On the symmetry of minimizers

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Dedicated to Dorel Miheţ,
for his teaching, his friendship, and the inspiration he gave to me.

Abstract

For a large class of variational problems we prove that minimizers are symmetric whenever they are C^1.

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1 Introduction and main results

In this paper we study the symmetry of minimizers for general variational problems of the form

$$(\mathcal{P}) \qquad \begin{aligned} &\text{minimize } E(u) := \int_\Omega F(|x|, u(x), |\nabla u(x)|)\, dx \qquad \text{under } k \text{ constraints} \\ &Q_j(u) = \int_\Omega G_j(|x|, u(x), |\nabla u(x)|)\, dx = \lambda_j, \qquad j = 1, \dots, k. \end{aligned}$$

The solutions of many partial differential equations are obtained as minimizers for problems like $(\mathcal{P})$. Knowing in advance that such solutions are symmetric is very important for their theoretical study as well as for their numerical approximation. If the minimizers of $(\mathcal{P})$ are standing or solitary waves for an evolution equation, symmetry could be very useful to investigate the stability properties of such solutions. Note also that in many problems, symmetry is the first step in proving the uniqueness of special solutions.

Given the motivation above, many important particular cases of $(\mathcal{P})$ have already been considered in the literature. In [11, 12], O. Lopes has developed his reflection method - a very efficient tool to prove symmetries for minimizers of functionals $E_1(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F_1(|x|, u)\, dx$ under the constraint $Q(u) = \int_\Omega G(|x|, u)\, dx = constant$, where Ω is a domain invariant by rotations. This method is based on a device of "reflecting" a minimizer with respect to hyperplanes that "split the constraint in two" and on the use of a unique continuation principle for the Euler-Lagrange equations satisfied by minimizers. Note that the method can be used for vector-valued minimizers whose components eventually change sign and no additional assumptions are made on the functions F_1 and G (except the usual growth and smoothness assumptions that ensure the existence and the regularity of minimizers). Up to now this method has been used for problems involving only one constraint. Its main restriction is that it can be used only when the minimizers satisfy an Euler-Lagrange system for which a unique continuation theorem is available. However, we have to mention that the reflection method has been successfully used in [13] for minimizers of some nonlocal functionals of the form $E_2(u) = \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2 \, d\xi + \int_{\mathbf{R}^N} F_2(u) \, dx$. The class of functionals considered in [13] include the generalized Choquard functional, the Hamiltonian for the generalized Davey-Stewartson equation as well as functionals involving fractional powers of the Laplacian. Instead of unique continuation results, some new and quite unexpected integral identities for nonlocal operators were used to get symmetry results.

In a recent paper [4], F. Brock studies the symmetry of minimizers of the functional $\int_{\mathbf{R}^N} \sum_{i=1}^{n} |\nabla u_i|^p + F(|x|, u_1, \dots, u_n) \, dx$ under several constraints $\int_{\mathbf{R}^N} G_{i,j}(u_i) \, dx = c_{i,j}$. He uses two-points rearrangements and a variant of the strong maximum principle due to Pucci, Serrin and Zou ([16]) to prove symmetries. Assuming that F is nonincreasing in the first variable and that $\frac{\partial F}{\partial u_i}$ is nondecreasing in the variables u_k for $k \neq i$ (a cooperative condition), he shows that the superlevel sets $\{u_i > t\}$ for $t > 0$, respectively the sublevel sets $\{u_i < t\}$ for $t < 0$, are balls. Under more restrictive conditions (F strictly decreaing in the first variable or an assumption that depends on Lagrange multipliers associated to minimizers - assumption that could be quite difficult to check in applications, as already mentioned in [4]), he proves that any component of the minimizer is radially symmetric about 0, has constant sign and is monotone in $|x|$. Note that whenever the arguments in [4] lead to symmetry, they also imply monotonicity. On the other hand, in [4] there is an example of sign-changing minimizer for a particular functional of the type considered. It is remarkable that the results of F. Brock are valid for an arbitrary number of constraints. However, these constraints must have a special form (because they have to be preserved by rearrangements of functions). For instance, one cannot allow constraints of the form $\int_{\mathbf{R}^N} G(u_i, u_j) \, dx = constant$.

We have to mention that in a series of recent papers (see [2], [15], [17] and references therein), different new techniques were developed to study the symmetry of solutions for some classes of elliptic problems. These techniques are essentially based on foliated Schwarz rearrangements and on polarization of functions and can be used for sign-changing solutions. They also give some monotonicity properties.

The aim of the present paper is to prove symmetry of minimizers for problem $(\mathcal{P})$ under general assumptions. We use the device of reflecting minimizers with respect to hyperplanes introduced by O. Lopes, but we do not need unique continuation theorems. Instead, we use in an essential way the regularity of minimizers. (To our knowledge, symmetry results for minimizers that may be nonsmooth were obtained only in the case of convex functionals.) We are able to deal with several constraints, but each additional constraint produces the loss of one direction of symmetry; we will see later (Examples 6 and 7) that under the general conditions considered here, this is a very natural phenomenon.

In the sequel Ω denotes an open set in $\mathbf{R}^N$ invariant under rotations (and centered at the origin). It is not assumed that Ω is connected or bounded. We denote $A_\Omega = \{|x| \mid x \in \Omega\}$. We consider vector-valued minimizers $u : \Omega \longrightarrow \mathbf{R}^n$ of $(\mathcal{P})$ that belong to some function space $\mathcal{X}$. Throughout $F, G_1, \dots G_k$ are real-valued functions defined on $A_\Omega \times \mathbf{R}^m \times [0, \infty)$ in such a way that for any $v \in \mathcal{X}$, the functions $x \longmapsto F(|x|, v(x), |\nabla v(x)|)$ and $x \longmapsto G_j(|x|, v(x), |\nabla v(x)|)$, $1 \leq j \leq k$, belong to $L^1(\Omega)$.

Let V be an affine subspace of $\mathbf{R}^N$. For $x \in \mathbf{R}^N$ we denote by $p_V(x)$ the projection of x onto V and by $s_V(x)$ the symmetric point of x with respect to V, that is $s_V(x) = 2p_V(x) - x$. We say that a function f defined on $\mathbf{R}^N$ is *symmetric* with respect to V if $f(x) = f(s_V(x))$ for any x. We say that f is *radially symmetric* with respect to V if there exists a function $\tilde{f}$ defined on $V \times [0, \infty)$ such that $f(x) = \tilde{f}(p_V(x), |x - p_V(x)|)$.

Let Π be a hyperplane in $\mathbf{R}^N$ and let Π^+ and Π^- be the two half-spaces determined by Π. Given a function f defined on $\mathbf{R}^N$, we denote

$$f_{\Pi^+}(x) = \begin{cases} f(x) & \text{if} \quad x \in \Pi^+ \cup \Pi, \\ f(s_\Pi(x)) & \text{if} \quad x \in \Pi^-, \end{cases} \qquad \text{respectively}$$

$$(1) \qquad f_{\Pi^-}(x) = \begin{cases} f(x) & \text{if} \quad x \in \Pi^- \cup \Pi, \\ f(s_\Pi(x)) & \text{if} \quad x \in \Pi^+. \end{cases}$$

If f is defined on a rotation invariant subset Ω centered at the origin, $\Omega \neq \mathbf{R}^N$, the above definition makes sense only if Π contains the origin. We say that Π splits the constraints in two for a function $v \in \mathcal{X}$ if

$$(2) \quad \int_{\Omega \cap \Pi^+} G_j(|x|, v(x), |\nabla v(x)|)\, dx = \int_{\Omega \cap \Pi^-} G_j(|x|, v(x), |\nabla v(x)|)\, dx \qquad \text{for } j = 1, \dots, k.$$

We make the following assumptions.

A1. For any $v \in \mathcal{X}$ and any hyperplane Π containing the origin, we have $v_{\Pi^+}, v_{\Pi^-} \in \mathcal{X}$.

A2. Problem $(\mathcal{P})$ admits minimizers in $\mathcal{X}$ and any minimizer is a C^1 function on Ω.

We can now state our symmetry results.

Theorem 1. *Assume that $0 \leq k \leq N-2$ and* **A1, A2** *are satisfied. Let $u \in \mathcal{X}$ be a minimizer for problem $(\mathcal{P})$. There exists a $k-$dimensional vector subspace V in $\mathbf{R}^N$ such that u is radially symmetric with respect to V.*

If $\Omega = \mathbf{R}^N$ and the considered functionals are invariant by translations, Theorem 1 can be improved. More precisely, consider the following particular case of $(\mathcal{P})$:

$$(\mathcal{P}') \qquad \begin{gathered} \text{minimize } E(u) := \int_{\mathbf{R}^N} F(u(x), |\nabla u(x)|)\, dx \qquad \text{subject to } k \text{ constraints} \\ Q_j(u) = \int_{\mathbf{R}^N} G_j(u(x), |\nabla u(x)|)\, dx = \lambda_j, \qquad j = 1, \dots, k. \end{gathered}$$

In this case assumption **A1** is replaced by

A1.' For any $v \in \mathcal{X}$ and any affine hyperplane Π in $\mathbf{R}^N$ we have $v_{\Pi^+}, v_{\Pi^-} \in \mathcal{X}$.

The following result holds.

Theorem 2. *Assume that $1 \leq k \leq N-1$,* **A1'** *and* **A2** *are satisfied and there exists $j \in \{1, \dots, k\}$ such that $\lambda_j \neq 0$. Let $u \in \mathcal{X}$ be a minimizer for problem $(\mathcal{P}')$. There exists a $(k-1)-$dimensional affine subspace V in $\mathbf{R}^N$ such that u is radially symmetric with respect to V.*

If $(\mathcal{P}')$ involves only one constraint, Theorem 2 implies that any minimizer is radial with respect to some point.

