Analyticity and decay properties of the solitary waves to the Benney-Luke equation

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Abstract

We prove that the Benney-Luke solitary waves are analytic functions and decay at infinity with an optimal algebraic rate as well as their derivatives.

1 Introduction

In a recent paper [9] PEGO and QUINTERO studied the propagation of long water waves with small amplitude. They showed that in the presence of a surface tension, the propagation of such waves is governed by the following equation originally derived by BENNEY and LUKE (see [1]):

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \varepsilon(\Phi_t\Delta\Phi + (\nabla\Phi)_t^2) = 0 \ . \tag{1.1}$$

Here a and b are positive and satisfy $a - b = \sigma - \frac{1}{3}$ where σ is the Bond number, while the parameters ε and μ are supposed to be small.

Pego and Quintero looked for traveling-wave solutions of (1.1), that is solutions of the form

$$\Phi(x, y, t) = \frac{\sqrt{\mu}}{\varepsilon} u\left(\frac{x - ct}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}}\right) .$$

The scaling was introduced here to eliminate ε and μ. A traveling-wave profile u should satisfy the equation

$$\begin{aligned}(c^2 - 1)u_{xx} - u_{yy} + (a - bc^2)u_{xxxx} + (2a - bc^2)u_{xxyy} + au_{yyyy} - \\ c(3u_x u_{xx} + u_x u_{yy} + 2u_y u_{xy}) = 0 \ .\end{aligned} \tag{1.2}$$

The energy associated to u is

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^2} (1 + c^2)u_x^2 + u_y^2 + (a + bc^2)u_{xx}^2 + (2a + bc^2)u_{xy}^2 + au_{yy}^2 \ \ dxdy \ . \tag{1.3}$$

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It was proved in [9] by the means of the concentration-compactness method that if the wave speed c satisfies $c^2 < \min(1, \frac{a}{b})$, then there exist non-trivial finite energy solutions of (1.2) in a space $\mathcal{V}$, where $\mathcal{V}$ is the completion of $C_0^\infty(\mathbf{R}^2)$ for the norm

$$\| \varphi \|_{\mathcal{V}}^2 = \int_{\mathbf{R}^2} \varphi_x^2 + \varphi_y^2 + \varphi_{xx}^2 + 2\varphi_{xy}^2 + \varphi_{yy}^2 \ \ dxdy \ \ .$$

The Benney-Luke equation reduces formally to the Kadomtsev-Petviashvili (KP) equation after a suitable renormalization. Indeed, putting $\tau = \dfrac{\varepsilon t}{2}$, $X = x - t$, $Y = \varepsilon^{\frac{1}{2}} y$ and $\Phi(x, y, t) = f(X, Y, \tau)$, neglecting $O(\varepsilon)$ terms we find that $\eta = f_X$ satisfies the KP equation

$$(1.4) \qquad (\eta_\tau - (\sigma - \frac{1}{3})\eta_{XXX} + 3\eta\eta_X)_X + \eta_{YY} = 0 \ \ .$$

De Bouard and Saut proved (see [5]) that finite energy solitary waves exist for the KP equation when $\sigma > \frac{1}{3}$ (the KP-I case).

Moreover, let $\sigma > \frac{1}{3}$ (that is, $a > b$), $\varepsilon = 1 - c^2$ and let u_ε be the corresponding solution of (1.2) obtained in [9]. Then if $\varepsilon \longrightarrow 0$, there exists a sequence (ε_j) such that (u_{ε_j}) converges (after a suitable renormalization) to a distribution $v_0 \in \mathcal{D}'(\mathbf{R}^2)$ and $\partial_x v_0$ is a nontrivial solitary wave of the KP equation (see [9]).

It is known (see de Bouard and Saut [6]) that the solitary waves of the KP equation are smooth and decay at infinity with an optimal algebraic rate ($\dfrac{1}{r^2}$ in dimension 2).

It is then natural to ask whether the Benney-Luke solitary waves have the same properties. The aim of this paper is to give an answer to this question.

We suppose throughout that the parameters a, b, c appearing in (1.2) satisfy: $a > 0$ and if $b > 0$, then $c^2 < \min(1, \frac{a}{b})$.

Our method follows very closely the ideas developed in [6].

This paper is organized as follows: in the next section we prove that the Benney-Luke solitary waves are analytic functions. Section 3 contains our main result about the decay at infinity of such waves. We give an algebraic decay rate which is optimal for the solutions of (1.2) and their first order derivatives. In Section 4 we state some integral identities satisfied by these solitary waves. Some technical facts about the Fourier transform that we use in proofs are treated in an Appendix.

2 Analyticity

The aim of this section is to prove that any solution $u \in \mathcal{V}$ of (1.2) is an analytic function and tends to zero at infinity as well as all its derivatives. We begin with the following result:

Theorem 2.1 Let $u \in \mathcal{V}$ be a solution of (1.2). Then
a) $u \in W^{k,p}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ and all $p \in]2, \infty]$;
b) $u_x,\ u_y \in W^{k,p}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ and all $p \in]1, \infty]$.

Proof: We make extensively use of the following theorem on Fourier multipliers due to LIZORKIN:

Theorem 2.2 ([8]) Let $\Phi : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a C^n function for $|\xi_j| > 0$, $j = 1, \dots, n$. Assume that

$$\xi_1^{k_1} \dots \xi_n^{k_n} \frac{\partial^k \Phi}{\partial \xi_1^{k_1} \dots \partial \xi_n^{k_n}} \in L^\infty(\mathbf{R}^n) \ ,$$

with $k_i = 0$ or 1, $k = k_1 + \dots k_n = 0, 1, \dots, n$. Then $\Phi \in M_q(\mathbf{R}^n)$ for $1 < q < \infty$, i.e. Φ is a Fourier multiplier on $L^q(\mathbf{R}^n)$.

We have $u_x,\ u_y \in H^1(\mathbf{R}^2) \subset L^p(\mathbf{R}^2)$ for all $p \in [2, \infty[$ by the Sobolev imbedding theorem. The nonlinearity can be written as $\partial_x(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) + \partial_y(u_x u_y)$. Let $Q(\xi_1, \xi_2) = (1 - c^2)\xi_1^2 + \xi_2^2 + (a - bc^2)\xi_1^4 + (2a - bc^2)\xi_1^2\xi_2^2 + a\xi_2^4$. Equation (1.2) gives

$$Q(\xi_1, \xi_2)\widehat{u_x} = -\xi_1^2 c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) - \xi_1\xi_2 c\mathcal{F}(u_x u_y)$$

and

$$Q(\xi_1, \xi_2)\widehat{u_y} = -\xi_1\xi_2 c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) - \xi_2^2 c\mathcal{F}(u_x u_y) \ .$$

The Theorem 2.2 implies that $u_x,\ u_y \in L^p(\mathbf{R}^2)$ for all $p \in]1, \infty[$. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2$. We have:

$$Q(\xi_1, \xi_2)\widehat{D^\alpha u} = i\xi_1(i\xi)^\alpha c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) + i\xi_2(i\xi)^\alpha c\mathcal{F}(u_x u_y) \ .$$

By Theorem 2.2, $D^\alpha u \in L^p(\mathbf{R}^2)$ for all $p \in]1, \infty[$ if $|\alpha| = 2, 3$. In particular, $u_x,\ u_y \in W^{2,p}(\mathbf{R}^2) \subset C^1 \cap L^\infty(\mathbf{R}^2)$ and for $|\alpha| = 2$, $D^\alpha u \in W^{1,p}(\mathbf{R}^2) \subset C^0 \cap L^\infty(\mathbf{R}^2)$ by the Sobolev imbedding theorem applied for a $p > 2$.

The rest of the proof follows easily by induction. Supose that all the derivatives of u of order $1, 2, \dots, n-1$ are in $C^0 \cap L^\infty \cap L^p(\mathbf{R}^2)$ and the n^{th} order derivatives are in $L^p(\mathbf{R}^2)$ for all $p \in]1, \infty[$. Let $\alpha \in \mathbf{N}^2$ with $|\alpha| = n + 1$ and $\beta \leq \alpha$ with $|\alpha - \beta| = 2$. Then

$$Q(\xi_1, \xi_2)\widehat{D^\alpha u} = i\xi_1(i\xi)^{\alpha-\beta} c\mathcal{F}(D^\beta(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2)) + i\xi_2(i\xi)^{\alpha-\beta} c\mathcal{F}(D^\beta(u_x u_y)) \ .$$

Again by Theorem 2.2 we obtain $D^\alpha u \in L^p(\mathbf{R}^2)$ for all $p \in]1, \infty[$. The Sobolev imbedding theorem gives us $D^{\alpha'} u \in C^0 \cap L^\infty(\mathbf{R}^2)$ if $|\alpha'| = n$. This finishes the induction and the proof of part b).

Since u_x $u_y \in L^p(\mathbf{R}^2)$ for $p \in]1,\infty]$, Theorem 14.4, p. 295 of [3] yields $u \in L^q(\mathbf{R}^2)$ for all $q \in]2,\infty[$ and

$$\| u \|_{L^q} \leq C_p \| \nabla u \|_{L^p} \quad , \text{ where } \frac{1}{p} = \frac{1}{2} + \frac{1}{q} \ .$$

Hence $u \in W^{k,q}(\mathbf{R}^2)$ for all $k \in \mathbf{N}$ and all $q \in]2,\infty[$. Consequently u is a C^∞ function, it is bounded and tends to zero at infinity.

The Theorem 2.1 is proved. □

Remark 2.3 If $u \in \mathcal{V}$ is a nontrivial solution of (1.2), then u_x and u_y are not in $L^1(\mathbf{R}^2)$.

