GAPS AND FATOU THEOREMS FOR SERIES IN SCHAUDER BASIS OF HOLOMORPHIC FUNCTIONS

PATRICE LASSÈRE & NGUYEN THANH VAN

ABSTRACT. We prove a gap theorem and the "Fatou change-of-sign theorem" for expansions in common Schauder basis of holomorphic functions.

1. INTRODUCTION

Let $\Omega \subset \mathbb{C}$ be an open set and $K \subset \Omega$ a compact. A regular condenser in $\mathbb{C}$ is a pair $\mathcal{C} = (\Omega, K)$ such that $K = \widehat{K_\Omega}$ (the holomorphic hull of K in Ω) and the extremal subharmonic function

$$h_{\Omega,K}(z) := \sup\{u(z) \ : \ u \in \mathrm{SH}(\Omega) \ : \ u \leq 1, \ u_{/K} \leq 0\}$$

is continuous on Ω and has boundary value 1 on $\partial\Omega$.

Let $(F_n)_n$ be a common Schauder basis of $\mathcal{O}(\Omega)$ (space of holomorphic functions on Ω equiped with the compact-convergence topology) and $\mathcal{O}(K)$ (space of germs of holomorphic functions on K equiped with the inductive limit topology $\lim_{K\subset U} \mathcal{O}(U)$) (for the existence of such a basis see Nguyen T.V. [5] or Bagby [1]). $(F_n)_n$ is then a basis for $O(\Omega_\alpha)$ ($\forall\, \alpha \in]0,1[$) where $\Omega_\alpha = \{z \in \Omega \ : \ h_{\Omega,K}(z) < \alpha\}$.
We will prove the following

1.1 Theorem : *Suppose $\Omega \setminus K$ connected. Let $\alpha \in]0,1[$ and $f \in \mathcal{O}(\Omega_\alpha)$ which is not analytically continuable to Ω_β for any $\beta \in]\alpha,1[$. Let $\sum_{n\geq 0} c_n F_n$ be the expansion of f in the basis $(F_n)_n$ in $\mathcal{O}(\Omega_\alpha)$. Then there exists a sequence $\varepsilon = (\varepsilon_n)_n \in \{0,1\}^{\mathbb{N}}$ (resp. $\{-1,1\}^{\mathbb{N}}$) such that the function $f_\varepsilon = \sum_{n\geq 0} \varepsilon_n c_n F_n$ is not analytically continuable through any point of $\partial\Omega_\alpha$.*

1.2 Remark : We know (Mityagin [4]) that any basis is absolute, so $\sum_{n\geq 0} c_n F_n$ converges normally on every compact of Ω_α and so $f_\varepsilon \in \mathcal{O}(\Omega_\alpha)$.

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2. Preliminary results

2.1 Theorem : *(Siciak [8] Cor.3.4 and also Saint Raymond [7] lemma 4) Let $(f_n)_n$ be a sequence of holomorphic functions in an open set Ω of $\mathbb{C}^d$. Let D be the set of points $a \in \Omega$ such that the serie $\sum_{n\geq 0} f_n$ is normally convergent on a neighbourhood of a. Assume that $\overline{D} \subset \Omega$. Then there exists $\varepsilon = (\varepsilon_n)_n \in \{0,1\}^{\mathbb{N}}$ (resp. $\{-1,1\}^{\mathbb{N}}$) such that the function $f_\varepsilon = \sum_{n\geq 0} \varepsilon_n f_n$ cannot be continued analytically through any boundary point of D.*

Siciak proved this theorem for $\varepsilon \in \{0,1\}^{\mathbb{N}}$, but his proof works as well for $\varepsilon \in \{-1,1\}^{\mathbb{N}}$.

2.2 Definition : *Let $\mathcal{C} = (\Omega, K)$ be a regular condenser in $\mathbb{C}$. A sequence $(f_n)_n \subset \mathcal{O}(\Omega)$ is said $\mathcal{C}$-regular when there exists $R > 1$ such that*

$$\lim_{n\to\infty} \|f_n\|_\Delta^{1/n} = R^{\alpha_\Delta}$$

for every compact disc $\Delta \subset \Omega \setminus K$, with $\alpha_\Delta := \sup_{z\in\Delta} h_{\Omega,K}(z)$ and $\|.\|_\Delta$ is the sup norm on Δ.