In applications, assumptions **A1** or **A1'** are usually easy to check. On the contrary, assumption **A2** requires much more attention. In most applications, under suitable growth and smoothness assumptions on the functions $F, G_1, \ldots, G_k$, the functionals $E, Q_1, \ldots, Q_k$ are differentiable on $\mathcal{X}$ and the minimizers satisfy Euler-Lagrange equations (however, this is not always the case: see [1] for examples of minimizers that do not satisfy Euler-Lagrange equations). Very often the Euler-Lagrange equations are, in fact, quasilinear elliptic systems. Many efforts have been made during the last 50 years, since the pioneer work of de Giorgi, Nash and Moser, to study the regularity of solutions of such systems and there is a huge literature devoted to the subject. Important progress has been made and various sufficient conditions that guarantee the regularity of solutions have been given. It would exceed the scope of the present paper to resume these works, or even to give here a significant list of conditions that ensure the regularity of minimizers. For these issues (and also for historical notes) we refer the reader to the standard books [5, 7, 8, 9, 10, 14] and references therein.

In the next section we give the proofs of Theorems 1 and 2. We end this paper by some remarks and examples which show that, under the general conditions considered here, our results are optimal even for scalar-valued minimizers.

2 Proofs

Proof of Theorem 1. Consider first the case $1 \leq k \leq N-2$. For $v \in \mathbf{R}^N$, $v \neq 0$, denote $\Pi_v = \{x \in \mathbf{R}^N \mid x.v = 0\}$, $\Pi_v^+ = \{x \in \mathbf{R}^N \mid x.v > 0\}$ and $\Pi_v^- = \{x \in \mathbf{R}^N \mid x.v < 0\}$. For $j = 1, \ldots, k$, we define $\varphi_j : S^{N-1} \longrightarrow \mathbf{R}$ by

$$\varphi_j(v) = \int_{\Pi_v^+ \cap \Omega} G_j(|x|, u(x), |\nabla u(x)|)\, dx - \int_{\Pi_v^- \cap \Omega} G_j(|x|, u(x), |\nabla u(x)|)\, dx.$$

It is obvious that $\varphi_j(-v) = \varphi_j(v)$ and it follows immediately from Lebesgue's dominated convergence theorem that each φ_j is continuous on S^{N-1}. We will use the following well-known result (see, e.g., [18], Theorem 9 p. 266):

Borsuk-Ulam Theorem. *Given a continuous map $f : S^{n_1} \longrightarrow \mathbf{R}^{n_2}$ with $n_1 \geq n_2 \geq 1$, there exists $x \in S^{n_1}$ such that $f(x) = f(-x)$.*

Equivalently, any continuous odd map $f : S^{n_1} \longrightarrow \mathbf{R}^{n_2}$, $n_1 \geq n_2 \geq 1$, must vanish.

We use the Borsuk-Ulam theorem for the odd continuous map $\Phi = (\varphi_1, \ldots, \varphi_k) : S^{N-1} \longrightarrow \mathbf{R}^k$ and we infer that there exists $e_1 \in S^{N-1}$ such that $\Phi(e_1) = 0$, that is Π_{e_1} splits the constraints in two for the minimizer u.

Our aim is to show that u is symmetric with respect to Π_{e_1}. We denote $u_1 = u_{\Pi_{e_1}^-}$ and $u_2 = u_{\Pi_{e_1}^+}$ the two reflected functions obtained from u as in (1). By **A1** we have $u_1,\ u_2 \in \mathcal{X}$. Since Π_{e_1} splits the constraints in two, a simple change of variables shows that $\int_\Omega G_j(|x|, u_1(x), |\nabla u_1(x)|)\, dx = 2 \int_{\Pi_v^- \cap \Omega} G_j(|x|, u_1(x), |\nabla u_1(x)|)\, dx = \lambda_j$ for any $j \in \{1, \ldots, k\}$, that is u_1 satisfies the constraints. In the same way u_2 satisfies the constraints. Since u is a minimizer for $(\mathcal{P})$, we must have $E(u_1) \geq E(u)$ and $E(u_2) \geq E(u)$. On the other hand, we get

$$E(u_1) + E(u_2) = 2 \int_{\Pi_v^- \cap \Omega} F(|x|, u_1(x), |\nabla u_1(x)|)\, dx + 2 \int_{\Pi_v^+ \cap \Omega} F(|x|, u_1(x), |\nabla u_1(x)|)\, dx$$
$$= 2E(u).$$

Thus necessarily $E(u_1) = E(u_2) = E(u)$ and u_1, u_2 are also minimizers for problem $(\mathcal{P})$. Moreover, they are symmetric with respect to Π_{e_1}.

Now let us consider the minimizer u_1. We define $\psi_j : S^{N-1} \longrightarrow \mathbf{R}$ by

$$\psi_j(v) = \int_{\Pi_v^+ \cap \Omega} G_j(|x|, u_1(x), |\nabla u_1(x)|)\, dx - \int_{\Pi_v^- \cap \Omega} G_j(|x|, u_1(x), |\nabla u_1(x)|)\, dx.$$

As previously, it is not hard to see that ψ_j is a continuous odd mapping on S^{N-1}, $1 \le j \le k$. In particular, the restriction of $\Psi = (\psi_1, \dots, \psi_k)$ to $S^{N-1} \cap \Pi_{e_1}$ is a continuous odd mapping from this space to $\mathbf{R}^k$. Since $S^{N-1} \cap \Pi_{e_1}$ can be identified to S^{N-2} and $k \le N-2$, we may use the Borsuk-Ulam theorem again and we infer that there exists $e_2 \in S^{N-1} \cap \Pi_{e_1}$ such that $\Psi(e_2) = 0$, i.e. Π_{e_2} splits the constraints in two for the minimizer u_1. We denote $u_{1,1} = (u_1)_{\Pi_{e_2}^-}$ and $u_{1,2} = (u_1)_{\Pi_{e_2}^+}$ the functions obtained from u_1 by the reflection procedure (1). Arguing as previously, we infer that $u_{1,1}$ and $u_{1,2}$ belong to $\mathcal{X}$, satisfy the constraints and are minimizers for problem $(\mathcal{P})$. Moreover, they are symmetric with respect to Π_{e_1} and with respect to Π_{e_2}. Next we use the following:

Lemma 3. *Let $w \in \mathcal{X}$ be a minimizer for $(\mathcal{P})$. Assume that* **A1, A2** *are satisfied and there exists a vector subspace V of $\mathbf{R}^N$ of dimension $m \le N-2$ such that any hyperplane containing V splits the constraints in two for w. Then w is radially symmetric with respect to V.*

Proof. Let $\mathcal{B}_1 = \{b_1, \dots, b_m\}$ be an orthonormal basis in V. Fix a hyperplane Π containing V. We extend $\mathcal{B}_1$ to an orthonormal basis $\mathcal{B} = \{b_1, \dots, b_N\}$ in $\mathbf{R}^N$ in such a way that $\Pi = \Pi_{b_N} = b_N^{\perp}$. We denote by $(x_1, \dots, x_N)$ the coordinates of a point x with respect to $\mathcal{B}$. Let $w_1 = w_{\Pi_{b_N}^-}$ and $w_2 = w_{\Pi_{b_N}^+}$. Clearly $w_1, w_2 \in \mathcal{X}$ by **A1**. By the assumption of Lemma 3, Π_{b_N} splits the constraints in two for w and this implies that w_1 and w_2 satisfy the constraints. As before we have $E(w_1) \ge E(w)$, $E(w_2) \ge E(w)$ and $E(w_1) + E(w_2) = 2E(w)$, thus necessarily $E(w_1) = E(w_2) = E(w)$ and w_1, w_2 are also minimizers. By **A2** we have $w, w_1, w_2 \in C^1(\Omega)$. Since w_1 is symmetric with respect to the x_N variable, we have $\frac{\partial w_1}{\partial x_N}(x_1, \dots, x_{N-1}, 0) = 0$ whenever $(x_1, \dots, x_{N-1}, 0) \in \Omega$. But $w(x) = w_1(x)$ for $x_N < 0$, therefore

$$\begin{aligned} \frac{\partial w}{\partial x_N}(x_1, \dots, x_{N-1}, 0) &= \lim_{s \uparrow 0} \frac{\partial w}{\partial x_N}(x_1, \dots, x_{N-1}, s) \\ &= \lim_{s \uparrow 0} \frac{\partial w_1}{\partial x_N}(x_1, \dots, x_{N-1}, s) = \frac{\partial w_1}{\partial x_N}(x_1, \dots, x_{N-1}, 0) = 0 \end{aligned} \tag{3}$$

for $(x_1, \dots, x_{N-1}, 0) \in \Omega$, i.e. the derivative of w in the direction orthogonal to Π vanishes on $\Omega \cap \Pi$. Thus we have proved that for any hyperplane Π containing V, we have

$$\frac{\partial w}{\partial n}(x) = 0 \qquad \text{for any } x \in \Omega \cap \Pi, \text{ where } n \text{ is the unit normal to } \Pi. \tag{4}$$