Proof: We argue by contradiction. Suppose $u_x \in L^1(\mathbf{R}^2)$. Then $\widehat{u_x}$ is a continuous function. But $\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2)$ and $\mathcal{F}(u_x u_y)$ are also continuous functions and

$$\widehat{u_x}(\xi_1,\xi_2) = -\frac{\xi_1^2}{Q(\xi_1,\xi_2)} c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) - \frac{\xi_1\xi_2}{Q(\xi_1,\xi_2)} c\mathcal{F}(u_x u_y) \ . \tag{2.2}$$

For a fixed $\lambda \in \mathbf{R}$ we put $\xi_2 = \lambda\xi_1$ and let $\xi_1 \longrightarrow 0$ in (2.2). We obtain

$$\widehat{u_x}(0,0) = -\frac{c}{1-c^2+\lambda^2}\int_{\mathbf{R}^2}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2)\, dxdy - \frac{c\lambda}{1-c^2+\lambda^2}\int_{\mathbf{R}^2} u_x u_y \, dxdy \ .$$

Since this is true for all $\lambda \in \mathbf{R}$ we deduce that

$$\int_{\mathbf{R}^2}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2)\, dxdy = \int_{\mathbf{R}^2} u_x u_y \, dxdy = 0$$

which implies that u is constant, contrary to the assumption. The same argument applies to u_y. □

Remark 2.4 If $u \in \mathcal{V}$ is a nontrivial solution of (1.2) and $r^{\frac{1}{2}}u_x$, $r^{\frac{1}{2}}u_y \in L^2(\mathbf{R}^2)$ where $r = \sqrt{x^2+y^2}$ (we shall see in the next section that this is always the case), then u cannot belong to $L^2(\mathbf{R}^2)$.

Proof: Assume $u \in L^2(\mathbf{R}^2)$. Then $\widehat{u} \in L^2(\mathbf{R}^2)$ and

$$\widehat{u}(\xi_1,\xi_2) = \frac{i\xi_1}{Q(\xi_1,\xi_2)} c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) + \frac{i\xi_2}{Q(\xi_1,\xi_2)} c\mathcal{F}(u_x u_y) \ . \tag{2.3}$$

The fact that $r^{\frac{1}{2}}u_x$, $r^{\frac{1}{2}}u_y \in L^2(\mathbf{R}^2)$ implies that $g_1 = c\mathcal{F}(\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2)$ and $g_2 = c\mathcal{F}(u_x u_y)$ are C^1 functions. The equation (2.3) can be written as

$$\begin{aligned} &\widehat{u}(\xi_1,\xi_2) = \\ &\left[\frac{i\xi_1}{Q(\xi_1,\xi_2)}(g_1(\xi_1,\xi_2) - g_1(0,0)) + \frac{i\xi_2}{Q(\xi_1,\xi_2)}(g_2(\xi_1,\xi_2) - g_2(0,0))\right] + \\ &\left[\frac{i\xi_1}{Q(\xi_1,\xi_2)} g_1(0,0) + \frac{i\xi_2}{Q(\xi_1,\xi_2)} g_2(0,0)\right] \ . \end{aligned} \tag{2.4}$$

Since g_1 and g_2 are locally Lipschitz functions, the first term in the right hand side of (2.4) is bounded for $\xi \in B_{\mathbf{R}^2}(0,1)$. This forces

$$\frac{i\xi_1}{Q(\xi_1,\xi_2)} g_1(0,0) + \frac{i\xi_2}{Q(\xi_1,\xi_2)} g_2(0,0) \in L^2(B_{\mathbf{R}^2}(0,1)) \, .$$

But $g_1(0,0) = c \int_{\mathbf{R}^2} (\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) dxdy > 0$, so it suffices to show that

$$\frac{a\xi_1 + b\xi_2}{Q(\xi_1,\xi_2)} \notin L^2(B_{\mathbf{R}^2}(0,1))$$

if $a,\ b \in \mathbf{R}$, $a \neq 0$ to obtain a contradiction.

For ξ varying in a bounded set K there exists $m_K > 0$ such that $Q(\xi) \leq m_K|\xi|^2$. We make the change of variables $\xi_1' = a\xi_1 + b\xi_2$, $\xi_2' = \xi_2$, $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. We have:

$$\begin{aligned} \int_{B_{\mathbf{R}^2}(0,1)} \frac{(a\xi_1 + b\xi_2)^2}{Q(\xi_1,\xi_2)^2} d\xi &= \int_{AB_{\mathbf{R}^2}(0,1)} \frac{(\xi_1')^2}{Q(A^{-1}(\xi_1',\xi_2'))^2} |det(A)|^{-1} d\xi' \\ &\geq C \int_{AB_{\mathbf{R}^2}(0,1)} \frac{(\xi_1')^2}{|\xi'|^4} d\xi' = \infty \, . \ \square \end{aligned}$$

We prove now that any solution $u \in \mathcal{V}$ of (1.2) is an analytic function. The proof relies on the Paley-Wiener theory. We borrowed the ideas developed by Li and Bona in [7].

Let $u \in \mathcal{V}$ be a solution of (2.1). By Theorem 2.1 we have $|\xi|(1+|\xi|^2)^{\frac{m}{2}}\widehat{u} \in L^2(\mathbf{R}^2)$ for all m. We take $m > 1$ and apply thy Cauchy-Schwarz inequality to get

$$\int_{\mathbf{R}^2} |\xi||\widehat{u}(\xi)| d\xi \leq \left(\int_{\mathbf{R}^2} |\xi|^2 (1+|\xi|^2)^m |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbf{R}^2} (1+|\xi|^2)^{-m} d\xi \right)^{\frac{1}{2}} < \infty.$$

Hence $|\xi|\widehat{u} \in L^1(\mathbf{R}^2)$. Equation (2.3) gives us

$$\begin{aligned} |\xi|\widehat{u}(\xi_1,\xi_2) &= \frac{ic\xi_1|\xi|}{Q(\xi_1,\xi_2)} \left(\frac{3}{2}(i\xi_1\widehat{u}) \star (i\xi_1\widehat{u}) + \frac{1}{2}(i\xi_2\widehat{u}) \star (i\xi_2\widehat{u}) \right) \\ &\quad + \frac{ic\xi_2|\xi|}{Q(\xi_1,\xi_2)} (i\xi_1\widehat{u}) \star (i\xi_2\widehat{u}) \end{aligned}$$

from which we infer that

$$|\xi||\widehat{u}| \leq \frac{3c|\xi|^2}{Q(\xi_1,\xi_2)} (|\xi||\widehat{u}|) \star (|\xi||\widehat{u}|). \tag{2.5}$$

Let $M = \max_{i=2,3,4}\left(\sup_{(\xi_1,\xi_2)\neq(0,0)}\left(\frac{3c|\xi|^i}{Q(\xi_1,\xi_2)}\right)\right)$. Obviously $M<\infty$. We note $\psi(\xi_1,\xi_2) = M|\xi|\cdot|\widehat{u}(\xi_1,\xi_2)|$. Then $\psi\geq 0$, $\psi\in L^1(\mathbf{R}^2)$ and the inequality (2.5) gives

$$\psi \leq \psi\star\psi, \quad |\xi|\psi\leq\psi\star\psi \quad \text{and} \quad |\xi|^2\psi\leq\psi\star\psi. \tag{2.6}$$

For an integrable function f we define $\mathcal{C}_1 f = f$ and for $n>1$, $\mathcal{C}_n f(x) = (f\star(\mathcal{C}_{n-1}f))(x)$. We have

Lemma 2.5 The function ψ introduced above satisfies

$$|\xi|^k\psi \leq (\frac{k}{2}+1)^{k-1}\mathcal{C}_{2([\frac{k}{2}]+1)}\psi \tag{2.7}$$

where $[x]$ denotes the greatest integer less or equal than x.

Proof. We proceed by induction on k. From (2.6) it follows that (2.7) holds for $k=1,2,3$. Notice that the first of the inequalities (2.6) implies that $\mathcal{C}_p\psi\leq\mathcal{C}_r\psi$ if $p\leq r$. We suppose that (2.7) is valid up to order k and prove that it is valid for $k+2$. We have:

$$\begin{aligned}
|\xi|^{k+2}\psi(\xi) &\leq |\xi|^k(\psi\star\psi)(\xi) = \int_{\mathbf{R}^2}|\xi|^k\psi(\xi-\zeta)\cdot\psi(\zeta)d\zeta\\
&\leq \int_{\mathbf{R}^2}(|\xi-\zeta|+|\zeta|)^k\psi(\xi-\zeta)\cdot\psi(\zeta)d\zeta\\
&= \int_{\mathbf{R}^2}\sum_{i=0}^k C_k^i(|\xi-\zeta|^i\psi(\xi-\zeta))\cdot(|\zeta|^{k-i}\psi(\zeta))d\zeta\\
&= \sum_{i=0}^k C_k^i(|\cdot|^i\psi)\star(|\cdot|^{k-i}\psi)(\xi)
\end{aligned}$$

where $C_k^i = \frac{k!}{i!(k-i)!}$ is the binomial coefficient. Using the induction hypothesis, the last sum is majorized by

$$\begin{aligned}
&\sum_{i=0}^k C_k^i\left((\frac{i}{2}+1)^{i-1}\mathcal{C}_{2([\frac{i}{2}]+1)}\psi\right)\star\left((\frac{k-i}{2}+1)^{k-i-1}\mathcal{C}_{2([\frac{k-i}{2}]+1)}\psi\right)\\
&\quad= \sum_{i=0}^k C_k^i(\frac{i}{2}+1)^{i-1}(\frac{k-i}{2}+1)^{k-i-1}\mathcal{C}_{2([\frac{i}{2}]+[\frac{k-i}{2}]+2)}\psi\\
&\quad\leq \left(\sum_{i=0}^k C_k^i(\frac{i}{2}+1)^{i-1}(\frac{k-i}{2}+1)^{k-i-1}\right)\mathcal{C}_{2([\frac{k+2}{2}]+1)}\psi.
\end{aligned}$$

