $\varphi^d = \varphi^{-1}((-\infty, d])$ and for a given subset $S \subset E$ and $\rho > 0$ we denote $S_\rho = \{u \in E \mid dist(u,S) \leq \rho\}$. We shall make use of the following Quantitative Deformation Lemma of M. Willem:

Lemma 3.4.([13]) *Let X be a Banach space, $\varphi \in C^1(X, \mathbf{R})$, $S \subset X$, $c \in \mathbf{R}$, $\varepsilon, \delta > 0$ such that :*

$$||\varphi'(u)|| \geq \frac{8\varepsilon}{\delta}, \quad \forall u \in \varphi^{-1}([c-2\varepsilon, c+2\varepsilon] \cap S_{2\delta}). \tag{3.1}$$

Then there exists $\eta \in C([0,1] \times X, X)$ such that

i) $\eta(t,u) = u$ if $t=0$ or if $u \notin \varphi^{-1}([c-2\varepsilon, c+2\varepsilon] \cap S_{2\delta})$,

ii) $\eta(1, \varphi^{c+\varepsilon} \cap S) \subset \varphi^{c-\varepsilon}$,

iii) $\eta(t, \cdot)$ is an homeomorphism of X, $\forall t \in [0,1]$,

iv) $||\eta(t,u) - u|| \leq \delta, \quad \forall u \in X, \quad \forall t \in [0,1]$,

v) $\varphi(\eta(\cdot, u))$ is non-increasing on $[0,1]$, $\forall u \in X$,

vi) $\varphi(\eta(t,u)) < c, \quad \forall u \in \varphi^c \cap S_\delta, \quad \forall t \in (0,1]$.

Let $\Gamma = \{\gamma \in C(B_F(0,r), B_F(0,r)+\Omega) \mid \gamma_{|\partial B_F(0,r)} = id\}$ and

$$c = \inf_{\gamma \in \Gamma} \max_{x \in B_F(0,r)} \varphi(\gamma(x)). \tag{3.2}$$

Taking $\gamma_0 = id_{B_F(0,r)} \in \Gamma$, it follows from assumption i) in Theorem 3.1 that $c \leq 0$. We claim that $c \geq \mu_1$. Indeed, let p_F be the canonical projection from E onto F. For any $\gamma \in \Gamma$, $p_F \circ \gamma$ is a continuous mapping of $B_F(0,r)$ into itself and $p_F \circ \gamma_{|\partial B_F(0,r)} = id$, so that there exists $x_\gamma \in B_F(0,r)$ such that $p_F \circ \gamma(x_\gamma) = 0$, that is $\gamma(x_\gamma) \in \Omega$ (at this point we use the fact that F is finite-dimensional). From assumption iii) we have $\varphi(\gamma(x_\gamma)) \geq \mu_1$, so obviously $\max\limits_{x \in B_F(0,r)} \varphi(\gamma(x)) \geq \mu_1$, which proves the claim.

If $c = 0$, the infimum in (3.2) is achieved for $\gamma_0 = id_{B_F(0,r)}$. We claim that in this case there exists a critical point of φ in $S = \{x \in B_F(0,r) \mid \varphi(x) = 0\}$. Indeed, suppose that this is false. Since S is compact and $S \subset Int(B_F(0,r) + \Omega)$, there exists $\varepsilon_0 > 0$ such that

$$(3.3) \qquad ||\varphi'(x)|| \geq 16\varepsilon_0, \ \ \forall x \in S_{\varepsilon_0} \text{ and } dist(S, \partial(B_F(0,r)+\Omega)) > 2\varepsilon_0.$$

We may apply Lemma 3.4 to φ, S, $c = 0$, $\delta = \frac{1}{2}\varepsilon_0$ and $\varepsilon = \varepsilon_0^2$ and we obtain a continuous mapping $\eta : [0,1] \times E \longrightarrow E$ with properties i)-vi) in that Lemma. Define $\gamma_1 : B_F(0,r) \longrightarrow E$ by $\gamma_1(x) = \eta(1,x)$. By (3.3) and Lemma 3.4 i) and iii) it follows that $\gamma_1 \in \Gamma$ and from Lemma 3.4 ii) and v) we infer that $\gamma_1(x) \leq -\varepsilon$, $\forall x \in S$, so $\max\limits_{x \in B_F(0,r)} \varphi(\gamma_1(x)) < 0$, contrary to the assumption that $c = 0$.

Hence Theorem 3.1 is proved in the case $c = 0$. From now on we may assume that $c < 0$. Let $S = \{x + y \mid x \in B_F(0,r_1), y \in \Omega, dist(y, \partial\Omega) \geq \frac{\delta_0}{2}\}$. Let $0 < \tilde{\delta} < \frac{1}{4} dist(S, \partial(B_F(0,r)+\Omega))$. To prove Theorem 3.1, it suffices to show that for any $\varepsilon > 0$ such that $c + 2\varepsilon < 0$ and $c - 2\varepsilon > \mu_0$, there exists $z_\varepsilon \in S_{2\tilde{\delta}}$ such that

$$(3.4) \qquad c - 2\varepsilon \leq \varphi(z_\varepsilon) \leq c + 2\varepsilon \quad \text{ and } \quad ||\varphi'(z_\varepsilon)|| < \frac{8\varepsilon}{\tilde{\delta}}.$$

Suppose that this thesis is false. Consider h and δ_0 as given by assumption iv). Define $h_0 : \Omega(\delta_0) \longrightarrow [0,r]$ by

$$h_0(y) = \begin{cases} r & \text{if } dist(y,\partial\Omega) < \frac{\delta_0}{2} \\ \frac{2}{\delta_0}(h(y) - r) \cdot dist(y, \partial\Omega) + 2r - h(y) & \text{if } \frac{\delta_0}{2} \leq dist(y,\partial\Omega) \leq \delta_0. \end{cases}$$

It is clear that h_0 is continuous and $h_0(y) \geq h(y)$. Let

$$\begin{aligned} W &= (B_F(0,r)+\Omega)) \setminus \{x + y \mid y \in \Omega(\delta_0),\ ||x|| < h(y)\} \quad \text{and} \\ W_0 &= (B_F(0,r)+\Omega)) \setminus \{x + y \mid y \in \Omega(\delta_0),\ ||x|| < h_0(y)\} \end{aligned}$$

Observe that $z \in W_0$ and $\varphi(z) \geq c - 2\varepsilon$ implies $z \in S$. Define $\psi : W \longrightarrow W_0$ by

$$\psi(x+y) = \begin{cases} h_0(y)\frac{x}{||x||} + y & \text{if } y \in \Omega \text{ and } h(y) \leq ||x|| \leq h_0(y) \\ x + y & \text{otherwise.} \end{cases}$$

It is easy to see that ψ is continuous and in view of assumption iv) we have $\varphi(z) \geq \varphi(\psi(z))$, $\forall z \in W$.

If ε is such that $\mu_0 < c - 2\varepsilon$ and $c + 2\varepsilon < 0$, consider $\gamma \in \Gamma$ such that $\max\limits_{x \in B_F(0,r)} \varphi(\gamma(x)) < c + \varepsilon$. Since $\varphi(x+y) \geq 0 > c + \varepsilon$ if $y \in \Omega(\delta_0)$ and $||x|| < h(y)$, we have necessarily $\gamma(x) \in W$, $\forall x \in B_F(0,r)$. Let $\gamma_2 = \psi \circ \gamma$. Then $\gamma_2 \in \Gamma$ and $\max\limits_{x \in B_F(0,r)} \varphi(\gamma_2(x)) \leq \max\limits_{x \in B_F(0,r)} \varphi(\gamma(x)) < c + \varepsilon$.