We pass to spherical coordinates in the last $N-m$ variables in $\mathbf{R}^N$, i.e. we use variables $(r, \theta_1, \dots, \theta_{N-m-1})$ instead of $(x_{m+1}, \dots, x_N)$, where $r = \left(x_{N-m+1}^2 + \dots + x_N^2\right)^{\frac{1}{2}}$ and $\theta_1, \dots \theta_{N-m-1}$ are the angular variables. Then (4) is equivalent to $\frac{\partial w}{\partial \theta_j} = 0$ on Ω for $j = 1, \dots, N-m-1$. We infer that w does not depend on $\theta_1, \dots, \theta_{N-m+1}$, i.e. there exists some function $\tilde{w}$ depending only on $x_1, \dots, x_m, r$ such that $w(x_1, \dots, x_N) = \tilde{w}(x_1, \dots, x_m, r)$ on Ω and Lemma 3 is proved. $\square$

Now come back to the proof of Theorem 1. Clearly, any $x \in \mathbf{R}^N$ has a unique decomposition $x = x_1e_1 + x_2e_2 + x'$, where $x_1, x_2 \in \mathbf{R}$ and $x' \in \{e_1, e_2\}^\perp$. Since $u_{1,1}$ and $u_{1,2}$ are symmetric with respect to Π_{e_1} and with respect to Π_{e_2}, we have $u_{1,i}(x_1e_1+x_2e_2+x') = u_{1,i}(x_1e_1 - x_2e_2 + x') = u_{1,i}(-x_1e_1 - x_2e_2 + x')$. Let Π be a hyperplane containing $\{e_1, e_2\}^\perp$. It is obvious that the transform $x_1e_1 + x_2e_2 + x' \longmapsto -x_1e_1 - x_2e_2 + x'$ is a one-to-one correspondence between Π^+ and Π^- and a simple change of variables gives

$$\int_{\Pi^+\cap\Omega} G_j(|x|, u_{1,i}(x), |\nabla u_{1,i}(x)|)\, dx = \int_{\Pi^-\cap\Omega} G_j(|x|, u_{1,i}(x), |\nabla u_{1,i}(x)|)\, dx, \quad j = 1, \dots, k,$$

hence Π splits the constraints in two for $u_{1,i}$, $i = 1, 2$. By Lemma 3, we infer that $u_{1,i}$ are radially symmetric with respect to $\{e_1, e_2\}^\perp$, i.e. $u_{1,i}(x_1e_1+x_2e_2+x') = \tilde{u}_{1,i}(\sqrt{x_1^2 + x_2^2}, x')$ for some functions $\tilde{u}_{1,1}$ and $\tilde{u}_{1,2}$. On the other hand, $u_{1,1}(x) = u_1(x) = u_{1,2}(x)$ for any $x \in \Pi_{e_2} \cap \Omega$, that is $\tilde{u}_{1,1}(|x_1|, x') = \tilde{u}_{1,2}(|x_1|, x')$ whenever $x_1e_1 + x' \in \Omega$. We conclude that necessarily $\tilde{u}_{1,1} = \tilde{u}_{1,2}$ and $u_{1,1}(x) = u_1(x) = u_{1,2}(x)$ for any $x \in \Omega$, thus u_1 is radially symmetric with respect to $\{e_1, e_2\}^\perp$.

Similarly there exists $v_2 \in S^{N-1} \cap e_1^\perp$ such that Π_{v_2} splits the constraints in two for u_2 and we infer that u_2 is radially symmetric with respect to $\{e_1, v_2\}^\perp$. We use this information together with the fact that $u_1 = u = u_2$ on $\Omega \cap \Pi_{e_1}$ to prove the symmetry of u.

If v_2 is colinear to e_2, i.e. $v_2 = \pm e_2$, we may assume that $v_2 = e_2$. Using the symmetry of u_1, u_2 and the fact that $u_1 = u = u_2$ on $\Omega \cap \Pi_{e_1}$, we obtain as above that $u_1 = u_2 = u$ on Ω, hence u is radially symmetric with respect to $\{e_1, e_2\}^\perp$.

If v_2 and e_2 are not colinear, Span$\{e_1, e_2, v_2\}$ is a three-dimensional subspace. Let $\{e_4, \dots, e_N\}$ be an orthonormal basis in $\{e_1, e_2, v_2\}^\perp$. We choose e_3 and v_3 in such a way that $\mathcal{B} = \{e_1, e_2, e_3, \dots, e_N\}$ and $\mathcal{B}' = \{e_1, v_2, v_3, e_4, \dots, e_N\}$ are orthonormal basis in $\mathbf{R}^N$ with the same orientation. Then there exists $\theta \in (0, \pi) \cup (\pi, 2\pi)$ such that $v_2 = \cos\theta\, e_2 + \sin\theta\, e_3$ and $v_3 = -\sin\theta\, e_2 + \cos\theta\, e_3$. Given a point $x \in \mathbf{R}^N$, we denote by $(x_1, x_2, \dots, x_N)$ its coordinates with respect to $\mathcal{B}$. It is clear that $(x_1, y_2 = \cos\theta\, x_2 + \sin\theta\, x_3, y_3 = -\sin\theta\, x_2 + \cos\theta\, x_3, x_4, \dots, x_N)$ are the coordinates of x with respect to $\mathcal{B}'$.

Fix $re_3 + \sum_{j=4}^N x_je_j \in \Omega \cap e_1^\perp$ and denote

$$\varphi(t) = \varphi_{r,x_4,\dots,x_N}(t) = u(r\cos t\, e_2 + r\sin t\, e_3 + \sum_{j=4}^N x_je_j).$$

Clearly, φ is C^1 and $2\pi-$periodic on $\mathbf{R}$. Since the restriction of $u = u_1$ to $\Omega \cap e_1^\perp$ is symmetric with respect to $\mathbf{R}e_2$, we get

$$\varphi(t) = u(-r\cos t\, e_2 + r\sin t\, e_3 + \sum_{j=4}^N x_je_j) = \varphi(\pi - t). \tag{5}$$

The restriction of $u = u_2$ to $\Omega \cap e_1^\perp$ is also symmetric with respect to $\mathbf{R}v_2$, therefore

$$\begin{aligned} \varphi(t) &= u(r(\cos t\cos\theta + \sin t\sin\theta)\, v_2 + r(\sin t\cos\theta - \cos t\sin\theta)v_3 + \textstyle\sum_{j=4}^N x_je_j) \\ &= u_2(r\cos(t-\theta)\, v_2 + r\sin(t-\theta)\, v_3 + \textstyle\sum_{j=4}^N x_je_j) \\ &= u_2(-r\cos(t-\theta)\, v_2 + r\sin(t-\theta)\, v_3 + \textstyle\sum_{j=4}^N x_je_j) \qquad (6) \\ &= u_2(r\cos(\pi-(t-\theta))\, v_2 + r\sin(\pi-(t-\theta))\, v_3 + \textstyle\sum_{j=4}^N x_je_j) \\ &= \varphi(\pi + 2\theta - t) = \varphi(t - 2\theta) \qquad \text{by (5)}. \end{aligned}$$

Hence any of the functions $\varphi_{r,x_4,\dots,x_N}$ admits 2π and 2θ as periods. The following situations may occur:

Case 1: $\frac{\theta}{\pi} \in \mathbf{R} \setminus \mathbf{Q}$. The set $\{2n\theta + 2k\pi \mid n, k \in \mathbf{Z}\}$ is dense in $\mathbf{R}$ and any number in this set is a period for $\varphi_{r,x_4,\dots,x_N}$. Since $\varphi_{r,x_4,\dots,x_N}$ is continuous, we infer that it is constant. This is equivalent to $u(\sum_{j=2}^N x_j e_j) = u(\sqrt{x_2^2 + x_3^2}\, e_2 + \sum_{j=4}^N x_j e_j)$ whenever $\sum_{j=2}^N x_j e_j \in \Omega \cap e_1^\perp$. With the above notation, using the symmetry properties of u_1 and u_2 we have for any $x \in \Omega$,

$$u_1(x) = u_1(\sqrt{x_1^2 + x_2^2}\, e_2 + \sum_{j=3}^N x_j e_j) = u(\sqrt{x_1^2 + x_2^2 + x_3^2}\, e_2 + \sum_{j=4}^N x_j e_j)$$

and

$$u_2(x) = u_2(\sqrt{x_1^2 + y_2^2}\, v_2 + y_3 v_3 + \textstyle\sum_{j=4}^N x_j e_j) = u(\sqrt{x_1^2 + y_2^2}\, v_2 + y_3 v_3 + \sum_{j=4}^N x_j e_j)$$

$$= u(\sqrt{x_1^2 + y_2^2 + y_3^2}\, e_2 + \textstyle\sum_{j=4}^N x_j e_j) = u(\sqrt{x_1^2 + x_2^2 + x_3^2}\, e_2 + \sum_{j=4}^N x_j e_j).$$

Consequently $u = u_1 = u_2$ on Ω and u is radially symmetric with respect to $\{e_1, e_2, e_3\}^\perp$.