We use a specialization of the Abel identity (see [10], p. 26)

$$\sum_{i=0}^{k} C_k^i (x_1+i)^{i-1}(x_2+k-i)^{k-i-1} = \frac{1}{x_1x_2}(x_1+x_2)(x_1+x_2+k)^{k-1}$$

for $x_1 = x_2 = 2$ to obtain

$$\sum_{i=0}^{k} C_k^i (\frac{i}{2}+1)^{i-1}(\frac{k-i}{2}+1)^{k-i-1} = \frac{1}{2^{k-2}}\sum_{i=0}^{k} C_k^i (2+i)^{i-1}(2+k-i)^{k-i-1}$$
$$= \frac{(4+k)^{k-1}}{2^{k-2}} = 2(2+\frac{k}{2})^{k-1} \le \left(\frac{k+2}{2}+1\right)^{k+1}.$$

Hence (2.7) holds for $k+2$ and the Lemma is proved. □

Theorem 2.6 Let $u \in \mathcal{V}$ be a solution of (1.2). Then there exists $\sigma > 0$ and an holomorphic function U of two complex variables z_1, z_2 defined in the domain

$$\Omega_\sigma = \{(z_1,z_2) \in \mathbf{C}^2 \quad | \quad |Im(z_1)| < \sigma, \;\; |Im(z_2)| < \sigma\}$$

such that $U(x,y) = u(x,y)$ for all $(x,y) \in \mathbf{R}^2$.

Proof. It is easily seen from (2.3) that $|\widehat{u}(\xi)| \le \dfrac{C}{|\xi|}$ for $0 < |\xi| \le 1$ and $|\widehat{u}(\xi)| \le \dfrac{C}{|\xi|^3}$ for $|\xi| \ge 1$, so $\widehat{u} \in L^1(\mathbf{R}^2)$.

Keeping the notation introduced above and using Lemma 2.5 we infer that for $k \ge 1$,

$$\begin{aligned}
|\xi|^k|\widehat{u}(\xi)| &= \frac{1}{M}|\xi|^{k-1}\cdot M|\xi|\cdot|\widehat{u}(\xi)| = \frac{1}{M}|\xi|^{k-1}\psi(\xi) \\
&\le \frac{1}{M}\left(\frac{k-1}{2}+1\right)^{k-2}\mathcal{C}_{2([\frac{k-1}{2}]+1)}\psi(\xi) \\
&\le \frac{1}{M}\left(\frac{k-1}{2}+1\right)^{k-2}\mathcal{C}_{k+1}\psi(\xi) \\
&\le \frac{1}{M}\left(\frac{k-1}{2}+1\right)^{k-2}||\psi||_{L^2}\cdot||\mathcal{C}_k\psi||_{L^2} \\
&\le \frac{1}{M}\left(\frac{k-1}{2}+1\right)^{k-2}||\psi||_{L^2}^2\cdot||\psi||_{L^1}^{k-1}.
\end{aligned}$$

Put $a_k = \dfrac{(\frac{k-1}{2}+1)^{k-2}||\psi||_{L^2}^2 \cdot ||\psi||_{L^1}^{k-1}}{Mk!}$. Then

$$\frac{a_{k+1}}{a_k} = \frac{1}{2}||\psi||_{L^1} \cdot \left(\frac{k+2}{k+1}\right)^{k-1} \longrightarrow \quad \frac{e}{2}||\psi||_{L^1} \quad \text{as} \quad k \longrightarrow \infty.$$

Let $\sigma = \dfrac{2}{e||\psi||_{L^1}}$. The series $\sum_{k=1}^{\infty} a_k s^k$ converges absolutely for $|s| < \sigma$; we denote by $C(s)$ its sum. Fix $\sigma_1 \in]\, 0, \sigma\, [$ and choose $\sigma_2 \in]\, \sigma_1, \sigma\, [$. One has

$$e^{\sigma_2|\xi|}|\widehat{u}(\xi)| \leq \sum_{k=0}^{\infty} \frac{\sigma_2^k|\xi|^k}{k!}|\widehat{u}(\xi)| \leq |\widehat{u}(\xi)| + \sum_{k=1}^{\infty} \sigma_2^k a_k = |\widehat{u}(\xi)| + C(\sigma_2).$$

Hence

$$e^{\sigma_1|\xi|}|\widehat{u}(\xi)| \leq e^{-(\sigma_2-\sigma_1)|\xi|}|\widehat{u}(\xi)| + e^{-(\sigma_2-\sigma_1)|\xi|}C(\sigma_2).$$

It follows that $e^{\sigma_1|\cdot|}\widehat{u} \in L^1(\mathbf{R}^2)$ for all $\sigma_1 < \sigma$. We define the function

$$U(z_1, z_2) = \frac{1}{2\pi} \int_{\mathbf{R}^2} e^{i(z_1\xi_1 + z_2\xi_2)}\widehat{u}(\xi_1, \xi_2)d\xi_1 d\xi_2.$$

By the Paley-Wiener Theorem, U is well defined and analytic in Ω_σ and the Plancherel's Theorem implies that $U(x, y) = u(x, y)$ for all $(x, y) \in \mathbf{R}^2$. This proves the Theorem 2.6. □

3 Decay properties

We prove in this section that all the solutions in $\mathcal{V}$ of (1.2) decay at infinity as $\frac{1}{r}$ and their derivatives decay as $\frac{1}{r^2}$.

From (2.3) we deduce that

$$\text{(3.1)} \quad u = ic\mathcal{F}^{-1}\left(\frac{\xi_1}{Q(\xi_1, \xi_2)}\right) \star (\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) + ic\mathcal{F}^{-1}\left(\frac{\xi_2}{Q(\xi_1, \xi_2)}\right) \star (u_x u_y) \quad .$$

(2.2) gives us

$$\text{(3.2)} \quad u_x = -c\mathcal{F}^{-1}\left(\frac{\xi_1^2}{Q(\xi_1, \xi_2)}\right) \star (\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) - c\mathcal{F}^{-1}\left(\frac{\xi_1\xi_2}{Q(\xi_1, \xi_2)}\right) \star (u_x u_y)$$

and similarly

$$\text{(3.3)} \quad u_y = -c\mathcal{F}^{-1}\left(\frac{\xi_1\xi_2}{Q(\xi_1, \xi_2)}\right) \star (\frac{3}{2}u_x^2 + \frac{1}{2}u_y^2) - c\mathcal{F}^{-1}\left(\frac{\xi_2^2}{Q(\xi_1, \xi_2)}\right) \star (u_x u_y) \quad .$$

As we have mentioned in Introduction, our method was inspired by the work of de Bouard and Saut [6]. This idea had already been used by Bona and

Li (see [4]). It is based on the study of the convolution equations (3.1), (3.2), (3.3).

We begin with an integral estimate.

Theorem 3.1 Let $u \in \mathcal{V}$ be a solution of (1.2). Then

$$\int_{\mathbf{R}^2} (x^2+y^2)|\nabla^2 u|^2 dxdy < \infty \tag{3.4}$$

and

$$\int_{\mathbf{R}^2} (x^2+y^2)|\nabla^3 u|^2 dxdy < \infty \; . \tag{3.5}$$

Proof: Fix a function $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi(x) = |x|$ for $|x| > 1$, $\varphi(0) = 0$, φ decrease on $]-\infty, 0]$ and increase on $[0, \infty[$. We put

$$\chi_n(x) = e^{-\varphi(\frac{x}{n})} \; .$$

We multiply (1.2) by $x^2\chi_n(x)u_{xx}$ and integrate over $\mathbf{R}^2$. Using several integrations by parts we have

$$\begin{aligned}
&\int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xx}u_{xxxx}\; dxdy = \\
&= -\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xx}u_{xxx}\; dxdy - \int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xxx}^2 dxdy \\
&= \frac{1}{2}\int_{\mathbf{R}^2} \partial_{xx}^2(\chi_n(x)x^2)u_{xx}^2\; dxdy - \int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xxx}^2 dxdy \; ;
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xx}u_{yy}\; dxdy = \\
&= -\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_x u_{yy}\; dxdy + \chi_n(x)x^2 u_x u_{xyy} dxdy \\
&= \int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xy}u_y\; dxdy + \int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xy}^2 dxdy \\
&= -\frac{1}{2}\int_{\mathbf{R}^2} \partial_{xx}^2(\chi_n(x)x^2)u_y^2\; dxdy + \int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xy}^2 dxdy \; ;
\end{aligned}$$

$$\int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xx}u_{xxyy}\; dxdy = -\int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xxy}^2 dxdy \; ;$$

$$\begin{aligned}
&\int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xx}u_{yyyy}\; dxdy = \\
&= \int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xxyy}u_{yy} dxdy \\
&= -\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xyy}u_{yy} + \chi_n(x)x^2 u_{xyy}^2\; dxdy \\
&= \frac{1}{2}\int_{\mathbf{R}^2} \partial_{xx}^2(\chi_n(x)x^2)u_{yy}^2\; dxdy - \int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xyy}^2 dxdy \; ;
\end{aligned}$$

$$
\begin{aligned}
&\int_{\mathbf{R}^2} \chi_n(x)x^2 u_{xx}u_x u_{yy}\ dxdy = \\
&= -\frac{1}{2}\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_x^2 u_{yy} + \chi_n(x)x^2u_x^2u_{yyx}\ dxdy \\
&= \int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xy}u_xu_y\ dxdy + \int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2u_x\ dxdy\ .
\end{aligned}
$$

