A characterization of essentially strictly convex functions on reflexive Banach spaces

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Abstract

We call a function $J : X \to \mathbb{R} \cup \{+\infty\}$ "adequate" whenever its tilted version by a continuous linear form $x \mapsto J(x) - \langle x^*, x\rangle$ has a unique (global) minimizer on X, for appropriate $x^* \in X^*$. In this note we show that this induces the essentially strict convexity of J. The proof passes through the differentiability property of the Legendre-Fenchel conjugate J^* of J, and the relationship between the essentially strict convexity of J and the Gâteaux-differentiability of J^*. It also involves a recent result from the area of the (closed convex) relaxation of variational problems. As a by-product of the main result derived, we express the subdifferential of the (generalized) Asplund function associated with a couple of functions (f, h) with $f \in \Gamma(X)$ cofinite and $h : X \to \mathbb{R} \cup \{+\infty\}$ weakly lower-semicontinuous. We do this in terms of (generalized) proximal set-valued mappings defined via (g, h). The theory is applied to Bregman-Tchebychev sets and functions for which some new results are established.

Keywords: Reflexive Banach space, weakly lower-semicontinuous function, essentially smooth function, essentially strictly convex function, closed convex relaxation, Asplund function, proximal set-valued mapping.

1. Introduction to the context of the work

Let us consider an extended real-valued (not necessarily convex) function $J : X \to \mathbb{R} \cup \{+\infty\}$, not identically equal to $+\infty$ (such a function is called proper), defined on a reflexive Banach space $(X, \|.\|)$. We denote by $(X^*, \|.\|_*)$ the topological dual space of X endowed with the dual norm $\|.\|_*$ of $\|.\|$, and by

$\Gamma(X)$ (resp. $\Gamma(X^*)$) the set of convex lower-semicontinuous proper functions on X (resp. X^*). All the notations we are going to use are usual in the context of Functional convex analysis, Set-valued and variational analysis. For example, by $\partial J : X \rightrightarrows X^*$ we mean the subdifferential (set-valued) mapping of J. Actually, that is the inverse set-valued mapping of ∂J which will be of interest to us, namely

$$\begin{aligned} MJ := (\partial J)^{-1} : X^* \rightrightarrows X \\ x^* \mapsto MJ(x^*) = \{x \in X | x^* \in \partial J(x)\}. \end{aligned} \tag{1}$$

To be a bit more explicit, given a continuous linear form x^* on X, $MJ(x^*)$ is the set of (global) minimizers of the function $x \in X \mapsto J(x) - \langle x^*, x\rangle$, the tilted version of J. As a general rule, we have

$$MJ(x^*) \subset \partial J^*(x^*), \text{ hence } \overline{\text{co}} MJ(x^*) \subset \partial J^*(x^*) \tag{2}$$

for all $x^* \in X^*$, where J^* stands for the Legendre-Fenchel conjugate of J. Since the underlying space X has been assumed to be reflexive, we know (see [As, Theorem 2] for example) that J^* is Gâteaux-differentiable on a dense G_δ-subset of int(dom J^*). Furthermore, the inclusion (2) is made meaningful when J is weakly lower-semicontinuous on X, as shown in the next statement.

Proposition 1. *Suppose J is weakly lower-semicontinuous on a reflexive Banach space X. Then $MJ(x^*)$ is nonempty for all $x^* \in$ int(dom J^*). In other words,*

$$\emptyset \neq \overline{\text{co}} MJ(x^*) \subset \partial J^*(x^*) \textit{ for all } x^* \in \text{int}(\text{dom}\, J^*). \tag{3}$$

Proof. The only point we have to prove is that $MJ(x^*)$ is nonempty whenever x^* is taken in int(dom J^*).