Case 2: $\frac{\theta}{\pi} = \frac{k}{n}$ where k, n are relatively prime integers, k is odd and n is even, say $k = 2k_1 + 1$ and $n = 2n_1$. Then $\pi = 2n_1\theta - 2k_1\pi$ is also a period for $\varphi_{r,x_4,\dots,x_N}$ and this implies

$$u(\sum_{j=2}^N x_j e_j) = u(-x_2 e_2 - x_3 e_3 + \sum_{j=4}^N x_j e_j) \qquad \text{whenever } \sum_{j=2}^N x_j e_j \in \Omega \cap e_1^\perp. \tag{7}$$

From the symmetry of u_1 and (7) we get for $x_1 \le 0$:

$$\begin{aligned} &u(\textstyle\sum_{j=1}^N x_j e_j) = u(\sqrt{x_1^2 + x_2^2}\, e_2 + \sum_{j=3}^N x_j e_j) \\ &= u(-\sqrt{x_1^2 + x_2^2}\, e_2 - x_3 e_3 + \textstyle\sum_{j=4}^N x_j e_j) = u(x_1 e_1 - x_2 e_2 - x_3 e_3 + \sum_{j=4}^N x_j e_j). \end{aligned} \tag{8}$$

Using the symmetry of u_2 and (7), we infer that (8) also holds for $x_1 \ge 0$. Let Π be a hyperplane containing $\{e_1, e_4, \dots, e_N\}$. It is clear that the mapping $\sum_{j=1}^N x_j e_j \longmapsto x_1 e_1 - x_2 e_2 - x_3 e_3 + \sum_{j=4}^N x_j e_j$ is a linear isometry between Π^+ and Π^-. Then (8) and a simple change of variables show that

$$\int_{\Pi^+ \cap \Omega} G_\ell(|x|, u(x), |\nabla u(x)|)\, dx = \int_{\Pi^- \cap \Omega} G_\ell(|x|, u(x), |\nabla u(x)|)\, dx,$$

for $\ell = 1, \dots, k$, i.e. Π splits the constraints in two for u. Since u is a minimizer, by Lemma 3 we infer that u is radially symmetric with respect to $\operatorname{Span}\{e_1, e_4, \dots, e_N\}$. In particular, the restriction of u to $\Omega \cap e_1^\perp$ is radially symmetric with respect to $\operatorname{Span}\{e_4, \dots, e_N\}$. As in case 1, this implies that u is radially symmetric with respect to $\operatorname{Span}\{e_4, \dots, e_N\}$.

Case 3: $\frac{\theta}{\pi} = \frac{k}{n}$ where k, n are relatively prime integers, k is even and n is odd, say $k = 2k_1$ and $n = 2n_1 + 1$. Then $\theta = 2k_1\pi - 2n_1\theta$ is a period for $\varphi_{r,x_4,\dots,x_N}$. By (5) we get $\varphi_{r,x_4,\dots,x_N}(t) = \varphi_{r,x_4,\dots,x_N}(\pi - t) = \varphi_{r,x_4,\dots,x_N}(\theta + \pi - t)$. This means that for $\sum_{j=2}^N x_j e_j \in \Omega$ we have

$$u(\sum_{j=2}^N x_j e_j) = u(-(x_2 \cos\theta + x_3 \sin\theta) e_2 + (-x_2 \sin\theta + x_3 \cos\theta) e_3 + \sum_{j=4}^N x_j e_j). \tag{9}$$

In other words, for fixed $x'' \in \operatorname{Span}\{e_4, \dots, e_N\}$, the function $x_2e_2 + x_3e_3 \longmapsto u(x_2e_2 + x_3e_3 + x'')$ is symmetric with respect to $\mathbf{R}w$, where $w = \cos(\frac{\theta+\pi}{2})e_2 + \sin(\frac{\theta+\pi}{2})e_3$. Note that the symmetry of $\operatorname{Span}\{e_1, e_2, e_3\}$ with respect to $\mathbf{R}w$ is a linear isometry of matrix $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\theta & -\sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$ with respect to the basis $\{e_1, e_2, e_3\}$. We show that for any $x \in \Omega$ we have

$$u(x) = u(Sx), \tag{10}$$

where $Sx = -x_1e_1 - (x_2\cos\theta + x_3\sin\theta)e_2 + (-x_2\sin\theta + x_3\cos\theta)e_3 + \sum_{j=4}^N x_je_j$. It suffices to consider the case $x_1 \le 0$. By using the symmetry of u_1, u_2 and (9) we get

$$u(x) = u_1(x) = u(\sqrt{x_1^2 + x_2^2}\, e_2 + \textstyle\sum_{j=3}^N x_je_j)$$

$$= u(-(\sqrt{x_1^2 + x_2^2}\cos\theta + x_3\sin\theta)e_2 + (-\sqrt{x_1^2 + x_2^2}\sin\theta + x_3\cos\theta)e_3 + \textstyle\sum_{j=4}^N x_je_j)$$

and

$$u(Sx) = u_2(Sx) = u_2(-x_1e_1 - x_2v_2 + x_3v_3 + \textstyle\sum_{j=4}^N x_je_j)$$

$$= u_2(-\sqrt{x_1^2 + x_2^2}\, v_2 + x_3v_3 + \textstyle\sum_{j=4}^N x_je_j)$$

$$= u(-\sqrt{x_1^2 + x_2^2}(\cos\theta\, e_2 + \sin\theta\, e_3) + x_3(-\sin\theta\, e_2 + \cos\theta\, {}_3) + \textstyle\sum_{j=4}^N x_je_j),$$

hence $u(x) = u(Sx)$. Let Π be a vector hyperplane containing $w, e_4, \dots, e_N$. It is easy to see that S is a linear isometry of $\mathbf{R}^N$ mapping $\Omega \cap \Pi^-$ onto $\Omega \cap \Pi^+$. Using (10) and a change of variables, we find that Π splits the constraints in two for u. By Lemma 3 we infer that u is radially symmetric with respect to $\operatorname{Span}\{w, e_4, \dots, e_N\}$.

In fact, since u_1 is radially symmetric with respect to $\operatorname{Span}\{e_3, e_4, \dots, e_N\}$ and $\operatorname{Span}\{w, e_4, \dots, e_N\}$, it can be proved that u_1 is radially symmetric with respect to $\operatorname{Span}\{e_4, \dots, e_N\}$. Similarly u_2 is radially symmetric with respect to $\operatorname{Span}\{e_4, \dots, e_N\}$ and then it is clear that u has the same property. We omit the proof because we will not make use of this observation.

Case 4: $\frac{\theta}{\pi} = \frac{k}{n}$ where k, n are relatively prime odd integers, say $k = 2k_1 + 1$ and $n = 2n_1 + 1$. Then $\theta - \pi = 2k_1\pi - 2n_1\theta$ is a period for $\varphi_{r,x_4,\dots,x_N}$. By (5) we have $\varphi_{r,x_4,\dots,x_N}(t) = \varphi_{r,x_4,\dots,x_N}(\pi - t) = \varphi_{r,x_4,\dots,x_N}(\theta - t)$, that is

$$u(x) = u((x_2\cos\theta + x_3\sin\theta)e_2 + (x_2\sin\theta - x_3\cos\theta)e_3 + \sum_{j=4}^N x_je_j) \tag{11}$$

for any $x = \sum_{j=2}^N x_je_j \in \Omega \cap e_1^\perp$. Proceeding as in case 3, we prove that u is radially symmetric with respect to $\operatorname{Span}\{w', e_4, \dots, e_N\}$, where $w' = \cos\frac{\theta}{2}\, e_2 + \sin\frac{\theta}{2}\, e_3$. (In fact, it can be proved that u is radially symmetric with respect to $\operatorname{Span}\{e_4, \dots, e_N\}$).

Note that in either case it follows that u is symmetric with respect to Π_{e_1}. Thus we have proved that whenever $e_1 \in S^{N-1}$ satisfies $\Phi(e_1) = 0$, u is symmetric with respect to Π_{e_1}. Assume that $e_1, \dots, e_\ell \in S^{N-1}$ are mutually orthogonal, satisfy $\Phi(e_1) = \ldots = \Phi(e_\ell) = 0$ and $\ell \le N - k - 1$. It is clear that $S_\ell = S^{N-1} \cap \{e_1, \dots, e_\ell\}^\perp$ can be identified to $S^{N-\ell-1}$ and the restriction of Φ to S_ℓ is an odd, continuous function from S_ℓ to

$\mathbf{R}^k$. Using the Borsuk-Ulam theorem we infer that there exists $e_{\ell+1} \in S_\ell$ such that $\Phi(e_{\ell+1}) = 0$. By induction it follows that there exist $N-k$ mutually orthogonal vectors $e_1, \ldots, e_{N-k} \in S^{N-1}$ such that $\Phi(e_1) = \ldots = \Phi(e_{N-k}) = 0$. We complete this set to an orthonormal basis $\{e_1, \ldots, e_N\}$ in $\mathbf{R}^N$. We already know that u is symmetric with respect to any of the hyperplanes $\Pi_{e_1}, \ldots, \Pi_{e_{N-k}}$. In particular, for $x = \sum_{j=1}^N x_j e_j \in \Omega$ we have

$$(12) \qquad u(x) = u(-x_1 e_1 + \sum_{j=2}^{N} x_j e_j) = \ldots = u(-\sum_{j=1}^{N-k} x_j e_j + \sum_{j=N-k+1}^{N} x_j e_j).$$

Let Π be a (vector) hyperplane containing $e_{N-k+1}, \ldots, e_N$. It is clear that the mapping $\sum_{j=1}^N x_j e_j \longmapsto -\sum_{j=1}^{N-k} x_j e_j + \sum_{j=N-k+1}^N x_j e_j$ is a linear isometry between Π^+ and Π^-. Using (12), we infer that Π splits the constraints in two for u. By Lemma 3, u is radially symetric with respect to Span$\{e_{N-k+1}, \ldots, e_N\}$.