Finally we get

(3.6)

$$
\begin{aligned}
&\int_{\mathbf{R}^2} \chi_n(x)x^2[(1-c^2)u_{xx}^2 + u_{xy}^2 + (a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2]\ dxdy \\
&\quad +3c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}^2u_x\ dxdy + c\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xy}u_xu_y\ dxdy \\
&\quad +c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2u_x\ dxdy + 2c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}u_{xy}u_y\ dxdy \\
&\quad = \int_{\mathbf{R}^2} \partial_{xx}^2(\chi_n(x)x^2)\left[\frac{a-bc^2}{2}u_{xx}^2 + \frac{a}{2}u_{yy}^2 + \frac{1}{2}u_y^2\right]\ dxdy\ .
\end{aligned}
$$

Since $\chi_n'(x) = -\frac{1}{n}\varphi'\left(\frac{x}{n}\right)e^{-\varphi(\frac{x}{n})}$, there exists a constant $k>0$ such that $|x\chi_n'(x)| \le k\chi_n(x)^{\frac{1}{2}}$ for all $x \in \mathbf{R}$ and $n \ge 1$. We have

$$
\begin{aligned}
&|\partial_x(\chi_n(x)x^2)u_{xy}u_xu_y| \\
&\le |\chi_n'(x)x^2u_{xy}u_xu_y| + 2|\chi_n(x)xu_{xy}u_xu_y| \\
&\le k\chi_n(x)^{\frac{1}{2}}|xu_{xy}u_xu_y| + 2\chi_n(x)|xu_{xy}u_xu_y| \\
&\le \frac{k+2}{2}\left[\chi_n(x)x^2u_{xy}^2 + u_y^2\right]|u_x|\ .
\end{aligned}
$$

and

$$
2|(\chi_n(x)x^2)u_{xx}u_{xy}u_y| \le \chi_n(x)x^2(u_{xx}^2+u_{xy}^2)|u_y|\ .
$$

Let $\varepsilon \in\ ]\,0,1\,[$. Since u_x and u_y tend to 0 as $r \longrightarrow \infty$, there exists $R_\varepsilon > 0$ such that $|u_x(x,y)| < \varepsilon$ and $|u_y(x,y)| < \varepsilon$ if $|(x,y)| > R_\varepsilon$. Then

$$
\begin{aligned}
\left| c\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xy}u_xu_y\ dxdy \right| &\le c\int_{\mathbf{R}^2} \frac{k+2}{2}\left[\chi_n(x)x^2u_{xy}^2 + u_y^2\right]|u_x|\ dxdy \\
&\le c\int_{B(0,R_\varepsilon)} \frac{k+2}{2}\left[\chi_n(x)x^2u_{xy}^2 + u_y^2\right]|u_x|\ dxdy \\
&\quad +c\varepsilon\frac{k+2}{2}\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2 + u_y^2\ dxdy \\
&\le C(\varepsilon) + c\cdot\frac{k+2}{2}\cdot\varepsilon\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2\ dxdy
\end{aligned}
$$

where $C(\varepsilon)$ is a constant depending on ε. Similar estimates hold for $\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}^2u_x\ dxdy$, $\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2u_x\ dxdy$ and $\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}u_{xy}u_y\ dxdy$. We take ε sufficiently small to obtain

$$\Bigg| 3c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}^2u_x\ dxdy + c\int_{\mathbf{R}^2} \partial_x(\chi_n(x)x^2)u_{xy}u_xu_y\ dxdy$$
$$+c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xy}^2u_x\ dxdy + 2c\int_{\mathbf{R}^2} \chi_n(x)x^2u_{xx}u_{xy}u_y\ dxdy\Bigg|$$
$$\leq C + \frac{1}{2}\int_{\mathbf{R}^2} \chi_n(x)x^2((1-c^2)u_{xx}^2 + u_{xy}^2)\ dxdy\ .$$

where C is a constant.

Combining the last inequality with (3.6) we get

$$\begin{aligned} &\frac{1}{2}\int_{\mathbf{R}^2} \chi_n(x)x^2[(1-c^2)u_{xx}^2 + u_{xy}^2]\ dxdy \\ &+\int_{\mathbf{R}^2} \chi_n(x)x^2[(a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2]\ dxdy \\ &\leq C + \int_{\mathbf{R}^2} \partial_{xx}^2(\chi_n(x)x^2)\left[\frac{a-bc^2}{2}u_{xx}^2 + \frac{a}{2}u_{yy}^2 + \frac{1}{2}u_y^2\right]\ dxdy\ . \end{aligned} \tag{3.7}$$

When $n \longrightarrow \infty$ the left hand side of (3.7) tends to

$$\int_{\mathbf{R}^2} x^2\left[\frac{1-c^2}{2}u_{xx}^2 + \frac{1}{2}u_{xy}^2 + (a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2\right]\ dxdy$$

by the monotone convergence theorem, while the right hand side tends to

$$C + \int_{\mathbf{R}^2} (a-bc^2)u_{xx}^2 + au_{yy}^2 + u_y^2\ dxdy < \infty$$

by Lebesgue's theorem on dominated convergence. Hence

$$\int_{\mathbf{R}^2} x^2(u_{xx}^2 + u_{xy}^2 + u_{xxx}^2 + u_{xxy}^2 + u_{xyy}^2)\ dxdy < \infty\ . \tag{3.8}$$

We multiply (1.2) by $\chi_n(y)y^2u_{xx}$ and integrate over $\mathbf{R}^2$ to get, after several integrations by parts,

(3.9)

$$\int_{\mathbf{R}^2} \chi_n(y)y^2[(1-c^2)u_{xx}^2 + u_{xy}^2 + (a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2]\, dxdy$$
$$-3c\int_{\mathbf{R}^2} \chi_n(y)y^2 u_{xx}^2 u_x\, dxdy + c\int_{\mathbf{R}^2} \partial_y(\chi_n(y)y^2)u_{xx}u_x u_y\, dxdy$$
$$-c\int_{\mathbf{R}^2} \chi_n(y)y^2(u_{xy}^2 u_x + 2u_{xx}u_{xy}u_y)\, dxdy$$
$$= \int_{\mathbf{R}^2} \partial_{yy}^2(\chi_n(y)y^2)\left[\frac{2a-bc^2}{2}u_{xx}^2 + 2au_{xy}^2 + u_x^2\right]\, dxdy$$
$$-\frac{a}{2}\int_{\mathbf{R}^2} \partial_{yyyy}^4(\chi_n(y)y^2)u_x^2\, dxdy\ .$$

As previously, there exists a constant $C > 0$ such that the last three terms in the left side of (3.9) are dominated by

$$C + \frac{1}{2}\int_{\mathbf{R}^2} \chi_n(y)y^2((1-c^2)u_{xx}^2 + u_{xy}^2)\, dxdy\ .$$

Then we have

$$\frac{1}{2}\int_{\mathbf{R}^2} \chi_n(y)y^2[(1-c^2)u_{xx}^2 + u_{xy}^2]\, dxdy$$
$$+\int_{\mathbf{R}^2} \chi_n(y)y^2[(a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2]\, dxdy$$
$$\le C + \int_{\mathbf{R}^2} \partial_{yy}^2(\chi_n(y)y^2)\left[u_x^2 + \frac{2a-bc^2}{2}u_{xx}^2 + 2au_{xy}^2\right]\, dxdy \tag{3.10}$$
$$-\frac{a}{2}\int_{\mathbf{R}^2} \partial_{yyyy}^4(\chi_n(y)y^2)u_x^2\, dxdy\ .$$

Passing to the limit as $n \longrightarrow \infty$ in (3.10) and using the monotone convergence theorem for the left side and Lebesgue's dominated convergence theorem for the right side one obtains:

$$\int_{\mathbf{R}^2} y^2\left[\frac{1-c^2}{2}u_{xx}^2 + \frac{1}{2}u_{xy}^2 + (a-bc^2)u_{xxx}^2 + (2a-bc^2)u_{xxy}^2 + au_{xyy}^2\right]\, dxdy$$
$$\le C + \int_{\mathbf{R}^2} 2u_x^2 + (2a-bc^2)u_{xx}^2 + 4au_{xy}^2\, dxdy < \infty\ .$$

Thus

$$\int_{\mathbf{R}^2} y^2(u_{xx}^2 + u_{xy}^2 + u_{xxx}^2 + u_{xxy}^2 + u_{xyy}^2)\, dxdy < \infty\ . \tag{3.11}$$

Multiplying the equation (1.2) by $\chi_n(x)x^2u_{yy}$ (respectively by $\chi_n(y)y^2u_{yy}$), integrating by parts and proceeding as above we obtain

$$\int_{\mathbf{R}^2} x^2(u_{xy}^2 + u_{yy}^2 + u_{xxy}^2 + u_{xyy}^2 + u_{yyy}^2)\, dxdy < \infty \tag{3.12}$$

and

$$(3.13) \qquad \int_{\mathbf{R}^2} y^2(u_{xy}^2 + u_{yy}^2 + u_{xxy}^2 + u_{xyy}^2 + u_{yyy}^2)\, dxdy < \infty \, .$$

Theorem 3.1 follows from (3.8), (3.11), (3.12) and (3.13). □

Lemma 3.2 We have $ru_x \in L^\infty(\mathbf{R}^2)$ and $ru_y \in L^\infty(\mathbf{R}^2)$.