2.3 Fundamental Lemma : *Let $\mathcal{C}$ be a regular condenser, let $(f_n) \subset \mathcal{O}(\Omega)$ be a $\mathcal{C}$-regular sequence and $(c_n)_n$ be a sequence of complex numbers satisfying*

$$\limsup_{n\to\infty} |c_n|^{1/n} = R^{-\alpha},$$

then there exists $\varepsilon = (\varepsilon_n)_n \in \{0,1\}^{\mathbb{N}}$ (resp. $\{-1,1\}^{\mathbb{N}}$) such that the function $f_\varepsilon = \sum_{n\geq 0} \varepsilon_n c_n f_n \in \mathcal{O}(\Omega_\alpha)$ is not analytically continuable through any point of $\partial\Omega_\alpha$.

Proof : Let

$$D = \{z \in \Omega \ : \ \sum_{n\geq 0} c_n f_n \ \text{ converges normally in a neighborhood of } z\}.$$

Following the above theorem of Siciak, it suffices to prove that $D = \Omega_\alpha$. Evidently $\Omega_\alpha \subset D$. Let $a \in \Omega \setminus \overline{\Omega_\alpha}$, $\Delta = \overline{D}(a,r) \subset \Omega \setminus \overline{\Omega_\alpha}$. The hypothesis on $(f_n)_n$ implies :

$$\forall\, \rho \in]0,1[,\ \exists\, A(\rho) = A \quad \text{such that} \quad \|f_n\|_\Delta \geq A\rho^n R^{n\alpha_\Delta}, \quad \forall\, n \in \mathbb{N}.$$

The hypothesis on $(c_n)_n$ implies :

$$\forall\, \tau \in]0,1[,\ \exists\, B(\tau) = B,\ \exists\, (n_k)_k \nearrow \infty \quad \text{such that} \quad |c_{n_k}| \geq B\tau^{n_k} R^{-\alpha n_k}, \quad \forall\, k \in \mathbb{N}.$$

If we choose ρ and τ such that $r := \rho\tau R^{\alpha_\Delta - \alpha} > 1$, then

$$\|c_{n_k} f_{n_k}\|_\Delta \geq ABr^{n_k} \nearrow \infty,$$

and $a \notin D$. So $D \supset \overline{\Omega_\alpha}$. Because $\overset{\circ}{\overline{\Omega_\alpha}} = \Omega_\alpha$, we have $D = \Omega_\alpha$.

3. Proof of the theorem :

Recall (Nguyen [5], Bagby [1]) that there exists a common basis $(B_n)_n$ of $\mathcal{O}(\Omega)$ and $\mathcal{O}(K)$ satisfying

$$\lim_{n\to\infty} \|B_n\|_{\partial\Omega_\alpha}^{1/n} = R^\alpha, \quad \forall\, \alpha \in]0,1[,$$

where $R = \exp(2\pi/C)$ and C is the capacity of the condenser $\mathcal{C}$:

$$C := \inf\left\{ \int\int_{\Omega\setminus K} |\nabla u(z)|^2 dxdy,\ u \in \mathcal{C}^1(\Omega\setminus K), u_{/\partial\Omega} \equiv 1,\ u_{/\partial K} \equiv 0 \right\}$$

By the simultaneous quasi-equivalence theorem of Dragilev (Dragilev [2], Nguyen-Lassère [6]) there exists $(\alpha_n)_n \subset \mathbb{C}^\star$ and a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that the basis $(G_n)_n$ where $G_n = \alpha_n F_{\pi(n)}$ is equivalent to $(B_n)_n$ in $\mathcal{O}(\Omega_\alpha)$, $\forall\, \alpha \in]0,1[$. This means that the relations $T(B_n) := G_{\pi_n},\ n \in \mathbb{N}$ define a simultaneous isomorphism T for all the spaces $\mathcal{O}(\Omega_\alpha)$, $(\alpha \in]0,1[)$. One deduces easily from this that

$$(\bigstar) \qquad \lim_{n\to\infty} \|G_n\|_{\partial\Omega_\alpha}^{1/n} = R^\alpha, \quad \forall\, \alpha \in]0,1[.$$