We apply Lemma 3.4 for the functional φ, the set S, c, ε and $\tilde{\delta}$ and we get $\eta \in C([0,1] \times E, E)$ with properties i)-vi) in that Lemma. Let $\gamma_3(x) = \eta(1, \gamma_2(x))$, $x \in B_F(0,r)$. Since $\varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap \gamma_2(B_F(0,r)) \subset \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap W_0 \subset S$ and $dist(S, \partial(B_F(0,r)+\Omega)) > 2\tilde{\delta}$, we infer from Lemma 3.4 i) and iv) that $\gamma_3(x) \in B_F(0,r)+\Omega, \forall x \in B_F(0,r)$ and $g_{3|\partial B_F(0,r)} = id$, hence $\gamma_3 \in \Gamma$. From Lemma 3.4, ii) it follows that $\max\limits_{x \in B_F(0,r)} \varphi(\gamma_3(x)) < c - \varepsilon$, contrary to (3.2). This contradiction proves Theorem 3.1. $\square$

4 Application to the functional E_c

We have already introduced the functionals E and E_c in Introduction. In this section we study the behaviour of the functional E_c near the ground state u_0 of (1.8) given by Theorem 1.1 and we prove that E_c admits a nontrivial critical point if c is sufficiently small. Let us verify first that E and E_c are well-defined on $\mathbf{H}$ and of class C^2 if $N \geq 4$. It is clear that the mapping $(u_1, u_2) \longmapsto V((r_0-u_1)^2+u_2^2)$ is of class $C^2(\mathbf{R}^2)$. We have $\sigma = \frac{4}{N-2} \leq 2$ because $N \geq 4$. Taking into account that for $\alpha > \beta$, $|u|^\alpha \leq C|u|^\beta$ for $|u|$ small, respectively $|u|^\beta \leq C|u|^\alpha$ for $|u|$ large, the following estimates hold:

$$\begin{aligned}
|V((r_0-u_1)^2+u_2^2)| &\leq C|-2r_0u_1+u_1^2+u_2^2|^2\chi_{\{u_1^2+u_2^2\leq 4r_0^2\}} \\
&\quad +C|u_1^2+u_2^2|^{\frac{\sigma}{2}+1}\chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\leq C'(|u_1|^2+|u_1|^{2+\sigma}+|u_2|^{2+\sigma}),
\end{aligned}$$

$$\begin{aligned}
|\frac{\partial}{\partial u_1}V((r_0-u_1)^2+u_2^2)| &= |2F((r_0-u_1)^2+u_2^2)(r_0-u_1)| \\
&\leq C|-2r_0u_1+u_1^2+u_2^2|\chi_{\{u_1^2+u_2^2\leq 4r_0^2\}} \\
&\quad +C((r_0-u_1)^2+u_2^2)^{\frac{\sigma}{2}}|r_0-u_1|\chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\leq C'(|u_1|+|u_1|^{1+\frac{\sigma}{2}}+|u_2|^{1+\frac{\sigma}{2}})+C'(|u_1|^{1+\sigma}+|u_2|^{1+\sigma}),
\end{aligned}$$

$$\begin{aligned}
|\frac{\partial}{\partial u_2}V((r_0-u_1)^2+u_2^2)| &= |-2F((r_0-u_1)^2+u_2^2)u_2| \\
&\leq C|-2r_0u_1+u_1^2+u_2^2|\cdot|u_2|\chi_{\{u_1^2+u_2^2\leq 4r_0^2\}} \\
&\quad +C((r_0-u_1)^2+u_2^2)^{\frac{\sigma}{2}}|u_2|\chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\leq C'(|u_1|^{\frac{2\sigma}{2+\sigma}}+|u_1|^\sigma+|u_2|^\sigma)|u_2|,
\end{aligned}$$

$$\begin{aligned}
|\frac{\partial^2}{\partial u_1^2}V((r_0-u_1)^2+u_2^2)| &= |-4F'((r_0-u_1)^2+u_2^2)(r_0-u_1)^2-2F((r_0-u_1)^2+u_2^2)| \\
&\leq C\chi_{\{u_1^2+u_2^2\leq 4r_0^2\}}+C(((r_0-u_1)^2+u_2^2)^{\frac{\sigma}{2}}\chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\leq C'(1+|u_1|^\sigma+|u_2|^\sigma)
\end{aligned}$$

$$\begin{aligned}
|\frac{\partial^2}{\partial u_1\partial u_2}V((r_0-u_1)^2+u_2^2)| &= |4F'((r_0-u_1)^2+u_2^2)(r_0-u_1)u_2| \\
&\leq C|u_2|\chi_{\{u_1^2+u_2^2\leq 4r_0^2\}} \\
&\quad +C((r_0-u_1)^2+u_2^2)^{-1+\frac{\sigma}{2}}|r_0-u_1||u_2|\chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\leq C'|u_2|^{\frac{\sigma}{2}}+C'(|u_1|^\sigma+|u_2|^\sigma),
\end{aligned}$$

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial u_2^2} V((r_0-u_1)^2+u_2^2)\right| &= |-4F'((r_0-u_1)^2+u_2^2)u_2^2 - 2F((r_0-u_1)^2+u_2^2)| \\
&\le C(|u_2|^2 + |-2r_0u_1+u_1^2+u_2^2|)\chi_{\{u_1^2+u_2^2\le 4r_0^2\}} \\
&\quad + C((r_0-u_1)^2+u_2^2)^\sigma \chi_{\{u_1^2+u_2^2>4r_0^2\}} \\
&\le C'(|u_2|^\sigma + |u_1|) + C'(|u_1|^\sigma + |u_2|^\sigma) \\
&\le C''(|u_1|^{\frac{2\sigma}{2+\sigma}} + |u_1|^\sigma + |u_2|^\sigma).
\end{aligned}
$$

From these estimates it follows that I is a C^2-functional from $(L^2 \cap L^{2+\sigma}(\mathbf{R}^N)) \times L^{2+\sigma}(\mathbf{R}^N)$ to $\mathbf{R}$. In view of the Sobolev embedding, I is of class C^2 on $\mathbf{H} = H^1(\mathbf{R}^N) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ and consequently so are E and E_c.

In order to apply Theorem 3.1 to the functional E_c near u_0, we are interested in the geometry of the level sets of E and E_c in a neighbourhood of u_0. We can get some basic information about the behaviour of E and E_c near u_0 by studying the differential $E'(u_0, 0)$.

We have already seen that u_0 is a critical point of E, that is $d_{u_1}E(u_0,0) = 0$ and $d_{u_2}E(u_0,0) = 0$. An easy calculation gives $d^2_{u_1,u_1}E(u_0,0).(v,v) = 2\langle Av, v\rangle$, where A is the operator introduced in (1.9), and $d^2_{u_1,u_2}E(u_0,0) = 0$. We have:

Lemma 4.1. $d^2_{u_2,u_2}E(u_0,0).(v,v) = 2\int_{\mathbf{R}^N}(r_0-u_0)^2\left|\nabla\left(\frac{v}{r_0-u_0}\right)\right|^2 dx.$

Proof. In view of Theorem 1.1, the linear mapping $v \longmapsto (r_0-u_0)v$ is a continuous isomorphism of $\mathcal{D}^{1,2}(\mathbf{R}^N)$ and its inverse is $w \longmapsto \frac{w}{r_0-u_0}$. Using equation (1.8) satisfied by u_0 and integrating by parts we get

$$
\begin{aligned}
&\int_{\mathbf{R}^N} F((r_0-u_0)^2)(r_0-u_0)^2v^2dx = \int_{\mathbf{R}^N}(\Delta u_0)(r_0-u_0)v^2dx \\
&= -\int_{\mathbf{R}^N}(r_0-u_0)\Delta(r_0-u_0)v^2dx \\
&= \int_{\mathbf{R}^N}|\nabla(r_0-u_0)|^2v^2dx + 2\int_{\mathbf{R}^N}(r_0-u_0)v\nabla(r_0-u_0).\nabla v\,dx,
\end{aligned}
$$

so we obtain

$$
\begin{aligned}
&d^2_{u_2,u_2}E(u_0,0).((r_0-u_0)v,(r_0-u_0)v) \\
&= 2\int_{\mathbf{R}^N}|\nabla((r_0-u_0)v)|^2dx + \int_{\mathbf{R}^N}\frac{\partial^2}{\partial u_2^2}(V((r_0-u_1)^2+u_2^2))_{|u_1=u_0,u_2=0}.(r_0-u_0)^2v^2dx \\
&= 2\int_{\mathbf{R}^N}|\nabla((r_0-u_0)v)|^2dx - 2\int_{\mathbf{R}^N}F((r_0-u_0)^2)(r_0-u_0)^2v^2dx \\
&= 2\int_{\mathbf{R}^N}(r_0-u_0)^2|\nabla v|^2dx.
\end{aligned}
$$

This proves Lemma 4.1. □

Let $H(v) = \int_{\mathbf{R}^N}(r_0-u_0)^2|\nabla\left(\frac{v}{r_0-u_0}\right)|^2dx$. Note that $H(v)^{\frac{1}{2}}$ defines a norm on $\mathcal{D}^{1,2}(\mathbf{R}^N)$ equivalent to the usual norm $||v||_{\mathcal{D}^{1,2}} = \left(\int_{\mathbf{R}^N}|\nabla v|^2dx\right)^{\frac{1}{2}}$.