The case $k = 0$ is much simpler. Problem $(\mathcal{P})$ consists in minimizing E on $\mathcal{X}$ without constraints. Assume that u is a minimizer. Let Π be a hyperplane containing the origin and let u_{Π^-}, u_{Π^+} be the two functions obtained from u as in (1). By **A1** we have $u_{\Pi^-}, u_{\Pi^+} \in \mathcal{X}$, thus $E(u_{\Pi^-}) \geq E(u)$ and $E(u_{\Pi^+}) \geq E(u)$. On the other hand, $E(u_{\Pi^-}) + E(u_{\Pi^+}) = 2E(u)$, thus necessarily $E(u_{\Pi^-}) = E(u_{\Pi^+}) = E(u)$ and u_{Π^-}, u_{Π^+} are also minimizers. As in the proof of Lemma 3, this implies $\frac{\partial u}{\partial n}(x) = 0$ for any $x \in \Omega \cap \Pi$, where n is the unit normal to Π. Then passing to spherical coordinates, as in Lemma 3, we see that u does not depend on the angular variables, i.e. u is a radial function. $\square$

Proof of Theorem 2. For $v \in S^{N-1}$ and $t \in \mathbf{R}$ we denote by $\Pi_{v,t}$ the affine hyperplane $\{x \in \mathbf{R}^N \mid (x - tv).v = 0\}$ and by $\Pi_{v,t}^+ = \{x \in \mathbf{R}^N \mid (x - tv).v > 0\}$, respectively $\Pi_{v,t}^- = \{x \in \mathbf{R}^N \mid (x - tv).v < 0\}$ the two half-spaces determined by $\Pi_{v,t}$. It is clear that $\Pi_{-v,-t}^- = \Pi_{v,t}^+$. For $j = 1, \ldots, k$, we define $\tilde{\psi}_j : S^{N-1} \times \mathbf{R} \longrightarrow \mathbf{R}$ by

$$\tilde{\psi}_j(v,t) = \int_{\Pi_{v,t}^+} G_j(u(x), |\nabla u(x)|)\, dx - \int_{\Pi_{v,t}^-} G_j(u(x), |\nabla u(x)|)\, dx.$$

Since $G_j(u, |\nabla u|) \in L^1(\mathbf{R}^N)$, it is a simple consequence of Lebesgue's dominated convergence theorem that $\tilde{\psi}_j$ is continuous on $S^{N-1} \times \mathbf{R}$. It is obvious that $\tilde{\psi}_j(-v,-t) = -\tilde{\psi}_j(v,t)$.

We claim that $\lim_{t \to \infty} \tilde{\psi}_j(v,t) = -\int_{\mathbf{R}^N} G_j(u(x), |\nabla u(x)|)\, dx = -\lambda_j$ uniformly with respect to $v \in S^{N-1}$. Indeed, fix $\varepsilon > 0$. There exists $R > 0$ such that

$$\int_{\mathbf{R}^N \setminus B(0,R)} |G_j(u(x), |\nabla u(x)|)|\, dx < \frac{\varepsilon}{2}.$$

For any $v \in S^{N-1}$ and $t > R$ we have $\Pi_{v,t}^+ \subset \mathbf{R}^N \setminus B(0,R)$, therefore

$$\left| \tilde{\psi}_j(v,t) + \int_{\mathbf{R}^N} G_j(u(x), |\nabla u(x)|)\, dx \right| = 2 \left| \int_{\Pi_{v,t}^+} G_j(u(x), |\nabla u(x)|)\, dx \right| < \varepsilon$$

and the claim is proved. It is clear that $\lim_{t \to -\infty} \tilde{\psi}_j(v,t) = \lambda_j$ uniformly in $v \in S^{N-1}$.

We denote $P = (0, \ldots, 0, 1) \in \mathbf{R}^{N+1}$, $S = (0, \ldots, 0, -1) \in \mathbf{R}^{N+1}$ and we define $\psi_j : S^N \longrightarrow \mathbf{R}$ by

$$\psi_j(x_1, \ldots, x_N, x_{N+1}) = \tilde{\psi}_j \left(\frac{(x_1, \ldots, x_N)}{|(x_1, \ldots, x_N)|}, \frac{x_{N+1}}{1 - |x_{N+1}|} \right)$$

if $(x_1, \dots, x_N, x_{N+1}) \notin \{P, S\}$, respectively $\psi_j(P) = -\lambda_j$ and $\psi_j(S) = \lambda_j$. Then ψ_j is an odd, continuous function on S^N.

Consider first the case $1 \le k \le N-2$. It follows from Theorem 1 that there exist two orthogonal vector subspaces V_1 and V_2 such that $\dim(V_1) = k$, $V_1 \oplus V_2 = \mathbf{R}^N$ and u is radially symmetric with respect to V_1. The set $\mathbf{S} = \{(y_1, \dots, y_N, y_{N+1}) \in S^N \mid (y_1, \dots, y_N) \in V_1\}$ can be identified to S^k. Since the restriction of $\Psi = (\psi_1, \dots, \psi_k)$ to $\mathbf{S} \simeq S^k$ is continuous, odd, $\mathbf{R}^k$−valued, by the Borsuk-Ulam theorem we infer that there exists $y^* = (y_1^*, \dots, y_N^*, y_{N+1}^*) \in \mathbf{S}$ such that $\psi(y^*) = 0$. We cannot have $y^* = S$ or $y^* = P$ because $\psi(S) = -\psi(P) = (\lambda_1, \dots, \lambda_N) \neq 0$. Denote $e_k = \frac{(y_1^*, \dots, y_N^*)}{|(y_1^*, \dots, y_N^*)|}$ and $t = \frac{y_{N+1}^*}{1 - |y_{N+1}^*|}$. Then $e_k \in V_1$, $|e_k| = 1$ and $\tilde{\psi}_j(e_k, t) = 0$ for $j = 1, \dots, k$, i.e. $\Pi_{e_k,t}$ splits the constraints in two for u. Choose e_i, $i = 1, \dots, N$, $i \neq k$ in such a way that $\{e_1, \dots, e_{k-1}, e_k\}$ and $\{e_{k+1}, \dots, e_N\}$ are orthonormal basis in V_1, respectively in V_2. Denote $u_*(x) = u(x - te_k)$. It is clear that u_* is a minimizer for $(\mathcal{P}')$, it is radially symmetric with respect to V_1 and the hyperplane $e_k^\perp = \Pi_{e_k,0}$ splits the constraints in two for u_*. Arguing exactly as in the proof of Theorem 1, we see that u_* is symmetric with respect to $e_k^\perp$. Using this fact and the radial symmetry with respect to V_1, we get

$$u_*(\sum_{i=1}^{N} x_i e_i) = u_*(\sum_{i=1}^{k} x_i e_i - \sum_{i=k+1}^{N} x_i e_i) = u_*(\sum_{i=1}^{k-1} x_i e_i - \sum_{i=k}^{N} x_i e_i). \tag{13}$$

By (13) we infer that any (vector) hyperplane containing $e_1, \dots, e_{k-1}$ splits the constraints in two for u_*. Then Lemma 3 implies that u_* is radially symmetric with respect to $\text{Span}\{e_1, \dots, e_{k-1}\}$, consequently u is radially symmetric with respect to the affine subspace $te_k + \text{Span}\{e_1, \dots, e_{k-1}\}$.

Now consider the case $k = N-1$. As above, there exists $y^* = (y_1^*, \dots, y_N^*, y_{N+1}^*) \in S^N \setminus \{S, P\}$ such that $\psi(y^*) = 0$. Denoting $e_1 = \frac{(y_1^*, \dots, y_N^*)}{|(y_1^*, \dots, y_N^*)|}$ and $t_1 = \frac{y_{N+1}^*}{1 - |y_{N+1}^*|}$, this means that Π_{e_1,t_1} splits the constraints in two for u. Let $u_1 = u_{\Pi^-_{e_1,t_1}}$ and $u_2 = u_{\Pi^+_{e_1,t_1}}$. It is clear that u_1, u_2 are also minimizers for $(\mathcal{P}')$. Since $\{(y_1, \dots, y_{N+1}) \in S^N \mid (y_1, \dots, y_N) \perp e_1\}$ is homeomorphic to S^{N-1} and there are exactly $N-1$ constraints, it is possible to restart the prevoius process with u_1 instead of u. We infer that there exists $e_2 \in e_1^\perp$, $|e_2| = 1$ and $t_2 \in \mathbf{R}$ such that Π_{e_2,t_2} splits the constraints in two for u_1. Putting $u_{1,1} = (u_1)_{\Pi^-_{e_2,t_2}}$ and $u_{1,2} = (u_1)_{\Pi^+_{e_2,t_2}}$, we see that $u_{1,1}$ and $u_{1,2}$ are minimizers for $(\mathcal{P}')$ and are symmetric with respect to Π_{e_1,t_1} and Π_{e_2,t_2}. It follows that $\tilde{u}_{1,1} = u_{1,1}(\cdot - t_1e_1 - t_2e_2)$ and $\tilde{u}_{1,2} = u_{1,2}(\cdot - t_1e_1 - t_2e_2)$ minimize $(\mathcal{P}')$ and are symmetric with respect to $e_1^\perp$ and $e_2^\perp$. Therefore any (vector) hyperplane in $\mathbf{R}^N$ containing $\{e_1, e_2\}^\perp$ splits the constraints in two for $\tilde{u}_{1,1}$ and for $\tilde{u}_{1,2}$ and using Lemma 3 we infer that $\tilde{u}_{1,1}$ and $\tilde{u}_{1,2}$ are radially symmetric with respect to $\{e_1, e_2\}^\perp$. Since $\tilde{u}_{1,1} = \tilde{u}_{1,2}$ on $\Pi_{e_2,0} = e_2^\perp$, we have necessarily $\tilde{u}_{1,1} = \tilde{u}_{1,2}$ on $\mathbf{R}^N$. Therefore $u_1 = \tilde{u}_{1,1}(\cdot + t_1e_1 + t_2e_2)$ is radially symmetric with respect to the affine subspace $t_1e_1 + t_2e_2 + \{e_1, e_2\}^\perp$.