Proof: From (1.2) we deduce that

$$(3.14) \qquad Q(\xi_1,\xi_2)\widehat{u_x} = ic\xi_1\mathcal{F}(3u_xu_{xx} + u_xu_{yy} + 2u_yu_{xy})$$

and

$$(3.15)\ . \qquad Q(\xi_1,\xi_2)\widehat{u_y} = ic\xi_2\mathcal{F}(3u_xu_{xx} + u_xu_{yy} + 2u_yu_{xy})$$

We note $h_i = \mathcal{F}^{-1}(\dfrac{\xi_i}{Q(\xi_1,\xi_2)})$ and $g = 3u_xu_{xx} + u_xu_{yy} + 2u_yu_{xy}$. The previous equations can be written as

$$u_x = ich_1 \star g \ \text{ and } \ u_y = ich_2 \star g$$

Then

$$(3.16) \qquad |ru_x| \leq C|rh_1| \star |g| + C|h_1| \star |rg| \, .$$

We claim that $rg \in W^{1,p}(\mathbf{R}^2)$ for all $p \in [1,2]$. Indeed, Theorems 3.1 and 2.1 imply that $rg \in L^p(\mathbf{R}^2)$ for all $p \in [1,2]$. Moreover, since $r\nabla^3 u \in L^2(\mathbf{R}^2)$ we have (denoting by D one of the operators ∂_x or ∂_y):

$$D(rDuD^2u) = (Dr)DuD^2u + rD^2uD^2u + rDuD^3u \in L^p(\mathbf{R}^2)$$

for $1 \leq p \leq 2$. Thus $D(rg) \in L^p(\mathbf{R}^2)$ and so $rg \in W^{1,p}(\mathbf{R}^2)$.

It is clear now that $|rg| \in W^{1,p}(\mathbf{R}^2)$.

By Lemma A1 in Appendix we have $rh_i \in L^\infty(\mathbf{R}^2)$. Then $h_i \in L^2_w(\mathbf{R}^2)$ and using the generalized Young's theorem we deduce

$$|h_i| \star |rg| \in L^q(\mathbf{R}^2) \ \text{ if } \ 2 < q < \infty$$

and

$$D(|h_i| \star |rg|) = |h_i| \star (D|rg|) \in L^q(\mathbf{R}^2) \ \text{ if } \ 2 < q < \infty \, .$$

So $|h_i| \star |rg| \in W^{1,q}(\mathbf{R}^2)$ for $2 < q < \infty$. The Sobolev imbedding theorem gives us $|h_i| \star |rg| \in L^\infty(\mathbf{R}^2)$. But $|rh_i| \star |g|$ is also in $L^\infty(\mathbf{R}^2)$ because $rh_i \in L^\infty(\mathbf{R}^2)$ and $g \in L^1(\mathbf{R}^2)$. Using (3.16) we obtain the desired conclusion. □

We note

$$k_1 = \mathcal{F}^{-1}\left(\frac{\xi_1^2}{Q(\xi_1,\xi_2)}\right) \ , \quad k_2 = \mathcal{F}^{-1}\left(\frac{\xi_1\xi_2}{Q(\xi_1,\xi_2)}\right) \ , \quad k_3 = \mathcal{F}^{-1}\left(\frac{\xi_2^2}{Q(\xi_1,\xi_2)}\right) \ .$$

Lemma 3.3 $\widehat{k_i} \in H^s(\mathbf{R}^2)$ for $0 \leq s < 1$ and $k_i \in L^q(\mathbf{R}^2)$ if $1 < q \leq 2$, $i = 1,2,3$.

Proof: The proof is essentially the same as the proof of Lemma 3.4 in [6]. For the sake of completeness, we give it here.

It is easy to verify that $\widehat{k_i} \in L^2(\mathbf{R}^2)$ and

$$|\nabla \widehat{k_i}| \leq C\frac{|\xi_1| + |\xi_2|}{Q(\xi_1,\xi_2)} \in L^q(\mathbf{R}^2)$$

if $1 \leq q < 2$. Hence $\widehat{k_i}$ belongs to the homogeneous Sobolev space $\dot{W}^{1,q}(\mathbf{R}^2)$, $1 \leq q < 2$. By Theorem 6.5.1 in [2], $\dot{W}^{1,q}(\mathbf{R}^2) \subset \dot{H}^s(\mathbf{R}^2)$ for $s = 2(1-\frac{1}{q})$. So $\widehat{k_i} \in \dot{H}^s(\mathbf{R}^2)$ for any $s \in [0,1)$, $i = 1,2,3$. Since $\widehat{k_i} \in L^2(\mathbf{R}^2)$ we have $\widehat{k_i} \in H^s(\mathbf{R}^2)$, $s \in [0,1)$, $i = 1,2,3$.

Let $q \in (1,2]$ be given. Let $\frac{1}{\alpha} = \frac{1}{q} - \frac{1}{2}$, $\alpha \in (2,\infty]$. We choose $s \in [0,1)$ such that $s\alpha > 2$. Then we have:

$$\begin{aligned}\| k_i \|_{L^q} &\leq \| (1+r^2)^{\frac{s}{2}} k_i \|_{L^2} \cdot \| \frac{1}{(1+r^2)^{\frac{s}{2}}} \|_{L^\alpha} \\ &= \| \widehat{k_i} \|_{H^s} \cdot \| \frac{1}{(1+r^2)^{\frac{s}{2}}} \|_{L^\alpha} \qquad < \infty\end{aligned}$$

Thus $k_i \in L^q(\mathbf{R}^2)$ for all $q \in (1,2]$, $i = 1,2,3$ and the lemma is proved. □

We may state now our main result.

Theorem 3.4 Let $u \in \mathcal{V}$ be a solution of (1.2). Then
a) $r^2 D^\alpha u \in L^\infty(\mathbf{R}^2)$ for all $\alpha \in \mathbf{N}^2$, $|\alpha| \geq 1$;
b) $ru \in L^\infty(\mathbf{R}^2)$.

In view of the remarks 2.3 and 2.4, the estimates given by Theorem 3.4 for u, u_x and u_y are optimal.

Proof: We note

$$\varphi_1 = \frac{3}{2}u_x^2 + \frac{1}{2}u_y^2 \ , \qquad \varphi_2 = u_x u_y \ .$$

The equations (3.2) and (3.3) can be written as

$$u_x = -ck_1 \star \varphi_1 - ck_2 \star \varphi_2 \ ,$$

$$u_y = -ck_2 \star \varphi_1 - ck_3 \star \varphi_2 \; .$$

Let us prove first that $r^{1+\delta}u_x$ and $r^{1+\delta}u_y$ are in $L^\infty(\mathbf{R}^2)$ if $\delta \in [0,1)$. It clearly suffices to show that $r^{1+\delta}(k_i \star \varphi_j) \in L^\infty(\mathbf{R}^2)$. We have:

$$|r^{1+\delta}(k_i \star \varphi_j)| \le C|r^{1+\delta}k_i| \star |\varphi_j| + C|k_i| \star |r^{1+\delta}\varphi_j| \; . \tag{3.17}$$

By Lemma A2 in Appendix (and the remark A3) we have $r^{1+\delta}k_i \in L^\infty(\mathbf{R}^2)$. But $\varphi_j \in L^1(\mathbf{R}^2)$ and so

$$|r^{1+\delta}k_i| \star |\varphi_j| \in L^\infty(\mathbf{R}^2) \; .$$

By Lemma 3.2 and Theorem 2.1,

$$|r^{1+\delta}\varphi_j| \le |(1+r)^2\varphi_j| \cdot |\frac{1}{(1+r)^{1-\delta}}| \in L^p(\mathbf{R}^2)$$

for all $p > \dfrac{2}{1-\delta}$. Since $k_i \in L^q(\mathbf{R}^2)$ for $1 < q \le 2$, we obtain (choosing $p > \dfrac{2}{1-\delta}$ and $q = \dfrac{p-1}{p}$):

$$|k_i| \star |r^{1+\delta}\varphi_j| \in L^\infty(\mathbf{R}^2) \; .$$

Thus the right side of (3.17) is bounded and so $r^{1+\delta}u_x$, $r^{1+\delta}u_y \in L^\infty(\mathbf{R}^2)$ for all $\delta \in [0,1)$.

We have:

$$|r^2 k_i \star \varphi_j| \le C|r^2 k_i| \star |\varphi_j| + C|k_i| \star |r^2\varphi_j| \; .$$

Clearly, $|r^2k_i| \star |\varphi_j| \in L^\infty(\mathbf{R}^2)$ because $r^2k_i \in L^\infty(\mathbf{R}^2)$ by Lemma A2 and $\varphi_j \in L^1(\mathbf{R}^2)$.

Since $|r^{1+\delta}\nabla u| \in L^\infty(\mathbf{R}^2)$ one obtains $r^2\varphi_j \in L^p(\mathbf{R}^2)$ for all $p \in]1,\infty]$. But $k_i \in L^q(\mathbf{R}^2)$ for $1 < q \le 2$ and so $|k_i| \star |r^2\varphi_j| \in L^\infty(\mathbf{R}^2)$. Thus r^2u_x, $r^2u_y \in L^\infty(\mathbf{R}^2)$.

The rest of part a) follows easily by induction. Keeping the notations of Lemma 3.2 we have $r^2 g \in L^p(\mathbf{R}^2)$ for all $p \in]1,\infty]$. If $\alpha \in \mathbf{N}^2$ and $|\alpha| = 2$, then $Q(\xi_1,\xi_2)\widehat{D^\alpha u} = -c \cdot \xi^\alpha \widehat{g}$, so $D^\alpha u$ can be written as

$$D^\alpha u = -c \cdot k_i \star g$$

for an $i \in \{1,2,3\}$. Hence $|r^2 D^\alpha u| \le C(|r^2k_i| \star |g| + |k_i| \star |r^2 g|) \in L^\infty(\mathbf{R}^2)$. Suppose now that $r^2 D^\alpha u \in L^\infty(\mathbf{R}^2)$ if $1 \le |\alpha| \le n$. Let $\gamma \in \mathbf{N}^2$ with $|\gamma| = n+1$. Let $\beta \in \mathbf{N}^2$, $\beta \le \gamma$ and $|\gamma - \beta| = 2$. Then $Q(\xi_1,\xi_2)\widehat{D^\gamma u} = -c \cdot \xi^{\gamma-\beta}\widehat{D^\beta g}$. Hence

$$D^\gamma u = -c \cdot k_i \star (D^\beta g)$$

for an $i \in \{1,2,3\}$. By hypothesis $r^2 D^\beta g \in L^p(\mathbf{R}^2)$ for all $p \in]1,\infty]$ and we deduce as above that $r^2 D^\gamma g \in L^\infty(\mathbf{R}^2)$.

b) We write (3.1) in the form

$$u = ich_1 \star \varphi_1 + ich_2 \star \varphi_2 \; . \tag{3.18}$$

As previously we prove that $r\varphi_i \in W^{1,p}(\mathbf{R}^2)$, $p \in]1,\infty]$ and so $|r\varphi_i| \in W^{1,p}(\mathbf{R}^2)$ for $p \in]1,\infty]$.