Because we have not good estimates of $E(u_0+u_1,0) = E(u_0)$ for $u_1 \in Ker(A)$, we work for the moment only on the space $(\mathbf{R}e_1 + Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ and we show that the restriction of E_c to this space admits a critical point near u_0 for c small. It will be seen later that this is in fact a critical point of E_c on the whole $\mathbf{H}$.

Since E is of class C^2 and $E'(u_0,0)=0$, $d^2_{u_1,u_2}E(u_0,0)=0$, using the Taylor expansion we may write for $u_1\in Y$, $u_2\in\mathcal{D}^{1,2}(\mathbf{R}^N)$ with $||(u_1,u_2)||_{\mathbf{H}}$ small and $t\in\mathbf{R}$, t small

$$E(u_0+u_1+te_1,u_2)=E(u_0,0)+\langle A(u_1+te_1),(u_1+te_1)\rangle+H(u_2)+h(t,u_1,u_2) \tag{4.1}$$

and

$$\begin{aligned} d_{u_1}E(u_0+u_1+te_1,u_2) &= d^2_{u_1,u_1}E(u_0,0)(u_1+te_1,\cdot) \\ &\quad +d^2_{u_1,u_2}E(u_0,0)(\cdot,u_2)+L(t,u_1,u_2) \\ &= 2A(u_1+te_1)+L(t,u_1,u_2) \end{aligned} \tag{4.2}$$

where $h:\mathbf{R}\times Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)\longrightarrow\mathbf{R}$, $L:\mathbf{R}\times Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)\longrightarrow H^{-1}(\mathbf{R}^N)$, $|h(t,u_1,u_2)|=o(|t|^2+||(u_1,u_2)||^2_{\mathbf{H}})$ and $||L(t,u_1,u_2)=o(|t|+||(u_1,u_2)||_{\mathbf{H}})$ as $(t,u_1,u_2)\longrightarrow(0,0,0)$.

For each $\varepsilon>0$ consider $t_\varepsilon,\ r_\varepsilon>0$ such that

$$\begin{aligned} &|h(t,u_1,u_2)|\le\varepsilon(|t|^2+||(u_1,u_2)||^2_{\mathbf{H}}) \quad \text{and} \\ &||L(t,u_1,u_2)||\le\varepsilon(|t|+||(u_1,u_2)||_{\mathbf{H}}) \end{aligned} \tag{4.3}$$

if $|t|\le t_\varepsilon$ and $||(u_1,u_2)||_{\mathbf{H}}\le r_\varepsilon$. For $|t|\le t_\varepsilon$ we have

$$E(u_0+te_1,0)-E(u_0,0)=t^2\langle Ae_1,e_1\rangle+h(t,0,0)\le-\lambda_1t^2+\varepsilon t^2. \tag{4.4}$$

If $u_1\in Y$ and $u_2\in\mathcal{D}^{1,2}(\mathbf{R}^N)$, it follows from Lemmas 2.4 and 4.1 that there exist two positive constants $\gamma_1,\ \gamma_2$ such that

$$\gamma_1||(u_1,u_2)||^2_{\mathbf{H}}\le\langle Au_1,u_1\rangle+H(u_2)\le\gamma_2||(u_1,u_2)||^2_{\mathbf{H}}. \tag{4.5}$$

Next, we show that E is "small" in a cone $\{te_1+u_1+iu_2\in(\mathbf{R}e_1+Y)\times\mathcal{D}^{1,2}(\mathbf{R}^N)\mid ||(u_1,u_2)||_{\mathbf{H}}\le kt,\ t\in[-\tilde t_\varepsilon,\tilde t_\varepsilon]\ \}$ and is "large" in a cone $\{te_1+u_1+iu_2\in(\mathbf{R}e_1+Y)\times\mathcal{D}^{1,2}(\mathbf{R}^N)\mid t\le l||(u_1,u_2)||_{\mathbf{H}},\ ||(u_1,u_2)||_{\mathbf{H}}\le\tilde r_\varepsilon\ \}$, where k and l do not depend on ε.

Let $\varepsilon\le\min(1,\frac{\lambda_1}{4},\frac{\gamma_1}{4})$. Let $k=\sqrt{\frac{\lambda_1}{4(1+\gamma_2)}}$. If $|t|\le\min(t_\varepsilon,\frac{r_\varepsilon}{k})$ and $||(u_1,u_2)||_{\mathbf{H}}\le k|t|$, by (4.1) and (4.3) we have

$$\begin{aligned} &E(u_0+u_1+te_1,u_2)-E(u_0,0) \\ &\le-\lambda_1t^2+\gamma_2||(u_1,u_2)||^2_{\mathbf{H}}+\varepsilon(t^2+||(u_1,u_2)||^2_{\mathbf{H}}) \\ &\le-\lambda_1t^2+\gamma_2k^2t^2+\varepsilon(1+k^2)t^2\le-\tfrac{\lambda_1}{2}t^2. \end{aligned} \tag{4.6}$$

Let $l=\frac14\sqrt{\frac{\gamma_1}{1+\lambda_1}}$. If $||(u_1,u_2)||_{\mathbf{H}}\le\min(r_\varepsilon,\frac{t_\varepsilon}{l})$ and $|t|\le l||(u_1,u_2)||_{\mathbf{H}}$ we have

$$\begin{aligned} &E(u_0+u_1+te_1,u_2)-E(u_0,0) \\ &\ge-\lambda_1t^2+\gamma_1||(u_1,u_2)||^2_{\mathbf{H}}-\varepsilon(t^2+||(u_1,u_2)||^2_{\mathbf{H}}) \\ &\ge||(u_1,u_2)||^2_{\mathbf{H}}(\gamma_1-\lambda_1l^2-\varepsilon l^2-\varepsilon) \\ &\ge\tfrac{\gamma_1}{2}||(u_1,u_2)||^2_{\mathbf{H}}. \end{aligned} \tag{4.7}$$

From now on, we consider throughout that $0<\varepsilon<\min(1,\frac{\lambda_1}{4},\frac{\gamma_1}{4},\frac{3l\lambda_1}{4||e_1||_{H^1}})$. The next lemma says that assumption iv) in Theorem 3.1 is satisfied.

Lemma 4.2. *There exists $c_0 > 0$ such that for any $c \in [-c_0, c_0]$ and any $(u_1, u_2) \in Y \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ with $||(u_1, u_2)||_{\mathbf{H}} \leq \min(r_\varepsilon, \frac{t_\varepsilon}{l})$ the function*

$$t \longmapsto E_c(u_0 + u_1 + te_1, u_2)$$

is increasing on $[-t_\varepsilon, -l||(u_1, u_2)||_{\mathbf{H}}]$ and decreasing on $[l||(u_1, u_2)||_{\mathbf{H}}, t_\varepsilon]$.