Similarly we prove that there exist $v_2 \in e_1^\perp$, $|v_2| = 1$ and $s_2 \in \mathbf{R}$ such that u_2 is radially symmetric with respect to the affine subspace $t_1e_1 + s_2v_2 + \{e_1, v_2\}^\perp$. Of course, nothing guarantees *à priori* that $(e_2, t_2) = \pm(v_2, s_2)$. The following situations may occur:

Case 1: e_2 and v_2 are colinear. Then we may assume that $e_2 = v_2$. There are two subcases:

a) $t_2 = s_2$. Then $u_1(\cdot - t_1e_1 - t_2e_2)$ and $u_2(\cdot - t_1e_1 - t_2e_2)$ are both radially symetric with respect to $\{e_1, e_2\}^\perp$ and are equal on $e_1^\perp$. We conclude that $u_1(\cdot - t_1e_1 - t_2e_2) = u_2(\cdot - t_1e_1 - t_2e_2)$, thus $u = u_1 = u_2$ is radially symmetric with respect to $t_1e_1 + t_2e_2 + \{e_1, e_2\}^\perp$.

b) $t_2 \neq s_2$, say $s_2 > t_2$. The symmetry of u_1 and u_2 imply that there exist some functions $\tilde{u}_1$, $\tilde{u}_2$ defined on $[0,\infty) \times \{e_1, e_2\}^\perp$ such that

$$
\begin{aligned}
u_1(x_1e_1 + x_2e_2 + x') &= \tilde{u}_1(\sqrt{(x_1 - t_1)^2 + (x_2 - t_2)^2}\,, x') \\
u_2(x_1e_1 + x_2e_2 + x') &= \tilde{u}_2(\sqrt{(x_1 - t_1)^2 + (x_2 - s_2)^2}\,, x')
\end{aligned}
\tag{14}
$$

for any $x_1, x_2 \in \mathbf{R}$ and $x' \in \{e_1, e_2\}^\perp$. Since $u_1 = u_2$ on $\Pi_{e_1,t_1} = t_1e_1 + e_1^\perp$, it follows that

$$
\tilde{u}_1(|x_2 - t_2|\,, x') = \tilde{u}_2(|x_2 - s_2|\,, x') \tag{15}
$$

for any $x_2 \in \mathbf{R}$ and $x' \in \{e_1, e_2\}^\perp$. In particular, (15) implies that for fixed $x' \in \{e_1, e_2\}^\perp$, $\tilde{u}_1(\cdot, x')$ and $\tilde{u}_2(\cdot, x')$ are periodic of period $a = 2(s_2 - t_2)$. Passing to cylindrical coordinates $x_1 = t_1 + r\cos\theta$, $x_2 = t_2 + r\sin\theta$, x' and using Fubini's theorem we have

$$
\begin{aligned}
\int_{\Pi^-_{e_1,t_1}} G_j(u(x), |\nabla u(x)|)\,dx &= \int_{\Pi^-_{e_1,t_1}} G_j(u_1(x), |\nabla u_1(x)|)\,dx \\
&= \int_0^\infty \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{\{e_1,e_2\}^\perp} G_j(\tilde{u}_1(r,x'), |\nabla \tilde{u}_1(r,x')|)\,dx'\,d\theta\,r\,dr \\
&= \pi \int_0^\infty \int_{\{e_1,e_2\}^\perp} G_j(\tilde{u}_1(r,x'), |\nabla \tilde{u}_1(r,x')|)\,dx'\,r\,dr.
\end{aligned}
\tag{16}
$$

Let $h_j(r) = \displaystyle\int_{\{e_1,e_2\}^\perp} G_j(\tilde{u}_1(r,x'), |\nabla \tilde{u}_1(r,x')|)\,dx'$. The function h_j is well-defined for a.e. $r \geq 0$, measurable, periodic of period a, and $\pi \displaystyle\int_0^\infty r h_j(r)\,r = \lambda_j/2$. By periodicity we have $\displaystyle\int_{na}^{(n+1)a} r h_j(r)\,dr = na \int_0^a h_j(r)\,dr + \int_0^a r h_j(r)\,dr$, thus $\displaystyle\int_0^{na} r h_j(r)\,dr = \frac{n(n-1)}{2} a \int_0^a h_j(r)\,dr + n \int_0^a r h_j(r)\,dr$. It follows that necessarily $\displaystyle\int_0^a h_j(r)\,dr = 0$ and $\displaystyle\int_0^a r h_j(r)\,dr = 0$ and this implies $\displaystyle\int_0^\infty r h_j(r)\,dr = 0$, i.e. $\lambda_j = 0$ for any j, contrary to the assumptions of Theorem 2. Consequently the case 1 b) may never occur.

Case 2: e_2 and v_2 are not colinear. It is then clear that the space $\mathrm{Span}\{e_1, e_2, v_2\}$ is 3−dimensional (thus $N \geq 3$). Let $\{e_4, \dots, e_N\}$ be an orthonormal basis of $\{e_1, e_2, v_2\}^\perp$. We choose e_3 and v_3 in such a way that $\mathcal{B} = \{e_1, \dots, e_N\}$ and $\mathcal{B}' = \{e_1, v_2, v_3, e_4, \dots, e_N\}$ are orthonormal basis in $\mathbf{R}^N$ with the same orientation. There exists $\theta \in (0,\pi) \cup (\pi, 2\pi)$ such that $v_2 = \cos\theta\, e_2 + \sin\theta\, e_3$ and $v_3 = -\sin\theta\, e_2 + \cos\theta\, e_3$. Since $\sin\theta \neq 0$, there exist some $\alpha, \beta \in \mathbf{R}$ such that $t_2e_2 + \alpha e_3 = s_2v_2 + \beta v_3$. Let $y = t_1e_1 + t_2e_2 + \alpha e_3$. We denote $u^* = u(\cdot - y)$, $u_1^* = u_1(\cdot - y)$ and $u_2^* = u_2(\cdot - y)$. It is obvious that u^*, u_1^* and u_2^* are minimizers for $(\mathcal{P}')$, u_1^* is radially symmetric with respect to $\mathrm{Span}\{e_3, \dots, e_N\}$, u_2^* is radially symmetric with respect to $\mathrm{Span}\{v_3, e_4, \dots, e_N\}$, $u^* = u_1^*$ on $\Pi^-_{e_1,0} \cup \Pi_{e_1,0}$ and $u^* = u_2^*$ on $\Pi^+_{e_1,0} \cup \Pi_{e_1,0}$. Proceeding as in the proof of Theorem 1 we show that either u^* is radially symmetric with respect to $\mathrm{Span}\{e_4, \dots, e_N\}$, or there exists $w \in \mathrm{Span}\{e_2, e_3\}$, such that u^* is radially symmetric with respect to $\mathrm{Span}\{w, e_4, \dots, e_N\}$. In any case it follows that u is radially symmetric with respect to an affine subspace of dimension at most $k - 1 = N - 2$. This completes the proof of Theorem 2. □

3 Remarks and examples

Remark 4. If Ω is connected and a unique continuation principle is available for minimizers, the proofs in the preceding section can be considerably simplified. Moreover, it is possible to deal with $N-1$ constraints in Theorem 1, respectively with N constraints in Theorem 2 (but this is of quite limited interest in applications because we get only symmetry with respect to a hyperplane).

For example, consider the problem $(\mathcal{P}1)$ of minimizing

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)\,dx \qquad \text{in } H^1(\Omega, \mathbf{R}^m) \quad (\text{or in } H_0^1(\Omega, \mathbf{R}^m))$$

under the constraints $Q_j(u) = \int_\Omega G_j(u)\,dx = \lambda_j$, $1 \le j \le k$ and the following standard assumptions:

H1. $F, G_1, \dots, G_k \in C^2(\mathbf{R}^m, \mathbf{R})$, $F(0) = G_j(0) = 0$, $\nabla F(0) = \nabla G_j(0) = 0$, and

$$|\nabla F(u)| \le C|u|^p, \qquad |\nabla G_j(u)| \le C|u|^p \qquad \text{for } |u| \ge 1, \text{ where } p < \frac{N+2}{N-2}.$$

H2. If $u \in H^1(\Omega, \mathbf{R}^m)$ (respectively $u \in H_0^1(\Omega, \mathbf{R}^m)$) is nonconstant and $\sum_{j=1}^k \alpha_j \nabla G_j(u) = \sum_{j=1}^k \beta_j \nabla G_j(u)$ on Ω, then $\alpha_j = \beta_j$ for $j = 1, \dots, k$.