Due to the fact that x^* has been chosen in the interior of dom J^*, the function J^* is finite and continuous at x^*. Therefore, there is a neighborhood of x^* on which J^* is bounded from above. Said otherwise, there is $\alpha > 0$ and $r \in \mathbb{R}$ such that

$$J^* \leq i_{B^*(x^*,\alpha)} + r. \tag{4}$$

Here, $i_{B^*(x^*,\alpha)}$ denotes the indicator function of the closed ball (for the norm $\|.\|_*$) centered at x^* and of radius $\alpha > 0$.

By taking the Legendre-Fenchel conjugates in both sides of (4), we get at the following inequality:

$$J \geq J^{**} \geq \langle x^*, .\rangle + \alpha\|.\| - r.$$

It then follows that the tilted function $J - \langle x^*, .\rangle$ is 0-coercive on X (i.e., it goes to $+\infty$ when $\|x\| \to +\infty$). As a result, $J - \langle x^*, .\rangle$ is bounded from below and achieves its lower bound. We just have proved that $MJ(x^*)$ is nonempty.□

¿From what was said about the Gâteaux-differentiability of J^* and the Proposition just above, we immediately derive:

Proposition 2. *Let J (and X) be as in Proposition 1. Then the set-valued mapping MJ is single-valued ($MJ(x^*) = \{m_J(x^*)\}$) on a G_δ-dense subset of* int(dom J^*).

2. “Adequate” vs essentially strictly convex functions

In order to reinforce the property of MJ laid out in Proposition 2, we introduce hereafter a definition.

Definition. A proper extended real-valued function $J : X \to \mathbb{R} \cup \{+\infty\}$ is said to be “adequate” if

$$\begin{cases} \text{dom } MJ = \text{dom } (\partial J^*) \text{ is a nonempty open set ;} \\ MJ \text{ is single-valued on its domain.} \end{cases}$$

Remembering the explanation of $MJ(x^*)$ in Section 1, the definition above expresses that the tilted version of J, namely $J - \langle x^*, .\rangle$, has a unique (global) minimizer on X. Such a definition of “adequate” functions is related, although we do not assume a priori any convexity property of J, with the following concepts, introduced in [BBC], that extend to the infinite dimensional setting those similar given in [Ro, Section 26]. Let us recall these specific definitions:

- A function $\Phi \in \Gamma(X^*)$ is said to be *essentially smooth* if the subdifferential set-valued mapping $\partial\Phi$ is both single-valued and locally bounded on its domain.
- A function $J \in \Gamma(X)$ is said to be *essentially strictly convex* if J is strictly convex on every convex subset of $\text{dom}(\partial J)$ and $MJ = (\partial J)^{-1}$ is locally bounded on its domain.

In the definitions above, a locally bounded set-valued mapping $x \mapsto \Gamma(x)$ means that some neighborhood V of x has a bounded image $\Gamma(V)$.

¿From [BBC, Theorem 5.4], we know that $J \in \Gamma(X)$ is essentially strictly convex if and only if J^* is essentially smooth. A function $J \in \Gamma(X)$ which is both essentially strictly convex and essentially smooth is called a Legendre function. Let us now provide a standard example of what an “adequate” function is.

Proposition 3. *A weakly lower-semicontinuous function $J : X \to \mathbb{R} \cup \{+\infty\}$ for which J^* is essentially smooth is necessarily “adequate”.*

Proof. By Proposition 1 we have $\text{int}(\text{dom}J^*) \subset \text{dom}MJ$. Since J^* is essentially smooth, one has $\text{int}(\text{dom}J^*) = \text{dom}\partial J^*$, so $\text{dom}\partial J^* \subset \text{dom}MJ$. Conversely, it comes from (2) that $\text{dom}MJ \subset \text{dom}\partial J^*$, and finally $\text{dom}MJ = \text{dom}\partial J^* = \text{int}(\text{dom}J^*)$. Since ∂J^* is single-valued on its domain, Proposition 1 says that MJ is single-valued on $\text{int}(\text{dom}J^*)$ and J is therefore “adequate”.□

In particular, any weakly lower-semicontinuous function $J : X \to \mathbb{R} \cup \{+\infty\}$ whose conjugate J^* is Gâteaux-differentiable on X^* is “adequate”.