Proof. Using (4.2), (4.3) and the identities $\langle Au_1, e_1\rangle = 0$, $\langle Ae_1, e_1\rangle = -\lambda_1$, we obtain on $[-t_\varepsilon, -l||(u_1, u_2)||_{\mathbf{H}}]$:

$$\frac{d}{dt}E_c(u_0 + u_1 + te_1, u_2) = d_{u_1}E(u_0 + u_1 + te_1, u_2).e_1 - 2c\int_{\mathbf{R}^N} e_1 u_{2x_1}dx$$

$$= 2\langle A(u_1 + te_1), e_1\rangle + L(t, u_1, u_2)e_1 - 2c\int_{\mathbf{R}^N} e_1 u_{2x_1}dx$$

$$\geq -2\lambda_1 t - \varepsilon(|t| + ||(u_1, u_2)||_{\mathbf{H}})||e_1||_{H^1} - 2|c| \cdot ||u_2||_{\mathcal{D}^{1,2}}$$

$$\geq (2\lambda_1 - \varepsilon)|t| - (\varepsilon||e_1||_{H^1} + 2|c|)||(u_1, u_2)||_{\mathbf{H}}$$

$$\geq [(2\lambda_1 - \varepsilon)l - (\varepsilon||e_1||_{H^1} + 2|c|)] \cdot ||(u_1, u_2)||_{\mathbf{H}}.$$

Taking $c_0 = \frac{l\lambda_1}{2}$, since $\varepsilon < \min(\frac{\lambda_1}{4}, \frac{3l\lambda_1}{4||e_1||_{\mathbf{H}}})$, it is clear that the last quantity is positive for $|c| < c_0$. A similar estimate holds on $[l||(u_1, u_2)||_{\mathbf{H}}, t_\varepsilon]$. $\square$

Theorem 4.3. *There exists $c_1 > 0$ such that for all $c \in [-c_1, c_1]$, the functional $\varphi_c(u_1, u_2) = E_c(u_0 + u_1, u_2) - E_c(u_0, 0)$ restricted to $(\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ admits a critical point $(u_{1,c}, u_{2,c})$. Moreover, $(u_{1,c}, u_{2,c}) \longrightarrow (0, 0)$ as $c \longrightarrow 0$.*

Proof. Let $t_0 = \min(t_\varepsilon, \frac{r_\varepsilon}{k})$. Let $r_0 = \min(r_\varepsilon, \frac{t_\varepsilon}{l}, kt_0)$. Now fix $t \in (0, t_0]$ and let $r(t) = \min(r_0, kt)$. If c is sufficiently small, we show that the assumptions of Theorem 3.1 are satisfied for $F = \mathbf{R}e_1$, $G = Y \times \mathcal{D}^{1,2}(\mathbf{R}^N)$, $B_F(0, r) = [-t, t]e_1$, $\Omega = B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t))$, $\mu_0 = -\frac{\lambda_1}{4}t^2$, $\mu_1 = -\frac{\lambda_1}{8}t^2$, $\delta_0 = \frac{r(t)}{2}$ and $h(u_1, u_2) = l||(u_1, u_2)||_{\mathbf{H}}$.

If $\tau \in [-t, t]$, using (4.4) we have

$$\begin{aligned} &\varphi_c(\tau e_1, 0) = E_c(u_0 + \tau e_1, 0) - E_c(u_0, 0) \\ &= E(u_0 + \tau e_1, 0) - E(u_0, 0) \leq (-\lambda_1 + \varepsilon)\tau^2. \end{aligned} \tag{4.8}$$

Because $0 < \varepsilon < \frac{\lambda_1}{4}$, assumption i) is satisfied.

Since Q is bounded on bounded sets of $\mathbf{H}$, there exists $c(t) \in (0, c_0]$ such that for any c with $|c| \leq c(t)$,

$$|cQ(u_0 + u_1 \pm te_1, u_2)| < \frac{\lambda_1}{4}t^2 \text{ for } (u_1, u_2) \in B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t)) \quad \text{and} \tag{4.9}$$

$$|cQ(u_0 + u_1 + \tau e_1, u_2)| < \min(\frac{\lambda_1}{8}t^2, \frac{\gamma_1}{16}r(t)^2) \tag{4.10}$$

for $(u_1, u_2) \in B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t))$ and $|\tau| \leq l||(u_1, u_2)||_{\mathbf{H}}$.

If $|c| \leq c(t)$ and $(u_1, u_2) \in B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t))$, by (4.6), the choice of $r(t)$ and (4.9) we have

$$\begin{aligned} &\varphi_c(\pm te_1 + u_1, u_2) = E(u_0 + u_1 \pm te_1, u_2) - E(u_0, 0) \\ &+Q(u_0 + u_1 \pm te_1, u_2) \leq -\frac{\lambda_1}{2}t^2 + \frac{\lambda_1}{4}t^2 = -\frac{\lambda_1}{4}t^2, \end{aligned} \tag{4.11}$$

Since φ_c' is bounded on bounded sets of $\mathbf{H}$, assumption ii) is verified (see also Remark 3.3).

Using (4.7) and (4.10) we get for $(u_1, u_2) \in B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t))$

$$\begin{aligned}\varphi_c(u_1, u_2) &= E(u_0 + u_1, u_2) - E(u_0, 0) + cQ(u_0 + u_1, u_2)\\ &\geq \tfrac{\gamma_1}{2}||(u_1, u_2)||_{\mathbf{H}} - \tfrac{\lambda_1}{8}t^2 \geq -\tfrac{\lambda_1}{8}t^2,\end{aligned} \tag{4.12}$$

thus assumption iii) holds. It follows from Lemma 4.2 that hypothesis iv) is verified. Also, for $|c| \leq c(t)$, if $(u_1, u_2) \in Y \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ are such that $\frac{r(t)}{2} \leq ||(u_1, u_2)||_{\mathbf{H}} \leq r(t)$ and $|\tau| \leq l||(u_1, u_2)||_{\mathbf{H}}$, we have by (4.7) and (4.10)

$$\begin{aligned}\varphi_c(\tau e_1 + u_1, u_2) &= E_c(u_0 + u_1 + \tau e_1, u_2) - E_c(u_0, 0)\\ &\geq \tfrac{\gamma_1}{2}||(u_1, u_2)||^2_{\mathbf{H}} - \tfrac{\gamma_1}{16}r(t)^2 \geq \tfrac{\gamma_1}{16}r(t)^2,\end{aligned} \tag{4.13}$$

so that assumption v) is satisfied. Hence we may apply Theorem 3.1 and we obtain a Palais-Smale sequence $(u^n_{1,c}, u^n_{2,c})$ for the functional φ_c restricted to $(\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. Moreover, $(u^n_{1,c}, u^n_{2,c}) \in [-t, t]e_1 + B_{Y\times\mathcal{D}^{1,2}(\mathbf{R}^N)}(0, r(t))$ for any n. Since $(u^n_{1,c})$, $(u^n_{2,c})$ are bounded in $H^1(\mathbf{R}^N)$, respectively in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, we may extract a subsequence (still denoted $(u^n_{1,c})$, $(u^n_{2,c})$) such that

$$\begin{array}{lll} u^n_{1,c} \rightharpoonup u_{1,c} & \text{weakly in } H^1(\mathbf{R}^N) & \\ u^n_{1,c} \longrightarrow u_{1,c} & \text{a.e. and in } L^p_{loc}, & \forall p \in [1, 2+\sigma) \\ u^n_{2,c} \rightharpoonup u_{2,c} & \text{weakly in } \mathcal{D}^{1,2}(\mathbf{R}^N) & \\ u^n_{2,c} \longrightarrow u_{2,c} & \text{a.e. and in } L^p_{loc}, & \forall p \in [1, 2+\sigma). \end{array} \tag{4.14}$$

It is clear that $||u_{1,c}||_{H^1} \leq t + r(t)$ and $||u_{2,c}||_{\mathcal{D}^{1,2}} \leq r(t)$. Let $(v_1, v_2) \in (\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. By weak convergence it is obvious that

$$T'(u_0 + u^n_{1,c}, u^n_{2,c}).(v_1, v_2) \longrightarrow T'(u_0 + u_{1,c}, u_{2,c}).(v_1, v_2) \text{ as } n \longrightarrow \infty, \tag{4.15}$$

$$Q'(u_0 + u^n_{1,c}, u^n_{2,c}).(v_1, v_2) \longrightarrow Q'(u_0 + u_{1,c}, u_{2,c}).(v_1, v_2) \text{ as } n \longrightarrow \infty. \tag{4.16}$$

On the other hand, it follows from the estimates at the beginning of this section that

$$\begin{aligned}&F((r_0 - u_0 - u^n_{1,c})^2 + (u^n_{2,c})^2)(r_0 - u_0 - u^n_{1,c}) \text{ is bounded in } L^2 + L^{\frac{2+\sigma}{1+\sigma}}(\mathbf{R}^N) \text{ and}\\ &F((r_0 - u_0 - u^n_{1,c})^2 + (u^n_{2,c})^2)u^n_{2,c} \text{ is bounded in } L^{\frac{2+\sigma}{1+\sigma}}(\mathbf{R}^N).\end{aligned}$$