Suppose that u is a minimizer for $(\mathcal{P}1)$ and a hyperplane Π (with $0 \in \Pi$ if $\Omega \ne \mathbf{R}^N$) splits the contraints in two for u. As before, it follows easily that the functions u_{Π^-} and u_{Π^+} are minimizers for $(\mathcal{P}1)$. Thus u and u_{Π^-} satisfy the Euler-Lagrange equations

$$(17) \qquad -\Delta u + \nabla F(u) + \sum_{j=1}^k \alpha_j \nabla G_j(u) = 0 \qquad \text{in } \Omega, \quad \text{respectively}$$

$$(18) \qquad -\Delta u_{\Pi^-} + \nabla F(u_{\Pi^-}) + \sum_{j=1}^k \beta_j \nabla G_j(u_{\Pi^-}) = 0 \qquad \text{in } \Omega$$

for some $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathbf{R}$. By standard regularity theory we get u, $u_{\Pi^-} \in W^{2,q}(\Omega)$ for any $q \in [2, \infty)$. In particular, u, $u_{\Pi^-} \in C^{1,\alpha}(\Omega)$ for $\alpha \in [0,1)$, and u, u_{Π^-} as well as their derivatives are bounded on Ω. If u is constant on $\Omega \cap \Pi^-$, it follows form (17) and the unique continuation principle (see [11]) that u is constant on Ω. Otherwise, from (17) and (18) we obtain $\sum_{j=1}^k \alpha_j \nabla G_j(u) = \sum_{j=1}^k \beta_j \nabla G_j(u)$ on $\Omega \cap \Pi^-$ and by **H2** we infer that $\alpha_j = \beta_j$, $j = 1, \dots, k$. Denoting $w = u - u_{\Pi^-}$, (17) and (18) imply that w satisfies

$$-\Delta w + A(x) w = 0 \qquad \text{in } \Omega,$$

where $A \in L^\infty(\Omega, M_m(\mathbf{R}))$. Since $w = 0$ in $\Omega \cap \Pi^-$, by the unique continuation principle we find $w = 0$ in Ω, i.e. $u = u_{\Pi^-}$ and u is symmetric with respect to Π. Hence we have proved that u is symmetric with respect to any hyperplane that splits the constraints in two. The rest of the proof is as in the preceding section.

Note that a nondegeneracy hypothesis like **H2** is needed to use a unique continuation principle.

Remark 5. In Theorems 1 and 2, any supplementary constraint in the minimization problem produces the loss of one direction of symmetry for minimizers. Under the general assumptions made there, this loss of symmetry cannot be avoided, as it can be seen in the following simple examples.

Example 6. i) Let Ω be either a ball or an annulus in $\mathbf{R}^N$, centered at the origin. Consider $F, G \in C^2(\mathbf{R}, \mathbf{R})$ satisfying assumption **H1** in Remark 4 and such that the problem $(\mathcal{P}_1)$ of minimizing $E_1(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)\,dx$ in $H^1(\Omega)$ under the constraint $\int_\Omega G(u)\,dx = \lambda$ admits a nonconstant solution u_*. It has been shown in [12] that u_* cannot be radially symmetric about 0 (but, of course, u_* is radially symmetric with respect to a line passing through 0). Consider the problem

$$(\mathcal{P}_k) \qquad \begin{aligned} &\text{minimize } E_k(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u_1) + \ldots + F(u_k)\,dx, \\ &\text{under the constraints } \int_\Omega G(u_j)\,dx = \lambda, \qquad j = 1, \ldots, k, \end{aligned}$$

where $u = (u_1, \ldots, u_k) \in H^1(\Omega, \mathbf{R}^k)$. It is clear that $u = (u_1, \ldots, u_k)$ is a solution of $(\mathcal{P}_k)$ if and only if each u_j is a solution of $(\mathcal{P}_1)$. If $R_1, \ldots, R_k$ are rotations in $\mathbf{R}^N$, the function $u(x) = (u_*(R_1 x), \ldots, u_*(R_k x))$ is a solution of $(\mathcal{P}_k)$. We infer that there are minimizers of $(\mathcal{P}_k)$ that are not radially symmetric with respect to any $(k-1)-$dimensional vector subspace of $\mathbf{R}^N$.

ii) Consider two functions $F, G \in C^2(\mathbf{R}, \mathbf{R})$ satisfying assumption **H1** in Remark 4 and $\lambda \in \mathbf{R}^*$ such that the problem $(\mathcal{P}_1')$ consisting in minimizing $\tilde{E}_1(u) = \int_{\mathbf{R}^N} \frac{1}{2}|\nabla u|^2 + F(u)\,dx$ in $H^1(\mathbf{R}^N)$ under the constraint $\int_{\mathbf{R}^N} G(u)\,dx = \lambda$ admits a nonconstant solution $\tilde{u}$. It folows immediately from Theorem 2 that $\tilde{u}$ is radially symmetric with respect to a point; we may assume that it is radially symmetric about the origin. It is easy to see that $u = (u_1, \ldots, u_k) \in H^1(\mathbf{R}^N, \mathbf{R}^k)$ is a solution of the problem

$$(\mathcal{P}_k') \qquad \begin{aligned} &\text{minimize } \tilde{E}_k(u) = \int_{\mathbf{R}^N} \frac{1}{2}|\nabla u|^2 + F(u_1) + \ldots + F(u_k)\,dx, \\ &\text{under the constraints } \int_{\mathbf{R}^N} G(u_j)\,dx = \lambda, \qquad j = 1, \ldots, k, \end{aligned}$$

in $H^1(\mathbf{R}^N, \mathbf{R}^k)$ if and only if each u_j is a solution of $(\mathcal{P}_1')$. Therefore for any $y_1, \ldots, y_k \in \mathbf{R}^N$, the function $u = (u_1(\cdot + y_1), \ldots, u_k(\cdot + y_k))$ is a solution for $(\mathcal{P}_k')$. Obviously, this minimizer is radially symmetric with respect to some $(k-1)-$dimensional affine subspace but, in general, it is not radially symmetric with respect to any affine subspace of lower dimension.

In Example 6, the loss of symmetry comes from the fact that problems $(\mathcal{P}_k)$ and $(\mathcal{P}_k')$ are decoupled: they can be decomposed into k independent scalar problems, each of them being rotation (respectively translation) invariant. It is then natural to ask whether in general problems like $(\mathcal{P})$ or $(\mathcal{P}')$ the loss of directions of symmetry could exceed the number of components of minimizers. The answer is affirmative, as it can be seen in the next example which shows that, in general, the result of Theorem 2 is optimal even for scalar-valued minimizers.

Example 7. We construct here a minimization problem of the form $(\mathcal{P}')$ involving two constraints and whose real-valued minimizers are *not* radial with respect to a point (of course, these minimizers are axially symmetric). This example relies on the existence of a nonnegative minimizer with compact support for a problem involving one constraint. A similar construction has already been used in [4].

Let $f \in C(\mathbf{R}) \cap C^1(0, \infty)$ be a real-valued function satisfying the following conditions:

C1. $f(s) = 0$ on $(-\infty, 0]$ and $f(s) = s^\alpha$ for $s \in (0, 1]$, where $\alpha \in (0, 1)$.

C2. The function $F(s) := \int_0^s f(\tau)\,d\tau$ has compact support.

C3. There exists $\zeta > 0$ such that $F(\zeta) < 0$.

Let $N \geq 3$ and $\mathcal{X} = \mathcal{D}^{1,2}(\mathbf{R}^N) \cap L^{1+\alpha}(\mathbf{R}^N)$. We introduce the functionals $T(u) = \int_{\mathbf{R}^N} |\nabla u|^2 \, dx$ and $V(u) = \int_{\mathbf{R}^N} F(u(x)) \, dx$. It is clear that $F(u) \in L^1(\mathbf{R}^N)$ for any $u \in \mathcal{X}$ and T, V are well-defined, C^1 functionals on $\mathcal{X}$. We consider the minimization problem:

$$(\mathcal{M}_1) \qquad \text{minimize } T(u) \text{ in } \mathcal{X} \text{ subject to the constraint } V(u) = -1.$$

We denote $I = \inf\{T(u) \mid u \in \mathcal{X}, V(u) = -1\}$ and we proceed in several steps.

Step 1. We have $I > 0$ and problem $(\mathcal{M}_1)$ has a minimizer $u_* \in \mathcal{X}$. The proof of this fact is a straightforward modification of the proof of Theorem 2 in [3] or of the proof of Theorem 1 in [6], so we omit it.

Step 2. Any minimizer u of $(\mathcal{M}_1)$ is nonnegative, bounded, C^1, has compact support and satisfies the equation $-\Delta u + \beta_0 f(u) = 0$ in $\mathcal{D}'(\mathbf{R}^N)$, where $\beta_0 = \frac{N-2}{2N} I$.

Let $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. Then $u^+, u^- \in \mathcal{X}$, $V(u^+) = V(u) = -1$ and $T(u) = T(u^+) + T(u^-) \geq T(u^+)$. Since u is a minimizer, we must have $T(u^+) = T(u)$ and $T(u^-) = 0$, hence $u^- = 0$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, that is $u \geq 0$ a.e. Take $C > 0$ such that $\text{supp}(F) \subset [0, C]$ and denote $u_0 = \min(u, C)$, $u_C = \max(u - C, 0)$. It is obvious that $u_0, u_C \in \mathcal{X}$, $u = u_0 + u_C$, $V(u_0) = V(u) = -1$ and $T(u) = T(u_0) + T(u_C)$. As above we infer that $T(u_C) = 0$, consequently $u_C = 0$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$ and $u \leq C$ a.e.

Since T and V are C^1 functionals on $\mathcal{X}$, it is easy to see that u satisfies an Euler-Lagrange equation $T'(u) + 2\beta V'(u) = 0$ in $\mathcal{X}'$ for some $\beta \in \mathbf{R}$ and this implies

$$-\Delta u + \beta f(u) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{19}$$

Since $u \in L^\infty(\mathbf{R}^N)$ and f is continuous, by standard elliptic estimates it follows that $u \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in (1, \infty)$, thus $u \in C^{1,\gamma}_{loc}(\mathbf{R}^N)$ for $\gamma \in [0, 1)$. In particular, u is C^1.