Lemma : (Nguyen [5] page 228) *Let $\mathcal{C} = (\Omega, K)$ be a regular condenser with $\Omega \setminus K$ connected. Let $(G_n)_n$ be a sequence of holomorphic functions on Ω such that*

$$\lim_{n\to\infty} \|G_n\|_{\partial\Omega_\alpha}^{1/n} = R^\alpha, \quad \forall\, \alpha \in]0,1[,$$

then, for any compact disc $\Delta \subset \Omega \setminus K$ we have

$$\lim_{n\to\infty} \|G_n\|_{\Delta}^{1/n} = R^{\alpha_\Delta}, \quad \text{where} \quad \alpha_\Delta = \sup_{z\in\Delta} h_{\Omega,K}(z).$$

Let f as in the statement of the theorem. Then, in $\mathcal{O}(\Omega_\alpha)$, we have

$$f = \sum_{n\geq 0} c_n F_n = \sum_{n\geq 0} d_n G_n = \sum_{n\geq 0} d_n \alpha_n F_{\pi(n)}$$

the series converge normally on any compact of Ω_α because any Schauder basis of a Fréchet nuclear space is absolute (Mityagin, [4]). The sequence $(d_n)_n$ satisfies evidently

$$\limsup_{n\to\infty} |d_n|^{1/n} = R^{-\alpha}$$

as consequence of $(\bigstar)$ and the hypothesis $f \notin \mathcal{O}(\Omega_\beta)$, $\forall\, \beta > \alpha$.

By the fundamental lemma, $\exists\, \varepsilon' = (\varepsilon'_n)_n \in \{0,1\}^{\mathbb{N}}$ (resp. $\{-1,1\}^{\mathbb{N}}$) such that the function

$$f_{\varepsilon'} = \sum_{n\geq 0} \varepsilon'_n d_n G_n = \sum_n \varepsilon'_n d_n \alpha_n F_{\pi(n)} = \sum_n \varepsilon'_{\pi^{-1}(n)} c_n F_n$$

holomorphic in Ω_α, is not analytically continuable through any point of $\partial\Omega_\alpha$ ($c_n = d_{\pi^{-1}(n)}\alpha_{\pi^{-1}(n)}$ by uniqueness of coefficients and the different series, absolutly convergent, are commutatively convergent). To conclude, it suffices to put $\varepsilon_n = \varepsilon'_{\pi^{-1}(n)}$. $\square$.

Remark : The following example was suggested by the referee : Let $\Omega := \{\, z \in \mathbb{C} \ :\ |z^2-1| < 2\}$, $K := \{\, z \in \mathbb{C} \ :\ |z^2-1| \leq a\}$ where $0 < a < 1$. Here $h_{\Omega,K}(z) = \log^+ \frac{|z^2-1|}{a} / \log \frac{2}{a}$. If $\alpha = \frac{\log \frac{1}{a}}{\log \frac{2}{a}}$ then Ω_α is the union of two disjoints components D_1 (with $-1 \in D_1$) and D_2 (with $1 \in D_2$) whose closures intersect at the point 0. Let f_1, f_2 be two different entire functions (e.g. two different constant functions) and consider $f(z) := f_j(z),\ j = 1,2$. Then $f \in \mathcal{O}(\Omega_\alpha)$ and does not have analytic continuation to any Ω_β with $\alpha < \beta < 1$. By the theorem, there exists a sequence $\varepsilon \in \{-1,1\}$ (resp. $\varepsilon \in \{0,1\}$) such that the function $f_\varepsilon := \sum_{n\geq 0} \varepsilon_n c_n F_n$ does not have analytic continuation across any boundary point of Ω_α.

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Lassère Patrice & Nguyen Thanh Van : Laboratoire de Mathématiques E.Picard, UMR CNRS 5580, Université Paul Sabatier, 118 route de Narbonne, 31062 TOULOUSE.
lassere@picard.ups-tlse.fr & nguyen@picard.ups-tlse.fr