By Lemma A1 in Appendix, $h_i \in L^2_w(\mathbf{R}^2)$. The generalized Young's theorem implies

$$|h_i| \star |r\varphi_i| \in L^q(\mathbf{R}^2) \text{ if } q \in]2,\infty[$$

and

$$D(|h_i| \star |r\varphi_i|) = |h_i| \star (D|r\varphi_i|) \in L^q(\mathbf{R}^2),\ q \in]2,\infty[$$

hence $|h_i| \star |r\varphi_i| \in W^{1,q}(\mathbf{R}^2)$ for $q \in]2,\infty[$. By the Sobolev imbedding theorem, $|h_i| \star |r\varphi_i| \in L^\infty(\mathbf{R}^2)$. Clearly $|rh_i| \star |\varphi_i| \in L^\infty(\mathbf{R}^2)$ because $rh_i \in L^\infty(\mathbf{R}^2)$ and $\varphi_i \in L^1(\mathbf{R}^2)$. Thus we have

$$|ru| \le C \sum_{i=1,2} (|rh_i| \star |\varphi_i| + |h_i| \star |r\varphi_i|) \in L^\infty(\mathbf{R}^2) \; .$$

This finishes the proof of Theorem 3.4. $\Box$

4 Some identities

We derive here some identities of Pohozaev type satisfied by the Benney-Luke solitary waves. If $u \in \mathcal{V}$ is a solution of (1.2), multiplying (1.2) by xu_x (respectively by yu_y) and integrating over $\mathbf{R}^2$ we obtain, after a few integrations by parts,

$$\begin{aligned}\int_{\mathbf{R}^2} (1-c^2)u_x^2 - u_y^2 + 3(a-bc^2)u_{xx}^2 - au_{yy}^2 + (2a-bc^2)u_{xy}^2 \; dxdy \\ +2c\int_{\mathbf{R}^2} u_x^3 \; dxdy = 0\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\int_{\mathbf{R}^2} (1-c^2)u_x^2 - u_y^2 + (a-bc^2)u_{xx}^2 - 3au_{yy}^2 - (2a-bc^2)u_{xy}^2 \; dxdy \\ +c\int_{\mathbf{R}^2} (u_x^3 - u_xu_y^2) \; dxdy = 0 \; .\end{aligned} \tag{4.2}$$

Multiplying (1.2) by u and integrating one obtains immediately

$$\begin{aligned}\int_{\mathbf{R}^2} (1-c^2)u_x^2 + u_y^2 + (a-bc^2)u_{xx}^2 + au_{yy}^2 + (2a-bc^2)u_{xy}^2 \; dxdy + \\ \frac{3c}{2}\int_{\mathbf{R}^2} u_x^3 + u_y^2u_x \; dxdy = 0 \; .\end{aligned} \tag{4.3}$$

Combining (4.1), (4.2) and (4.3) we deduce

$$\int_{\mathbf{R}^2} (1-c^2)u_x^2 + u_y^2 \, dxdy = 2\int_{\mathbf{R}^2} (a-bc^2)u_{xx}^2 + au_{yy}^2 + (2a-bc^2)u_{xy} \, dxdy \ .$$

5 Appendix

We prove here some technical facts about the Fourier transform of a special kind of functions of two variables.

Lemma A1. Let $a, b, c, d, e > 0$ and let $Q(\xi_1, \xi_2)$ be the polynomial of two variables

$$Q(\xi_1, \xi_2) = a\xi_1^4 + b\xi_2^4 + c\xi_1^2\xi_2^2 + d\xi_1^2 + e\xi_2^2 \ .$$

If

$$(i) \qquad c^2 - 4ab > 0$$

then we have

a) $r\mathcal{F}^{-1}\left(\dfrac{\xi_1}{Q(\xi_1,\xi_2)}\right) \in L^\infty(\mathbf{R}^2)$;

b) $r\mathcal{F}^{-1}\left(\dfrac{\xi_2}{Q(\xi_1,\xi_2)}\right) \in L^\infty(\mathbf{R}^2)$.

where $r = \sqrt{x^2+y^2}$ and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Proof: We regard separately $Q(\xi_1, \xi_2)$ as a polynomial of second degree in ξ_1^2 (respectively in ξ_2^2) and calculate its discriminant in each case.

$$Q(\xi_1, \xi_2) = a\xi_1^4 + (c\xi_2^2 + d)\xi_1^2 + b\xi_2^4 + e\xi_2^2$$

$$\Delta_1(\xi_2) = (c^2 - 4ab)\xi_2^4 + 2(cd - 2ae)\xi_2^2 + d^2$$

$$Q(\xi_1, \xi_2) = b\xi_2^4 + (c\xi_1^2 + e)\xi_2^2 + a\xi_1^4 + d\xi_1^2$$

$$\Delta_2(\xi_1) = (c^2 - 4ab)\xi_1^4 + 2(ce - 2bd)\xi_1^2 + e^2 \ .$$

Remark that we always have

$$ce - 2bd > 0 \text{ or } cd - 2ae > 0 \ .$$

Indeed, suppose that $ce - 2bd \le 0$. Then $d \ge \dfrac{ce}{2b}$ implies $cd - 2ae \ge \dfrac{c^2e}{2b} - 2ae = \dfrac{e}{2b}(c^2 - 4ab) > 0$ by (i). So we may assume without loss of generality that $ce - 2bd > 0$ and $b = 1$. In this case $Q(\xi_1, \xi_2)$ can be written as a product

$$Q(\xi_1, \xi_2) = (\xi_2^2 + A^2(\xi_1))(\xi_2^2 + B^2(\xi_1))$$

where $A(\xi)$ and $B(\xi)$ are positive and

$$A^2(\xi) = \frac{1}{2}[c\xi^2 + e - \sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}] ,$$

$$B^2(\xi) = \frac{1}{2}[c\xi^2 + e + \sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}] .$$

It is easy to check that the functions A and B have the following properties:

(1) $A \in C(\mathbf{R}) \cap C^\infty(\mathbf{R} \setminus \{0\})$, $B \in C^\infty(\mathbf{R})$, $A(-\xi) = A(\xi)$, $B(-\xi) = B(\xi)$.

(2) There exist constants $C_1, C_2 > 0$ such that

$$C_1|\xi| \le A(\xi) \le C_2|\xi| \text{ and}$$
$$C_1(1+|\xi|) \le B(\xi) \le C_2(1+|\xi|) , \quad \forall \xi \in \mathbf{R}.$$

(3) There are $C_1, C_2 > 0$ verifying

$$C_1 \le A'(\xi) \le C_2 , \quad \forall \xi > 0$$
$$C_1\xi \le B'(\xi) \le C_2\xi , \quad \forall \xi \in [-1,1]$$
$$C_1 \le B'(\xi) \le C_2 , \quad \forall \xi \in [1,\infty[.$$

(4) There exists $C > 0$ such that

$$|A''(\xi)| \le \frac{C}{|\xi|} , \quad \forall \xi \in \mathbf{R} \setminus \{0\} \text{ and}$$
$$|B''(\xi)| \le \frac{C}{(1+|\xi|)} , \quad \forall \xi \in \mathbf{R} .$$

Putting $h_i = \mathcal{F}^{-1}\left(\frac{\xi_i}{Q(\xi_1,\xi_2)}\right)$ we have:

$$h_1(x,y) = \int_{\mathbf{R}^2} e^{ix\xi_1+iy\xi_2} \frac{\xi_1}{B^2(\xi_1)-A^2(\xi_1)} \left(\frac{1}{\xi_2^2+A^2(\xi_1)} - \frac{1}{\xi_2^2+B^2(\xi_1)}\right) d\xi_1 d\xi_2 .$$

But $\mathcal{F}^{-1}\left(\frac{1}{\xi^2+a^2}\right)(x) = -\frac{1}{2a}e^{-a|x|}$ if $Re(a) > 0$ and so we obtain

$$h_1(x,y) =$$

$$(5) \qquad = \int_{\mathbf{R}} e^{ix\xi} \frac{\xi}{B^2(\xi)-A^2(\xi)} \left[\frac{1}{2B(\xi)}e^{-B(\xi)|y|} - \frac{1}{2A(\xi)}e^{-A(\xi)|y|}\right] d\xi$$

$$= \int_{\mathbf{R}} e^{ix\xi} \cdot \frac{\xi\left[\frac{1}{2B(\xi)}e^{-B(\xi)|y|} - \frac{1}{2A(\xi)}e^{-A(\xi)|y|}\right]}{\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}} d\xi .$$

By (2),

$$\left| e^{ix\xi} \frac{\xi}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}} \left[\frac{1}{2B(\xi)} e^{-B(\xi)|y|} - \frac{1}{2A(\xi)} e^{-A(\xi)|y|} \right] \right|$$

$$\leq Ce^{-C_1|y||\xi|} \ ,$$

hence if $y \neq 0$,

$$|h_1(x,y)| \leq C \int_{\mathbf{R}} e^{-C_1|y||\xi|} \, d\xi = \frac{C}{|y|} \ . \tag{6}$$

To obtain an estimate of $|h_1(x,y)|$ in terms of $\dfrac{1}{|x|}$ we use the following elementary