In the sequel we will need Theorem A below that completes an earlier result by Soloviov [So, Theorem 1.2.1] (see also [Za, Theorem 9.3.2]). Its proof easily follows from [BBC, Theorems 5.4 and 5.6] and the above considerations.

Theorem A. *A weakly lower-semicontinuous function $J : X \to \mathbb{R} \cup \{+\infty\}$ on a reflexive Banach space X is essentially strictly convex if and only if J^* is essentially smooth.*

Our main result (the forthcoming Theorem 1) also relies on the next theorem whose main applications are in the relaxation of variational problems. It just says that the inclusion (3) in Proposition 1 is in fact an equality.

Theorem B. *Let $J : X \to \mathbb{R} \cup \{+\infty\}$ be a weakly lower-semicontinuous function on a reflexive Banach space. Then:*

$$\partial J^*(x^*) = \overline{\text{co}} MJ(x^*) \text{ for all } x^* \in \text{int}(\text{dom}\, J^*). \tag{5}$$

PROOF. Since $x^* \in \text{int}(\text{dom}\, J^*)$ we have from [LV, Theorem 1] or [HULV, Theorem 6] that

$$\partial J^*(x^*) = \text{argmin}(J - \langle x^*, .\rangle)^{**} = \bigcap_{\epsilon > 0} \overline{\text{co}}(\epsilon - \text{argmin}(J - \langle x^*, .\rangle)). \tag{5'}$$

Assume now that $x \notin \overline{\text{co}} MJ(x^*)$. By the Hahn-Banach Theorem, there exist $y^* \in X^*$, $r \in \mathbb{R}$, such that

$$i^*_{MJ(x^*)}(y^*) < r < \langle y^*, x\rangle,$$

and $MJ(x^*)$ is included in the (weakly) open half-space $[\langle y^*, .\rangle < r]$. Since $x^* \in \text{int}\,\text{dom} J^*$ and J is weakly lower-semicontinuous, all the sublevel sets $\{u \in X \,:\, J(u) - \langle x^*, u\rangle \leq s\}, s \in \mathbb{R}$, are weakly compact. So, the set-valued mapping $\epsilon \geq 0 \mapsto \epsilon - \text{argmin}(J - \langle x^*, .\rangle)$ is weakly upper-semicontinuous [Mo2, Proposition 11.c], and there exists $\epsilon > 0$ such that $\epsilon - \text{argmin}(J - \langle x^*, .\rangle) \subset [\langle y^*, .\rangle < r]$. We thus have $i^*_{\epsilon - \text{argmin}(J - \langle x^*, .\rangle)}(y^*) \leq r < \langle y^*, x\rangle$, and, by the Hahn-Banach Theorem again, $x \notin \overline{\text{co}}(\epsilon - \text{argmin}(J - \langle x^*, .\rangle))$. It then follows from (5') that $x \notin \partial J^*(x^*)$, and the proof is achieved.□

We now are in a position to state the main result of our note.

Theorem 1. *A weakly lower-semicontinuous function $J : X \to \mathbb{R} \cup \{+\infty\}$ defined on a reflexive Banach space X turns out to be "adequate" if and only if it is essentially strictly convex.*

PROOF. (SUFFICIENCY) Since J is essentially strictly convex, J^* is essentially smooth and, by Proposition 3, J is "adequate".