Passing again to a subsequence, we may assume that

$$\begin{aligned}&F((r_0 - u_0 - u^n_{1,c})^2 + (u^n_{2,c})^2)(r_0 - u_0 - u^n_{1,c}) \rightharpoonup f_1 \text{ weakly in } L^2 + L^{\frac{2+\sigma}{1+\sigma}}(\mathbf{R}^N)\\ &F((r_0 - u_0 - u^n_{1,c})^2 + (u^n_{2,c})^2)u^n_{2,c} \rightharpoonup f_2 \text{ weakly in } L^{\frac{2+\sigma}{1+\sigma}}(\mathbf{R}^N).\end{aligned}$$

In view of the estimates at the beginning of Section 4 and of the convergence $u^n_{1,c} \longrightarrow u_{1,c}$, $u^n_{2,c} \longrightarrow u_{2,c}$ in $L^p_{loc}(\mathbf{R}^N)$, $1 \leq p < 2+\sigma$, we have $F((r_0 - u_0 - u^n_{1,c})^2 + (u^n_{2,c})^2)(r_0 - u_0 - u^n_{1,c}) \longrightarrow F((r_0 - u_0 - u_{1,c})^2 + u^2_{2,c})(r_0 - u_0 - u_{1,c})$ and $F((r_0 - u_0 - u^n_{1,c})^2 + (u^n_{2,c})^2)u^n_{2,c} \longrightarrow F((r_0 - u_0 - u_{1,c})^2 + u^2_{2,c})u_{2,c}$ in $L^q_{loc}(\mathbf{R}^N)$, $1 \leq q < \frac{2+\sigma}{1+\sigma}$. By the uniqueness of the limit in $\mathcal{D}'(\mathbf{R}^N)$ we infer that $f_1 =$

$F((r_0 - u_0 - u_{1,c})^2 + u_{2,c}^2)(r_0 - u_0 - u_{1,c})$ and $f_2 = F((r_0 - u_0 - u_{1,c})^2 + u_{2,c}^2)u_{2,c}$. Now the weak convergence implies that

$$(4.17)\quad I'(u_0+u_{1,c}^n, u_{2,c}^n).(v_1, v_2) \longrightarrow 2\int_{\mathbf{R}^N} f_1 v_1 - f_2 v_2 dx = I'(u_0+u_{1,c}, u_{2,c}).(v_1, v_2).$$

Since $\lim_{n\to\infty} E_c'(u_0+u_{1,c}^n, u_{2,c}^n).(v_1, v_2) = 0$, from (4.15), (4.16) and (4.17) we infer that

$$(4.18)\quad E_c'(u_0 + u_{1,c}, u_{2,c}).(v_1, v_2) = 0 \quad \text{for all } (v_1, v_2) \in (\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N).$$

In conclusion, we have proved that for any $t \in (0, t_0]$ there exists $c(t) > 0$ such that for $|c| \le c(t)$, the restriction of φ_c to the space $(\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$ admits a critical point $(u_{1,c}, u_{2,c})$ and $||u_{1,c}||_{H^1} \le t + r(t)$, $||u_{2,c}||_{\mathcal{D}^{1,2}} \le r(t)$. The proof of Theorem 4.3 is completed. □

Theorem 4.4. *There exists $c_\star > 0$ such that for $|c| \le c_\star$, E_c admits a nontrivial critical point $u_c \in \mathbf{H}$. Moreover, $u_c \longrightarrow u_0$ as $c \longrightarrow 0$.*

Proof. Let $u_c = (u_0 + u_{1,c}, u_{2,c}) = u_0 + u_{1,c} + iu_{2,c}$ where $(u_{1,c}, u_{2,c})$ is given by Theorem 4.3. It follows from (4.18) that $E_c'(u_c) = 0$ on $(\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$, that is $d_{u_2}E_c(u_0+u_{1,c}, u_{2,c}) = 0$ on $\mathcal{D}^{1,2}(\mathbf{R}^N)$ and $d_{u_1}E_c(u_0+u_{1,c}, u_{2,c}) = 0$ on $\mathbf{R}e_1 \oplus Y = (Ker(A))^\perp \cap H^1(\mathbf{R}^N)$. All we have to do is to show that $d_{u_1}E_c(u_0+u_{1,c}, u_{2,c}) = 0$ on $Ker(A)$. For small c, this will be done thanks to the invariance of E_c by translations in $\mathbf{R}^N$. (Note also that $\frac{\partial u_0}{\partial x_i}$, $i = 1, \dots, N$ are in the kernel of A just because E is translation invariant).

It will be seen in the next section that $u_{1,c}$ and $u_{2,c}$ are in $H^2(\mathbf{R}^N)$, respectively in $\mathcal{D}^{1,2} \cap \mathcal{D}^{2,2}(\mathbf{R}^N)$, where $\mathcal{D}^{2,2}(\mathbf{R}^N) = \{v \in \mathcal{D}'(\mathbf{R}^N) \ |\nabla^2 v \in L^2(\mathbf{R}^N)\ \}$. Then for each $i \in \{1, \dots, N\}$, the mapping $t \longmapsto u_c(x_1, \dots, x_i + t, \dots, x_N)$ is C^1 from $\mathbf{R}$ to $\mathbf{H}$ and

$$(4.19)\qquad E_c(u_c(x_1, \dots, x_i + t, \dots, x_N)) = E_c(u_c), \quad \forall t \in \mathbf{R}.$$

Differentiating (4.19) at $t = 0$ we get

$$(4.20)\qquad E_c'(u_c).\frac{\partial u_c}{\partial x_i} = 0.$$

Because $d_{u_2}E_c(u_c) = 0$, (4.20) gives $d_{u_1}E_c(u_c).\left(\frac{\partial u_0}{\partial x_i} + \frac{\partial u_c}{\partial x_i}\right) = 0$. By Theorem 2.6 we have $H^1(\mathbf{R}^N) = \mathbf{R}e_1 + Y + Span\{\frac{\partial u_0}{\partial x_i},\ i = 1, \dots, N\}$, the sum being orthogonal in $L^2(\mathbf{R}^N)$. Note that $\frac{\partial u_0}{\partial x_i}$, $i = 1, \dots, N$ are orthogonal in $L^2(\mathbf{R}^N)$ and $\frac{\partial u_{1,c}}{\partial x_i} \longrightarrow 0$ in $L^2(\mathbf{R}^N)$ as $c \longrightarrow 0$. It follows that for c sufficiently small we also have $H^1(\mathbf{R}^N) = \mathbf{R}e_1 + Y + Span\{\left(\frac{\partial u_0}{\partial x_i} + \frac{\partial u_{1,c}}{\partial x_i}\right),\ i = 1, \dots, N\}$ and from (4.20) we deduce that $d_{u_1}E_c(u_c) = 0$ on $H^1(\mathbf{R}^N)$, as we need. Thus Theorem 4.4 is proved. □

Remark 4.5. Both the functional E_c and equation (1.2) are invariant by rotations in the $(x_2, \dots, x_N)$-variables. Therefore instead of working on $\mathbf{H}$, we could work on $\mathbf{H}_{1,rad} = \{u \in \mathbf{H} \ \mid\ u \text{ is radially symmetric in } (x_2, \dots, x_N)\}$. Our proofs remain valid without changes and we obtain a critical point $\tilde{u}_c$ of E_c on $\mathbf{H}_{1,rad}$ for $|c| \le c_\star$. Of course that in this case we know *à priori* that $E_c'(\tilde{u}_c).v = 0$ only for $v \in \mathbf{H}_{1,rad}$. Because the group G of rotations in $(x_2, \dots, x_N)$ acts isometrically on $\mathbf{H}$

and $Fix(G) = \mathbf{H}_{1,rad}$, from the Principle of Symmetric Criticality (see [9] or [13]) we obtain that in fact $\tilde{u}_c$ is a critical point of E_c on $\mathbf{H}$. Therefore we have the following:

Corollary 4.6. *If $|c| \le c_\star$, there exists a solution $\tilde{u}_c \in \mathbf{H}$ of (1.2) which is radially symmetric in the transverse variables $(x_2, \ldots, x_N)$. Moreover, $\tilde{u}_c \longrightarrow u_0$ in $\mathbf{H}$ as $c \longrightarrow 0$.*