It is standard to prove that u satisfies the Pohozaev identity $(N-2)T(u) + 2\beta N V(u) = 0$ (to see this, it suffices to multiply (19) by $\chi(\frac{x}{n}) \sum_{i=1}^N x_i \frac{\partial u}{\partial x_i}$, where $\chi \in C_c^\infty(\mathbf{R}^N)$ is such that $\chi \equiv 1$ on $B(0,1)$, to integrate by parts and then to pass to the limit as $n \longrightarrow \infty$). Since $V(u) = -1$ and $T(u) = I$, we find $\beta = \frac{N-2}{2N} I = \beta_0 > 0$.

Let $v(x) = u(\frac{x}{\sqrt{\beta_0}})$. Then $v \in C^1(\mathbf{R}^N)$, $v \geq 0$ and v satisfies the equation

$$-\Delta v + f(v) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

Moreover, we have $\displaystyle\int_0^1 \frac{1}{(F(s))^{\frac{1}{2}}} \, ds = (\alpha+1)^{\frac{1}{2}} \int_0^1 \frac{1}{s^{\frac{\alpha+1}{2}}} \, ds < \infty$. Thus we may use Theorem 2 p. 773 in [16] and we infer that v has compact support. Hence u has compact support.

Step 3. Any minimizer u of $(\mathcal{M}_1)$ is radially symmetric with respect to a point. Indeed, steps 1 and 2 show that $(\mathcal{M}_1)$ satisfies assumptions **A1'** and **A2** in Introduction, hence the radial symmetry of minimizers follows from Theorem 2. Note that the unique continuation principle is not valid for minimizers of $(\mathcal{M}_1)$, therefore the method in [11] cannot be used to prove their radial symmetry.

Step 4. Construction of nonradial minimizers for a minimization problem involving two constraints.

We introduce the functional $W(u) = \int_{\mathbf{R}^N} F(-u(x)) \, dx$. Clearly, W is well-defined and C^1 on $\mathcal{X}$. We consider the minimization problem:

$$(\mathcal{M}_2) \qquad \text{minimize } T(u) \text{ in } \mathcal{X} \text{ subject to the constraints } V(u) = -1 \text{ and } W(u) = -1.$$

We claim that $u \in \mathcal{X}$ is a solution of $(\mathcal{M}_2)$ if and only if u^+ and u^- are solutions of $(\mathcal{M}_1)$.

To see this, let u_* be a minimizer of $(\mathcal{M}_1)$, radially symmetric with respect to the origin. Let $R > 0$ be such that $\mathrm{supp}(u_*) \in B(0,R)$. For $y \in \mathbf{R}^N \setminus B(0,2R)$, we put $u_y(x) = u_*(x) - u_*(x+y)$. It is obvious that $V(u_y) = V(u_*) = -1$, $W(u_y) = V(u_*(\cdot + y)) = -1$ and $T(u_y) = T(u_*) + T(u_*(\cdot + y)) = 2I$.

For any $u \in \mathcal{X}$ satisfying $V(u) = W(u) = -1$ we have $V(u^+) = V(u) = -1$ and $V(u^-) = W(u) = -1$, hence $T(u^+) \geq I$ and $T(u^-) \geq I$, consequently $T(u) \geq 2I$. We conclude that for any $|y| \geq 2R$, u_y is a minimizer of $(\mathcal{M}_2)$. Moreover, a function $u \in \mathcal{X}$ can solve $(\mathcal{M}_2)$ if and only if $V(u^+) = V(u^-) = -1$ and $T(u^+) = T(u^-) = I$, i.e. if and only if u^+ and u^- solve $(\mathcal{M}_1)$.

As in step 2 we infer that all minimizers of $(\mathcal{M}_2)$ are C^1. Thus $(\mathcal{M}_2)$ satisfies the assumptions **A1'** and **A2** and Theorem 2 implies that all minimizers of $(\mathcal{M}_2)$ are axially symmetric. Since u_* is radial with respect to the origin, it is clear that any of the minimizers u_y is axially symmetric with respect to the line Oy, but is not radial about a point. Hence $(\mathcal{M}_2)$ admits nonradial minimizers.

In fact, with some extra work it can be proved that the suport of any minimizer of $(\mathcal{M}_1)$ is precisely a ball. If u is a minimizer of $(\mathcal{M}_2)$, $\mathrm{supp}(u) = \mathrm{supp}(u^+) \cup \mathrm{supp}(u^-)$ is the union of two balls with disjoint interiors. Therefore no minimizer of $(\mathcal{M}_2)$ can be radially symmetric.

In some particular cases, however, minimizers may have more symmetry than provided by Theorems 1 and 2, as it can be seen in the following example.

Example 8. Consider the problem of minimizing $E(u) = \int_{\mathbf{R}} \frac{1}{2}|u'(x)|^2 + F(u(x))\,dx$ in $H^1(\mathbf{R})$, under an arbitrary number of constraints $\int_{\mathbf{R}} G_j(u(x))\,dx = \lambda_j$, $1 \leq j \leq k$. We assume that the functions $F, G_1, \dots, G_j$ satisfy the assumption **H1** in Remark 4.

In this case Theorem 2 gives no information about the minimizers. However, if the problem above admits minimizers, any of them must be symmetric with respect to a point. Indeed, let u be a nonconstant minimizer. Then it satisfies an Euler-Lagrange equation

$$-u'' + F'(u) + \alpha_1 G_1'(u) + \ldots + \alpha_k G_k'(u) = 0 \qquad \text{in } \mathbf{R}. \tag{20}$$

It follows easily from (20) that $u \in C^2(\mathbf{R}, \mathbf{R})$. Since $u(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$, u achieves its maximum or its minimum at some point $a \in \mathbf{R}$ and consequently $u'(a) = 0$. Let $\tilde{u}(x) = u(2a - x)$. Then $\tilde{u}$ satisfies (20) and $\tilde{u}(a) = u(a)$, $\tilde{u}'(a) = u'(a) = 0$. Since the Cauchy problem associated to (20) has unique solution, we have $u = \tilde{u}$, i.e. u is symmetric about a. Moreover, we see that u must be symmetric with respect to any of its critical points. Since u cannot be periodic, we infer that there are no other critical points, thus u is monotonic on $(-\infty, a]$ and on $[a, \infty)$.

We have discussed in the first section an example of problem where arbitrarily many constraints were allowed and the symmetry properties of minimizers did not depend on the number of constraints (see [4]). This fact is due to the assumptions made on the nonlinear term (monotonicity in $|x|$ and cooperativity condition), that imply a strong coupling between the components of the minimizers and prevent situations like those in Examples 6 and 7 to occur.

Remark 9. Our results can be extended in an obvious way to minimization problems on cylinders. To be more specific, consider the problem $(\mathcal{P}_c)$ consisting in minimizing

$$E(u) = \int_A \int_\Omega F(|x|, y, u(x,y), |\nabla_x u(x,y)|, \nabla_y u(x,y), \dots, \nabla_y^\ell(x,y))\, dxdy$$

under the constraints

$$Q_j(u) = \int_A \int_\Omega G_j(|x|, y, u(x,y), |\nabla_x u(x,y)|, \nabla_y u(x,y), \dots, \nabla_y^\ell(x,y))\, dxdy, \quad j = 1, \dots, k,$$

where $x \in \Omega \subset \mathbf{R}^{N_1}$, $y \in A \subset \mathbf{R}^{N_2}$, Ω is an open set invariant by rotations in $\mathbf{R}^{N_1}$ and A is a measurable set in $\mathbf{R}^{N_2}$. We assume that problem $(\mathcal{P}_c)$ admits minimizers in a functional space $\mathcal{X}$ and the following assumptions hold:

A1$_c$. For any $w \in \mathcal{X}$ and any hyperplane Π in $\mathbf{R}^{N_1}$ containing the origin, we have $w_{(\Pi \times \mathbf{R}^{N_2})^-}, w_{(\Pi \times \mathbf{R}^{N_2})^+} \in \mathcal{X}$.

A2$_c$. For any minimizer $u \in \mathcal{X}$ and any $y \in A$, the function $u(\cdot, y)$ is C^1 on Ω.

Note that the minimization problem may involve derivatives of any order in y and we do not need more regularity of minimizers with respect to y than provided by the fact that $u \in \mathcal{X}$.

We have the following results, the proofs being similar to those of Theorems 1 and 2.

Theorem 1'. *Assume that u is a minimizer for problem $(\mathcal{P}_c)$ in $\mathcal{X}$, assumptions* $\mathbf{A1}_c$ *and* $\mathbf{A2}_c$ *are satisfied and* $0 \le k \le N_1 - 2$. *There exists a $k-$dimensional vector subspace V of $\mathbf{R}^{N_1}$ such that u is radially symmetric with respect to $V \times \mathbf{R}^{N_2}$.*

Theorem 2'. *Assume that $\Omega = \mathbf{R}^{N_1}$, $1 \le k \le N_1 - 1$ and the functions F, G_j in $(\mathcal{P}_c)$ do not depend on x. Assume also that* $\mathbf{A2}_c$ *is satisfied and* $\mathbf{A1}_c$ *holds for any affine hyperplane Π in $\mathbf{R}^{N_1}$. If u is a minimizer for problem $(\mathcal{P}_c)$ in $\mathcal{X}$, there exists a $(k-1)-$dimensional affine subspace $V \subset \mathbf{R}^{N_1}$ such that u is radially symmetric with respect to $V \times \mathbf{R}^{N_2}$.*

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