Lemma H. Let $I \subset \mathbf{R}$ be an interval (bounded or not) and let $f : I \longrightarrow \mathbf{R}$ be an integrable and monotone function. There exists an absolute constant $C > 0$ (we may take $C = 4\sqrt{2}\pi$) such that

$$\left| \int_I e^{ix\xi} f(\xi) \, d\xi \right| \leq \frac{C}{|x|} \cdot \sup_{\xi \in I} |f(\xi)| \ .$$

Let $f_{1,y}(\xi) = \dfrac{\xi}{A(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}} e^{-A(\xi)|y|}$. Note that $f_{1,y}$ is differentiable on $\mathbf{R} \setminus \{0\}$. If we prove that $f'_{1,y}$ has at most N zeros where N does not depend on y, then we can decompose $\mathbf{R}$ into (at most) $N+2$ intervals where $f_{1,y}$ is monotone. Applying Lemma H on each of these intervals we finally obtain

$$\left| \int_{\mathbf{R}} e^{ix\xi} f_{1,y}(\xi) \, d\xi \right| \leq \frac{C}{|x|} \cdot \sup_{\xi \in \mathbf{R}} |f_{1,y}(\xi)| \leq \frac{C}{|x|} \ . \tag{7}$$

Let us count now the zeros of $f'_{1,y}$. For $\xi \neq 0$ one obtains

$$f'_{1,y}(\xi) = \frac{e^{-A(\xi)|y|}}{A(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}} \times$$

$$\times \left[\frac{-(c^2-4a)\xi^4+e^2}{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2} - \frac{\xi A'(\xi)}{A(\xi)} - \xi|y|A'(\xi) \right] \ .$$

Thus $f'_{1,y}(\xi) = 0$ clearly implies that ξ is a solution of an equation

$$P(\xi)A^2(\xi) + R(\xi)A'(\xi)A(\xi) + S(\xi,|y|)A'(\xi)A^2(\xi) = 0 \ ,$$

where $P(\xi)$, $R(\xi)$ are polynomials in ξ and and $S(\xi,|y|)$ is a polynomial in two variables ξ, $|y|$. Multiplying this by $\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}$ we obtain

$$P_1(\xi)+R_1(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}$$

$$+\left(S_1(\xi,|y|)+S_2(\xi,|y|)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}\right)A(\xi)=0\ ,$$

where $P_1(\xi)$, $R_1(\xi)$ and $S_1(\xi,|y|)$, $S_2(\xi,|y|)$ are polynomials. Passing the last term on the right and taking the squares we deduce that ξ must satisfy

$$P_2(\xi)+R_2(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}$$

$$=S_3(\xi,|y|)+S_4(\xi,|y|)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}\ .$$

(here $P_2(\xi)$, $R_2(\xi)$, $S_3(\xi,|y|)$ and $S_4(\xi,|y|)$ are polynomials).

If we isolate $\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}$ and take again the squares, we find that

$$\Phi(\xi,|y|)=0\ ,$$

where $\Phi(\xi,|y|)$ is a polynomial in two variables. Let N be the degree of Φ in the first variable. It is clear now that for a fixed y, the last equation has at most N solutions; hence for each y, $f'_{1,y}$ has at most N zeros in $\mathbf{R}\setminus\{0\}$.

Exactly the same argument applies to

$$f_{2,y}(\xi)=\frac{\xi}{B(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}e^{-B(\xi)|y|}$$

and gives us the estimate

$$\left|\int_{\mathbf{R}} e^{ix\xi}f_{2,y}(\xi)\,d\xi\right|\leq\frac{C}{|x|}\ . \tag{8}$$

From (6), (7) and (8) we infer that

$$|h_1(x,y)|\leq\frac{C}{|x|+|y|}\ ,$$

that is, $rh_1\in L^\infty(\mathbf{R}^2)$.

b) One easily checks that if $Re(a)>0$ and $Re(b)>0$, then

$$\mathcal{F}\left(\frac{i}{2}sgn(x)\frac{1}{b^2-a^2}(e^{-a|x|}-e^{-b|x|})\right)(\xi)=\frac{\xi}{(\xi^2+a^2)(\xi^2+b^2)}$$

or equivalently

$$\mathcal{F}^{-1}\left(\frac{\xi}{(\xi^2+a^2)(\xi^2+b^2)}\right)(x)=\frac{i}{2}sgn(x)\frac{1}{b^2-a^2}(e^{-a|x|}-e^{-b|x|})\ .$$

Consequently, we have

$$h_2(x,y) =$$

$$= \int_{\mathbf{R}^2} e^{ix\xi_1+iy\xi_2} \frac{\xi_2}{(\xi_2^2+A^2(\xi_1))(\xi_2^2+B^2(\xi_1))}\, d\xi_1 d\xi_2$$

$$= \int_{\mathbf{R}} \frac{i}{2} sgn(y) e^{ix\xi} \frac{e^{-A(\xi)|y|}-e^{-B(\xi)|y|}}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\, d\xi \ .$$

If $y \neq 0$,

$$\begin{aligned} |h_2(x,y)| &\leq C\int_{\mathbf{R}} e^{-A(\xi)|y|}-e^{-B(\xi)|y|}\, d\xi \\ &\leq C\int_{\mathbf{R}} e^{-C_1|y||\xi|} \qquad (9) \\ &= \frac{C}{|y|} \ . \end{aligned}$$

If $x \neq 0$, we apply Lemma H to the functions

$$g_{1,y}(\xi) = \frac{e^{-A(\xi)|y|}}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}$$

and

$$g_{2,y}(\xi) = \frac{e^{-B(\xi)|y|}}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}$$

and reason as in part a) to obtain

$$h_2(x,y) \leq \frac{C}{|x|} \ . \qquad (10)$$

Inequalities (9) and (10) clearly give $h_2(x,y) \leq \dfrac{C}{|r|}$, which is the desired conclusion. □

Lemma A2. With the assumptions and the notations of Lemma A1, we have:

a) $r^2\mathcal{F}^{-1}\left(\dfrac{\xi_1^2}{Q(\xi_1,\xi_2)}\right) \in L^\infty(\mathbf{R}^2)$;

b) $r^2\mathcal{F}^{-1}\left(\dfrac{\xi_1\xi_2}{Q(\xi_1,\xi_2)}\right) \in L^\infty(\mathbf{R}^2)$;

c) $r^2\mathcal{F}^{-1}\left(\dfrac{\xi_2^2}{Q(\xi_1,\xi_2)}\right) \in L^\infty(\mathbf{R}^2)$.

Proof: a) As in the proof of Lemma A1, we write

$$k_1(x,y) = \mathcal{F}^{-1}\left(\frac{\xi_1^2}{Q(\xi_1,\xi_2)}\right) =$$

$$= \int_{\mathbf{R}^2} e^{ix\xi_1+iy\xi_2} \frac{\xi_1^2}{\sqrt{(c^2-4a)\xi_1^4+2(ce-2d)\xi_1^2+e^2}} \left(\frac{1}{\xi_2^2+A^2(\xi_1)} - \frac{1}{\xi_2^2+B^2(\xi_1)}\right) d\xi_1 d\xi_2$$

$$= \int_{\mathbf{R}} e^{ix\xi} \cdot \frac{\xi^2\left(\frac{1}{2B(\xi)}e^{-B(\xi)|y|} - \frac{1}{2A(\xi)}e^{-A(\xi)|y|}\right)}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\ d\xi\ ;$$

$$k_1(x,y) = T_B - T_A\ ,$$

where

$$T_B = \int_{\mathbf{R}} e^{ix\xi - B(\xi)|y|} \cdot \frac{\xi^2}{2B(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\ ,$$

$$T_A = \int_{\mathbf{R}} e^{ix\xi - A(\xi)|y|} \cdot \frac{\xi^2}{2A(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\ .$$

Integrating by parts we get

(11)

$$T_B = e^{ix\xi - B(\xi)|y|} \cdot \frac{\xi^2}{(ix - B'(\xi)|y|)2B(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\Bigg|_{-\infty}^{\infty} -$$

$$-\int_{-\infty}^{\infty} e^{ix\xi - B(\xi)|y|} \cdot \frac{\xi}{(ix - B'(\xi)|y|)B(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\ d\xi$$

$$-\int_{-\infty}^{\infty} e^{ix\xi - B(\xi)|y|} \cdot \frac{\xi^2 \cdot B''(\xi)|y|}{(ix - B'(\xi)|y|)^2 2B(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\ d\xi$$

$$+\int_{-\infty}^{\infty} e^{ix\xi - B(\xi)|y|} \cdot \frac{\xi^2 B'(\xi)}{(ix - B'(\xi)|y|)2B^2(\xi)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\ d\xi$$

$$+\int_{-\infty}^{\infty} e^{ix\xi - B(\xi)|y|} \cdot \frac{2\xi^3[(c^2-4a)\xi^2+ce-2d]}{(ix - B'(\xi)|y|)2B(\xi)(\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2})^3}\ d\xi\ .$$

The first term equals 0.