5 Regularity

In this section we show that the critical points obtained in Theorem 4.3 are in $H^2(\mathbf{R}^N) \times \mathcal{D}^{2,2}(\mathbf{R}^N)$ (thus completing the proof of Theorem 4.4) and we obtain some other regularity properties of the solutions of equation (1.2). We begin with the following simple lemma:

Lemma 5.1. Let $(u_1, u_2) \in \mathbf{H}$ satisfy $E_c'(u_1, u_2).(v_1, v_2) = 0$, $\forall (v_1, v_2) \in (\mathbf{R}e_1 \oplus Y) \times \mathcal{D}^{1,2}(\mathbf{R}^N)$. Then

$$d_{u_1} E_c(u_1, u_2) \in Ker(A) \quad \text{ and } \quad d_{u_2} E_c(u_1, u_2) = 0.$$

Proof. It is obvious that $d_{u_2} E_c(u_1, u_2) = 0$. Let p_1, p_2 be the orthogonal projections of $L^2(\mathbf{R}^N)$ onto $Ker(A)$, respectively onto $\mathbf{R}e_1 \oplus Y$. It is clear that $d_{u_1} E_c(u_1, u_2).p_2 v = 0$ for all $v \in H^1(\mathbf{R}^N)$. Hence for any $v \in H^1(\mathbf{R}^N)$ we have

$$\begin{aligned} |\langle d_{u_1} E_c(u_1, u_2), v \rangle_{H^{-1}, H^1}| &= |\langle d_{u_1} E_c(u_1, u_2), p_1 v \rangle_{H^{-1}, H^1}| \\ &\le C ||p_1 v||_{H^1} \\ &\le C' ||p_1 v||_{L^2} \quad \text{because } Ker(A) \text{ is finite-dimensional} \\ &\le C' ||v||_{L^2}. \end{aligned}$$

By density of $H^1(\mathbf{R}^N)$ in $L^2(\mathbf{R}^N)$ we infer that $d_{u_1} E_c(u_1, u_2)$ has an unique extension as a bounded linear functional on $L^2(\mathbf{R}^N)$, hence $d_{u_1} E_c(u_1, u_2) \in L^2(\mathbf{R}^N)$. Observe that $\mathbf{R}e_1 \oplus Y = H^1(\mathbf{R}^N) \cap Im(A)$ is dense in $Im(A)$ and $Im(A)^\perp = Ker(A)$ because A is self-adjoint. Since $\langle d_{u_1} E_c(u_1, u_2), v \rangle = 0$, $\forall v \in \mathbf{R}e_1 \oplus Y = H^1(\mathbf{R}^N) \cap Im(A)$, by density we infer that $d_{u_1} E_c(u_1, u_2) \in Ker(A)$. $\square$

Lemma 5.2. *Suppose that $N \ge 4$ and $F \in C^1([0, \infty))$ satisfies*

i) $F(r_0^2) = 0$ and

ii) $F(x) \le 0$ and $|F(x)| \le x^{\frac{\sigma}{2}}$ for large x.

Let $u = u_1 + iu_2$ with $u_1 \in H^1(\mathbf{R}^N)$ and $u_2 \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ be a solution of the equation

$$icu_{x_1} - \Delta u + F(|r_0 - u|^2)(r_0 - u) = f_1 + if_2. \tag{5.1}$$

We have:

a) If $f_1,\ f_2 \in L^2(\mathbf{R}^N) \cap L^{2+\sigma}(\mathbf{R}^N)$, then $u_1 \in H^2(\mathbf{R}^N)$ and $u_2 \in \mathcal{D}^{1,2} \cap \mathcal{D}^{2,2}(\mathbf{R}^N)$.

b) If $f_1,\ f_2 \in L^q(\mathbf{R}^N), \forall q \in [2, \infty)$, then $u_1 \in W^{2,q}(\mathbf{R}^N),\ u_2 \in \mathcal{D}^{1,q} \cap \mathcal{D}^{2,q}(\mathbf{R}^N)$ $\forall q \in [2, \infty)$ and $u_2 \in W^{2,q}(\mathbf{R}^N)$, $\forall q \ge 2 + \sigma$.

Proof. Equation (5.1) is equivalent to the system

$$-cu_{2x_1} - \Delta u_1 + F((r_0 - u_1)^2 + u_2^2)(r_0 - u_1) = f_1 \tag{5.2}$$

$$(5.3) \qquad c u_{1x_1} - \Delta u_2 - F((r_0-u_1)^2+u_2^2)u_2 = f_2.$$

We show first that $u_1 \in L^{q_1}(\mathbf{R}^N)$ and $u_2 \in L^{q_2}(\mathbf{R}^N)$ with $q_1, q_2 \geq 2+2\sigma$. This step was inspired by the proof of Theorem 2.3 in [5]. For $i = 1, 2$ and $n \in \mathbf{N}$, let

$$u_i^n(x) = \begin{cases} -n & \text{if} \quad u_i(x) < -n \\ u_i(x) & \text{if} \quad -n \leq u_i(x) \leq n \\ n & \text{if} \quad u_i(x) > n. \end{cases}$$

It is clear that $u_1^n \in H^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, $u_2^n \in \mathcal{D}^{1,2}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $\nabla u_i^n = \chi_{\{-n \leq u_i \leq n\}} \nabla u_i$, $i = 1, 2$. Let $h_p(s) = |s|^{p-2}s$, $p \geq 2$. Then $h_p(u_1^n) \in H^1(\mathbf{R}^N)$ and $h_p(u_2^n) \in \mathcal{D}^{1,2}(\mathbf{R}^N)$. Multiplying (5.3) by $h_p(u_2^n)$ and integrating we get

$$(5.4) \qquad \begin{aligned} &(p-1)\int_{\mathbf{R}^N} |\nabla u_2^n|^2 |u_2^n|^{p-2} dx = \int_{\mathbf{R}^N} f_2 |u_2^n|^{p-2} u_2^n dx \\ &+ \int_{\{-n \leq u_i \leq n\}} F((r_0-u_1)^2+u_2^2)|u_2|^p dx \\ &+ \int_{\{u_2<-n\}\cup\{u_2>n\}} F((r_0-u_1)^2+u_2^2)|u_2| n^{p-1} dx \\ &- c\int_{\mathbf{R}^N} u_{1x_1} |u_2^n|^{p-2} u_2^n dx. \end{aligned}$$

Denoting by $F_{max} = \max\limits_{x \in [0,\infty)} F(x)$, we have:

$$\int_{\{-n \leq u_i \leq n\}} F((r_0-u_1)^2+u_2^2)|u_2|^p dx \leq F_{max} \int_{\mathbf{R}^N} |u_2|^p dx;$$

$$\int_{\{u_2<-n\}\cup\{u_2>n\}} F((r_0-u_1)^2+u_2^2)|u_2| n^{p-1} dx \leq 0 \text{ if } n \text{ is sufficiently large,}$$

$$\begin{aligned} \Big|c\int_{\mathbf{R}^N} u_{1x_1} |u_2^n|^{p-2} u_2^n dx\Big| &= \Big| -c(p-1)\int_{\mathbf{R}^N} u_1 |u_2^n|^{p-2} u^n_{2\,x_1} dx\Big| \\ &= \Big|\tfrac{-2c(p-1)}{p^2}\int_{\mathbf{R}^N} u_1 |u_2^n|^{\frac{p}{2}-2} u_2^n \cdot \frac{\partial}{\partial x_1}\left(|u_2^n|^{\frac{p}{2}}\right) dx\Big| \\ &\leq \tfrac{2(p-1)}{p^2}\int_{\mathbf{R}^N} \Big|\frac{\partial}{\partial x_1}\left(|u_2^n|^{\frac{p}{2}}\right)\Big|^2 dx + C(p)\int_{\mathbf{R}^N} |u_1|^2 |u_2|^{p-2} dx. \end{aligned}$$

Using the identity $\int_{\mathbf{R}^N} |\nabla u|^2 |u|^{p-2} dx = \dfrac{4}{p^2}\int_{\mathbf{R}^N} \Big|\nabla\left(|u|^{\frac{p}{2}}\right)\Big|^2 dx$, (5.4) gives

$$(5.5) \qquad \begin{aligned} &\tfrac{4(p-1)}{p^2}\int_{\mathbf{R}^N} \Big|\nabla |u_2^n|^{\frac{p}{2}}\Big|^2 dx \leq \int_{\mathbf{R}^N} |f_2||u_2|^{p-1} dx + F_{max}\int_{\mathbf{R}^N} |u_2|^p dx \\ &+ \tfrac{2(p-1)}{p^2}\int_{\mathbf{R}^N} \Big|\frac{\partial}{\partial x_1}\left(|u_2^n|^{\frac{p}{2}}\right)\Big|^2 dx + C(p)\int_{\mathbf{R}^N} |u_1|^2 |u_2|^{p-2} dx. \end{aligned}$$

Note that the right hand side of (5.5) may be infinite. Since $f_2 \in L^{2+\sigma}(\mathbf{R}^N)$ and $u_1, u_2 \in L^{2+\sigma}(\mathbf{R}^N)$ by the Sobolev embedding, taking $p = 2+\sigma$ in (5.5) we get

$$(5.6) \qquad \frac{2(p-1)}{p^2}\int_{\mathbf{R}^N} \Big|\nabla |u_2^n|^{\frac{2+\sigma}{2}}\Big|^2 dx \leq K,$$

where K does not depend on n. Passing to the limit as $n \longrightarrow \infty$ in (5.6) and using Lebesgue's dominated convergence theorem we infer that $\nabla\left(|u_2|^{\frac{2+\sigma}{2}}\right) \in L^2(\mathbf{R}^N)$. By the Sobolev embedding we obtain $|u_2|^{\frac{2+\sigma}{2}} \in L^{2+\sigma}(\mathbf{R}^N)$, that is $u_2 \in L^{\frac{(2+\sigma)^2}{2}}(\mathbf{R}^N)$.

Multiplying (5.2) by $h_p(u_1^n)$ and integrating we get

$$\begin{aligned}
&(p-1)\int_{\mathbf{R}^N} |\nabla u_1^n|^2 |u_1^n|^{p-2} dx = \int_{\mathbf{R}^N} f_1 |u_1^n|^{p-2} u_1^n dx \\
&+\int_{\{-n\le u_1\le n\}} F((r_0-u_1)^2+u_2^2)|u_1|^p dx \\
&+\int_{\{u_1<-n\}\cup\{u_1>n\}} F((r_0-u_1)^2+u_2^2)|u_1|n^{p-1} dx \\
&-r_0\int_{\mathbf{R}^N} F((r_0-u_1)^2+u_2^2)|u_1^n|^{p-2}u_1^n dx + c\int_{\mathbf{R}^N} u_{2x_1}|u_1^n|^{p-2}u_1^n dx.
\end{aligned} \tag{5.7}$$

We have $|F((r_0-z_1)^2+z_2^2)| \le C|-2r_0z_1+z_1^2+z_2^2|$ for $z_1^2+z_2^2 \le 4r_0^2$ and $|F((r_0-z_1)^2+z_2^2)| \le C((r_0-z_1)^2+z_2^2)^{\frac{\sigma}{2}} \le C'(|z_1|^\sigma+|z_2|^\sigma)$ for $z_1^2+z_2^2 > 4r_0^2$.

If $\sigma \le 1$ (that is, $N \ge 6$), then $|F((r_0-z_1)^2+z_2^2)| \le C(|z_1|+|z_2|)$ for all z_1, z_2 and proceeding as above we infer that

$$\begin{aligned}
&\tfrac{4(p-1)}{p^2}\int_{\mathbf{R}^N}\left|\nabla|u_1^n|^{\frac{p}{2}}\right|^2 dx \le \int_{\mathbf{R}^N}|f_1||u_1|^{p-1}dx + F_{max}\int_{\mathbf{R}^N}|u_1|^p dx \\
&+C\int_{\mathbf{R}^N}(|u_1|+|u_2|)|u_1|^{p-1}dx \\
&+\tfrac{2(p-1)}{p^2}\int_{\mathbf{R}^N}\left|\frac{\partial}{\partial x_1}\left(|u_1^n|^{\frac{p}{2}}\right)\right|^2 dx + C(p)\int_{\mathbf{R}^N}|u_2|^2|u_1|^{p-2}dx.
\end{aligned} \tag{5.8}$$

Of course, the right side of (5.8) may be infinite. Because $u_1 \in L^2 \cap L^{2+\sigma}(\mathbf{R}^N)$ and $u_2 \in L^{2+\sigma} \cap L^{\frac{(2+\sigma)^2}{2}}(\mathbf{R}^N)$, it is easy to see that

$$\int_{\mathbf{R}^N}|u_1|^p dx < \infty \text{ and } \int_{\mathbf{R}^N}|f_1||u_1|^{p-1}dx < \infty \text{ for } 2 \le p \le 2+\sigma,$$

$$\int_{\mathbf{R}^N}|u_2||u_1|^{p-1}dx < \infty \text{ for } 2+\tfrac{\sigma}{2+\sigma} \le p \le 2+\sigma+\tfrac{\sigma}{2+\sigma},$$

$$\int_{\mathbf{R}^N}|u_2|^2|u_1|^{p-2}dx < \infty \text{ for } 2+\tfrac{2\sigma}{2+\sigma} \le p \le 2+\sigma+\tfrac{2\sigma}{2+\sigma}$$

Taking $p = 2+\sigma$ in (5.8) we obtain

$$\frac{2(p-1)}{p^2}\int_{\mathbf{R}^N}\left|\nabla|u_1^n|^{\frac{2+\sigma}{2}}\right|^2 dx \le K < \infty, \tag{5.9}$$

where K does not depend on n. Passing to the limit as $n \longrightarrow \infty$ in (5.9) and using again Lebesgue's dominated convergence theorem we get that $\nabla\left(|u_1|^{\frac{2+\sigma}{2}}\right) \in L^2(\mathbf{R}^N)$ and therefore $u_1 \in L^{\frac{(2+\sigma)^2}{2}}(\mathbf{R}^N)$ by the Sobolev embedding.

If $2 \ge \sigma \ge 1$, we have $|F((r_0-z_1)^2+z_2^2)| \le C(|z_1|+|z_1|^\sigma+|z_2|^\sigma)$ for all $z_1,\ z_2$

(note that $\sigma \le 2$ because $N \ge 4$) so that (5.7) gives

$$
\begin{aligned}
&\tfrac{4(p-1)}{p^2} \int_{\mathbf{R}^N} \left|\nabla |u_1^n|^{\frac{p}{2}}\right|^2 dx \le \int_{\mathbf{R}^N} |f_1||u_1|^{p-1} dx + F_{max} \int_{\mathbf{R}^N} |u_1|^p dx \\
&+C \int_{\mathbf{R}^N} (|u_1| + |u_1|^\sigma + |u_2|^\sigma)|u_1|^{p-1} dx \qquad (5.10)\\
&+\tfrac{2(p-1)}{p^2} \int_{\mathbf{R}^N} \left|\frac{\partial}{\partial x_1}\left(|u_1^n|^{\frac{p}{2}}\right)\right|^2 dx + C(p) \int_{\mathbf{R}^N} |u_2|^2 |u_1|^{p-2} dx.
\end{aligned}
$$

Since $u_1 \in L^2 \cap L^{2+\sigma}(\mathbf{R}^N)$ and $u_2 \in L^{2+\sigma} \cap L^{\frac{(2+\sigma)^2}{2}}(\mathbf{R}^N)$, it is clear that

$$
\begin{aligned}
&\int_{\mathbf{R}^N} |u_1|^p dx < \infty \text{ and } \int_{\mathbf{R}^N} |f_1||u_1|^{p-1} dx < \infty \text{ for } 2 \le p \le 2+\sigma, \\
&\int_{\mathbf{R}^N} |u_1|^{p+\sigma-1} dx < \infty \text{ for } 3-\sigma \le p \le 3, \\
&\int_{\mathbf{R}^N} |u_2|^\sigma |u_1|^{p-1} dx < \infty \text{ for } 2 + \tfrac{2-\sigma}{2+\sigma} \le p \le 2+\sigma+\tfrac{2-\sigma}{2+\sigma}, \\
&\int_{\mathbf{R}^N} |u_2|^2 |u_1|^{p-2} dx < \infty \text{ for } 2 + \tfrac{2\sigma}{2+\sigma} \le p \le 2+\sigma+\tfrac{2\sigma}{2+\sigma}.
\end{aligned}
$$

Therefore for $p \in [2 + \frac{2\sigma}{2+\sigma}, 3]$ we obtain

$$
\frac{2(p-1)}{p^2} \int_{\mathbf{R}^N} \left|\nabla |u_1^n|^{\frac{p}{2}}\right|^2 dx \le K < \infty, \qquad (5.11)
$$

with K independent of n. As previously we get that $\nabla\left(|u_1|^{\frac{p}{2}}\right) \in L^2(\mathbf{R}^N)$ and $u_1 \in L^{\frac{(2+\sigma)p}{2}}(\mathbf{R}^N)$ by the Sobolev embedding. In particular, for $p = 3$ we obtain $u_1 \in L^{\frac{3}{2}(2+\sigma)}(\mathbf{R}^N)$. Thus we have proved that $u_1 \in L^{q_1}(\mathbf{R}^N)$ and $u_2 \in L^{q_2}(\mathbf{R}^N)$ with $q_1,\ q_2 \ge 2+2\sigma$.