Suppose $y \neq 0$. If $\xi \in [-1,1] \setminus \{0\}$ by (3) we have

$$\left| \frac{\xi}{ix - B'(\xi)|y|} \right| = \frac{1}{\left| i\frac{x}{\xi} - \frac{B'(\xi)}{\xi}|y| \right|} \leq \frac{1}{\left| \frac{B'(\xi)}{\xi} \right| |y|} \leq \frac{1}{C_1|y|} \ .$$

If $\xi \in \mathbf{R} \setminus [-1,1]$, (3) gives us

$$\left| \frac{1}{ix - B'(\xi)|y|} \right| \leq \frac{1}{B'(\xi)|y|} \leq \frac{1}{C_1|y|} \ .$$

It is now easy to see that the absolute value of each of the four integrals above is less than

$$\frac{C}{|y|} \int_{-\infty}^{\infty} e^{-B(\xi)|y|} d\xi \leq \frac{C}{|y|^2} \ .$$

Hence

$$T_B \leq \frac{C}{|y|^2} \ . \tag{12}$$

Consider, for example, the first integral in (11). It can be written as

$$\int_{-\infty}^{\infty} e^{ix\xi} f_{x,y}(\xi) \, d\xi \ ,$$

where

$$f_{x,y}(\xi) = e^{-B(\xi)|y|} \cdot \frac{B'(\xi)\xi|y| + ix\xi}{(x^2 + B'(\xi)^2|y|^2)B(\xi)\sqrt{(c^2 - 4a)\xi^4 + 2(ce - 2d)\xi^2 + e^2}}$$

We argue as in the proof of Lemma A1, part a). The number of zeros of $\frac{d}{d\xi}(Re \ f_{x,y}(\xi))$ and $\frac{d}{d\xi}(Im \ f_{x,y}(\xi))$ is finite and does not depend on (x,y). Lemma H applies and we deduce that for $x \neq 0$,

$$\left| \int_{-\infty}^{\infty} e^{ix\xi} f_{x,y}(\xi) \, d\xi \right| \leq \frac{C}{|x|} \sup_{\xi \in \mathbf{R}} |f_{x,y}(\xi)| \leq \frac{C}{x^2} \ .$$

Using the same argument we obtain that the other three integrals in T_B are bounded (in absolute value) by $\frac{C}{x^2}$. Hence

$$|T_B| \leq \frac{C}{x^2} \ . \tag{13}$$

Inequalities (12) and (13) give

$$|T_B| \leq \frac{C}{r^2} \ .$$

In the same manner (integrating by parts on $(0,\infty)$ and observing that A is symmetric, i.e $A(-\xi)=A(\xi)$) we obtain

$$|T_A| \leq \frac{C}{r^2} \ ,$$

and so $|k_1(x,y)| \leq \dfrac{C}{r^2}$.

b) We have

$$k_2(x,y) = \mathcal{F}^{-1}\left(\frac{\xi_1\xi_2}{Q(\xi_1,\xi_2)}\right) =$$

(14)

$$\begin{aligned} &= \int_{\mathbf{R}^2} e^{ix\xi_1+iy\xi_2} \frac{\xi_1\xi_2}{(\xi_2^2+A^2(\xi_1))(\xi_2^2+B^2(\xi_1))}\, d\xi_2 d\xi_1 \\ &= \frac{i}{2}sgn(y)\int_{\mathbf{R}} e^{ix\xi} \frac{\xi(e^{-A(\xi)|y|}-e^{-B(\xi)|y|})}{\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\, d\xi \\ &= \frac{i}{2}sgn(y)(S_A-S_B) \ . \end{aligned}$$

Integrating by parts,

(15)

$$\begin{aligned} S_B = &-\int_{-\infty}^{\infty} e^{ix\xi-B(\xi)|y|}\cdot\frac{d}{d\xi}\left[\frac{\xi}{(ix-B'(\xi)|y|)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\right] d\xi \\ = &-\int_{-\infty}^{\infty} e^{ix\xi-B(\xi)|y|}\cdot\frac{1}{(ix-B'(\xi)|y|)\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\, d\xi \\ &-\int_{-\infty}^{\infty} e^{ix\xi-B(\xi)|y|}\cdot\frac{\xi B''(\xi)|y|}{(ix-B'(\xi)|y|)^2\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2}}\, d\xi \\ &+\int_{-\infty}^{\infty} e^{ix\xi-B(\xi)|y|}\cdot\frac{2\xi^2[(c^2-4a)\xi^2+ce-2d]}{(ix-B'(\xi)|y|)(\sqrt{(c^2-4a)\xi^4+2(ce-2d)\xi^2+e^2})^3}\, d\xi \ . \end{aligned}$$

We argue as previously and use again Lemma H for the real and the imaginary part of the functions appearing in the oscillatory integrals in S_B. If $x\neq 0$ we find that the first and the third of these integrals are bounded (in absolute value) by $\dfrac{C}{x^2}$, while the second is bounded by $\dfrac{C|y|}{|x|^3}$.

The term S_A is easier to handle because $|A'(\xi)| \geq C_1 > 0$ if $\xi \neq 0$. We obtain that $|S_A| \leq \dfrac{C}{x^2}$.

Finally,

$$|k_2(x,y)| \le \frac{C}{x^2} + \frac{C|y|}{|x|^3} \; . \tag{16}$$

By (14) we have

$$\begin{aligned} |k_2(x,y)| &\le C \int_{\mathbf{R}} |\xi| (e^{-A(\xi)|y|} - e^{-B(\xi)|y|}) \, d\xi \\ &\le C \int_{\mathbf{R}} |\xi| e^{-C_1|\xi|\cdot|y|} \, d\xi \qquad (17) \\ &= \frac{C'}{y^2} \; . \end{aligned}$$

From (16) and (17) we deduce that

$$\begin{aligned} |k_2(x,y)| &\le \min\left(\frac{C}{x^2} + \frac{C|y|}{|x|^3}, \frac{C'}{y^2}\right) \\ &\le C' \min\left(\frac{1}{x^2}, \frac{1}{y^2}\right) \\ &\le \frac{C''}{r^2} \; . \end{aligned}$$

This proves b).

c) We have

$$k_3(x,y) = \mathcal{F}^{-1}\left(\frac{\xi_2^2}{Q(\xi_1,\xi_2)}\right)(x,y) =$$

$$= \int_{\mathbf{R}^2} e^{ix\xi_1 + iy\xi_2} \cdot \frac{1}{B^2(\xi_1) - A^2(\xi_1)} \cdot \left(\frac{-A^2(\xi_1)}{\xi_2^2 + A^2(\xi_1)} + \frac{B^2(\xi_1)}{\xi_2^2 + B^2(\xi_1)}\right) d\xi_1 d\xi_2$$

$$= \frac{1}{2} \int_{\mathbf{R}} e^{ix\xi} \frac{A(\xi) e^{-A(\xi)|y|} - B(\xi) e^{-B(\xi)|y|}}{\sqrt{(c^2 - 4a)\xi^4 + 2(ce - 2d)\xi^2 + e^2}} \, d\xi \; .$$

If $y \ne 0$ then clearly

$$\begin{aligned} |k_3(x,y)| &\le C \int_{-\infty}^{\infty} A(\xi) e^{-A(\xi)|y|} + B(\xi) e^{-B(\xi)|y|} \, d\xi \\ &\le C' \int_{-\infty}^{\infty} |\xi| e^{-C_1|\xi|\cdot|y|} \, d\xi + C \int_{-1}^{1} e^{-C|y|} \, d\xi \qquad (18) \\ &\le \frac{C''}{y^2} \; . \end{aligned}$$

Integrating by parts we get

$$\int_{\mathbf{R}} e^{ix\xi - B(\xi)|y|} \cdot \frac{B(\xi)}{\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}}\, d\xi =$$

$$- \int_{-\infty}^{\infty} e^{ix\xi - B(\xi)|y|} \cdot \frac{B'(\xi)}{(ix - B'(\xi)|y|)\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}}$$

$$- \int_{-\infty}^{\infty} e^{ix\xi - B(\xi)|y|} \cdot \frac{B(\xi)B''(\xi)|y|}{(ix - B'(\xi)|y|)^2\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}}$$

$$+ \int_{-\infty}^{\infty} e^{ix\xi - B(\xi)|y|} \cdot \frac{2B(\xi)[(c^2-4a)\xi^3 + (ce-2d)\xi]}{(ix - B'(\xi)|y|)(\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2})^3}\, .$$

We use the same argument involving Lemma H as before and conclude that the last sum of integrals is bounded by $\dfrac{C}{x^2} + \dfrac{C|y|}{|x|^3}$.

Let us estimate the term of $k_3(x,y)$ containing $A(\xi)$. Integrating by parts on $(-\infty, 0)$ and on $(0,\infty)$ we have

$$\int_{\mathbf{R}} e^{ix\xi - A(\xi)|y|} \cdot \frac{A(\xi)}{\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}}\, d\xi =$$

$$= \frac{2A(0)A'(0+)}{x^2 + [A'(0+)]^2} -$$

$$- \int_{-\infty}^{\infty} e^{ix\xi - A(\xi)|y|} \cdot \frac{A'(\xi)}{(ix - A'(\xi)|y|)\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}}$$

$$- \int_{-\infty}^{\infty} e^{ix\xi - A(\xi)|y|} \cdot \frac{A(\xi)A''(\xi)|y|}{(ix - A'(\xi)|y|)^2\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2}}$$

$$+ \int_{-\infty}^{\infty} e^{ix\xi - A(\xi)|y|} \cdot \frac{2A(\xi)[(c^2-4a)\xi^3 + (ce-2d)\xi]}{(ix - A'(\xi)|y|)(\sqrt{(c^2-4a)\xi^4 + 2(ce-2d)\xi^2 + e^2})^3}\, .$$

Using the same method we obtain that the last sum of integrals is bounded by $\dfrac{C}{x^2}$. Finally,

$$|k_3(x,y)| \le \frac{C}{x^2} + \frac{C|y|}{|x|^3}\, . \tag{19}$$

Inequalities (18) and (19) give us $|k_3(x,y)| \le \dfrac{C}{r^2}$.

The Lemma A2 is proved. □

Remark A3. It is much easier to show that $|k_i(x,y)| \leq \dfrac{C}{r}$. The proof is similar to that of Lemma A1 and does not use integrations by parts. Hence $|k_i(x,y)| \leq \dfrac{C}{r^\alpha}$ for all $\alpha \in [1,2]$, $i = 1,2,3$.

Acknowledgement

I am indebted to professor JEAN-CLAUDE SAUT who brought the subject to my attention for his helpful suggestions and advice.

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