From the above estimates it follows that

$$
\begin{aligned}
&|F((r_0-u_1)^2+u_2^2)(r_0-u_1)| \\
&\le C(|u_1|+|u_2|^2)\chi_{\{u_1^2+u_2^2 \le 4r_0^2\}} + C(|u_1|^\sigma+|u_2|^\sigma)(|u_1|+|u_2|)\chi_{\{u_1^2+u_2^2 > 4r_0^2\}} \\
&\le C'(|u_1|^+|u_2|^{1+\frac{\sigma}{2}})\chi_{\{u_1^2+u_2^2 \le 4r_0^2\}} + C'(|u_1|^{1+\sigma}+|u_2|^{1+\sigma}) \in L^2(\mathbf{R}^N)
\end{aligned}
$$

and similarly $F((r_0-u_1)^2+u_2^2)u_2 \in L^2(\mathbf{R}^N)$. From (5.2) and (5.3) we infer now that $\Delta u_1 \in L^2(\mathbf{R}^N)$ and $\Delta u_2 \in L^2(\mathbf{R}^N)$, which imply that $u_1 \in H^2(\mathbf{R}^N)$ and $u_2 \in \mathcal{D}^{2,2}(\mathbf{R}^N)$. This proves a).

b) Suppose now that $f_1, f_2 \in L^q(\mathbf{R}^N)$ for all $q \in [2,\infty)$. Let $r \ge \frac{(2+\sigma)^2}{2}$ if $\sigma \le 1$, respectively $r \ge \frac{2}{3}(2+\sigma)$ if $\sigma > 1$ and $s \ge \frac{(2+\sigma)^2}{2}$ be such that $u_1 \in L^2 \cap L^r(\mathbf{R}^N)$ and $u_2 \in L^{2+\sigma} \cap L^s(\mathbf{R}^N)$.

It is easily seen that $\int_{\mathbf{R}^N} |u_1|^2 |u_2|^{p-2} dx < \infty$ if $2+\sigma \le p \le 2 + s(1-\frac{2}{r})$.Let $p_1 = \min(s, 2+s(1-\frac{2}{r}))$. From (5.5) it follows that $\nabla\left(|u_2|^{\frac{p_1}{2}}\right) \in L^2(\mathbf{R}^N)$, thus

$u_2 \in L^{\frac{2+\sigma}{2}p_1}(\mathbf{R}^N)$. We also have

$$\int_{\mathbf{R}^N} |u_1|^{p+\sigma-1} dx < \infty \text{ for } 3-\sigma \le p \le r+1-\sigma,$$

$$\int_{\mathbf{R}^N} |u_2|^{\sigma}|u_1|^{p-1} dx < \infty \text{ for } 2+\tfrac{2-\sigma}{2+\sigma} \le p \le 1+r(1-\tfrac{\sigma}{s}),$$

$$\int_{\mathbf{R}^N} |u_2||u_1|^{p-1} dx < \infty \text{ for } 2+\tfrac{\sigma}{2+\sigma} \le p \le 1+r(1-\tfrac{1}{s})$$

$$\int_{\mathbf{R}^N} |u_2|^2|u_1|^{p-2} dx < \infty \text{ for } 2+\tfrac{2\sigma}{2+\sigma} \le p \le 2+r(1-\tfrac{2}{s}).$$

In the case $\sigma \le 1$, we obtain from (5.8) that $\nabla\left(|u_1|^{\frac{p}{2}}\right) \in L^2(\mathbf{R}^N)$ if $2+\frac{2\sigma}{2+\sigma} \le p \le p_2 = \min(r, 1+r(1-\frac{1}{s}), 2+r(1-\frac{2}{s}))$, while in the case $\sigma > 1$ we obtain from (5.10) that $\nabla\left(|u_1|^{\frac{p}{2}}\right) \in L^2(\mathbf{R}^N)$ if $2+\frac{2\sigma}{2+\sigma} \le p \le p_2' = \min(r, 1+r(1-\frac{\sigma}{s}), 2+r(1-\frac{2}{s}), r+1-\sigma)$. By the Sobolev embedding, $u_1 \in L^{\frac{2+\sigma}{2}p_2}(\mathbf{R}^N)$ if $\sigma \le 1$, respectively $u_1 \in L^{\frac{2+\sigma}{2}p_2'}(\mathbf{R}^N)$ if $\sigma > 1$. Thus we obtained that $u_1 \in L^{r'}(\mathbf{R}^N)$ and $u_2 \in L^{s'}(\mathbf{R}^N)$, where $r' = \frac{2+\sigma}{2}p_2$ if $\sigma \le 1$, respectively $r' = \frac{2+\sigma}{2}p_2'$ if $\sigma > 1$ and $s' = \frac{2+\sigma}{2}p_1$. Repeating this argument it follows that $u_1 \in L^p(\mathbf{R}^N)$ for all $p \in [2,\infty)$ and $u_2 \in L^q(\mathbf{R}^N)$ for all $q \in [2+\sigma,\infty)$. Consequently $F((r_0-u_1)^2+u_2^2)(r_0-u_1), F((r_0-u_1)^2+u_2^2)u_2 \in L^p(\mathbf{R}^N)$ for all $p \in [2,\infty)$.

Since $u_1 \in H^2(\mathbf{R}^N)$ and $u_2 \in \mathcal{D}^{1,2} \cap \mathcal{D}^{2,2}(\mathbf{R}^N)$, we have $u_{1x_1}, u_{2x_1} \in H^1(\mathbf{R}^N) \subset L^2 \cap L^{2+\sigma}(\mathbf{R}^N)$. Using (5.2) and (5.3) we infer that $\Delta u_1, \Delta u_2 \in L^p(\mathbf{R}^N)$ for all $p \in [2, 2+\sigma]$, therefore $u_1 \in W^{2,p}(\mathbf{R}^N)$, $\forall p \in [2,2+\sigma]$, $u_2 \in \mathcal{D}^{1,p} \cap \mathcal{D}^{2,p}(\mathbf{R}^N)$, $\forall p \in [2,2+\sigma]$ and $u_2 \in W^{2,2+\sigma}(\mathbf{R}^N)$. Iterating this argument we obtain the conclusion in Lemma 5.2, b). $\square$

Remark 5.3. From Lemma 5.3 b) it follows in particular that $u_1, u_1 \in C^{1,\alpha}(\mathbf{R}^N)$ for all $\alpha \in [0,1)$, u_1, u_2 are bounded and tend to zero at infinity.

Finally, suppose that F is C^k and $f_1, f_2 \in W^{k,q}(\mathbf{R}^N)$ for all $q \in [2,\infty)$. Differentiating equation (5.2), respectively (5.3), we obtain $u_1 \in W^{k+2,q}(\mathbf{R}^N)$, $\forall q \in [2,\infty)$, $u_2 \in \mathcal{D}^{1,q} \cap \mathcal{D}^{k+2,q}(\mathbf{R}^N)$, $2 \le q < 2+\sigma$ and $u_2 \in W^{k+2,q}(\mathbf{R}^N)$, $2+\sigma \le q < \infty$.

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