(NECESSITY) Let us assume that the weakly lower-semicontinuous function $J : X \to \mathbb{R} \cup \{+\infty\}$ is "adequate" ; by Theorem A, we will succeed in our aims if we prove that J^* is essentially smooth. By definition of what an "adequate" J is, MJ is single-valued on its domain $\text{dom}\, MJ = \text{dom}(\partial J^*)$, which is nonempty and open. We thus have $\text{dom}\, MJ = \text{int}(\text{dom}\, J^*)$. As an application of Theorem B, we then get that ∂J^* is single-valued on $\text{dom}(\partial J^*) = \text{int}(\text{dom}\, J^*) \neq \emptyset$. It then follows from [BBC, Theorem 5.6] that J^* is essentially smooth.□

3. Generalized proximal set-valued mappings in reflexive Banach spaces

In this section we consider the case when

$$J = f + h$$

where $f \in \Gamma(X)$ is cofinite (this means that f^* is real-valued on X^* : $\mathrm{dom} f^* = X^*$) and $h : X \to \mathbb{R} \cup \{+\infty\}$ admits a continuous affine minorant : $\mathrm{dom} h^* \neq \emptyset$. We also assume that J is proper : $\mathrm{dom} f \cap \mathrm{dom} h \neq \emptyset$. Such a function J is not necessarily convex but it is cofinite. Indeed, picking $a^* \in \mathrm{dom} h^*$, there exists $r \in \mathbb{R}$ such that $J \geq f + \langle a^*, . \rangle - r$, and so, for any $x^* \in X^*$, one has $J^*(x^*) \leq f^*(x^* - a^*) + r \in \mathbb{R}$. On the other hand, since J is proper, J^* does not take the value $-\infty$. Consequently, J is cofinite. We now introduce the notation

$$MJ(x^*) = \mathrm{Prox}_h^f(x^*), x^* \in X^*, \tag{6}$$

and justify this choice by considering the case when $(X, \| \, \|)$ is a Hilbert space and $f = \frac{1}{2}\| \, \|^2$. We thus have

$$MJ(x^*) = \underset{u \in X}{\mathrm{argmin}} \left(\frac{1}{2}\|x^* - u\|^2 + h(u) \right), x^* \in X^* = X.$$

In other words, MJ is just the so-called Moreau proximal set-valued mapping associated with the function h. It should be emphasized however that we do not assume that h is convex. We also set

$$J^* = (f + h)^* = \Phi_h^f, \tag{7}$$

a continuous convex function that can be viewed as an extension of the Asplund function : indeed, taking $f = \frac{1}{2}\| \, \|^2, S \subset X, h = i_S$ (the indicator function of S) in a Hilbert space setting, and denoting by d_S the distance function to S, we get, for any $x^* \in X^* = X$,

$$\Phi_h^f(x^*) = \frac{1}{2}\left(\|x^*\|^2 - d_S^2(x^*)\right) = \Phi_S(x^*),$$

which is called the Asplund function associated with S ([HU]). In the same setting (i.e. $(X, \| \, \|)$ Hilbert and $f = \frac{1}{2}\| \, \|^2$) one has

$$\Phi_h^f = \frac{1}{2}\| \, \| - (h \Box \frac{1}{2}\| \, \|^2),$$

where $\Box$ denotes the infimal convolution operation, and $h \Box \frac{1}{2}\| \, \|^2$ is a Moreau envelope of the (non-necessarily convex) function h (see for example [Wa] Theorem 3.5 for the Euclidean case).

Proposition 4. *Let X be reflexive, $f \in \Gamma(X)$, f cofinite, and $h : X \to \mathbb{R} \cup \{+\infty\}$ a weakly lower-semicontinuous function such that* $\mathrm{dom} f \cap \mathrm{dom} h \neq \emptyset$, $\mathrm{dom} h^* \neq \emptyset$*. Then the set-valued mapping* Prox_h^f *defined in (6) is single-valued on a $G_\delta-$dense subset of X^* and the subdifferential of the continuous convex function Φ_h^f defined in (7) is given by*

$$\partial \Phi_h^f(x^*) = \overline{\mathrm{co}} \left(\mathrm{Prox}_h^f(x^*) \right), \forall x^* \in X^*. \tag{8}$$

PROOF. We know that the weakly lower-semicontinuous function $J = f + h$ is cofinite. It then suffices to apply Proposition 2 and Theorem B.□

In the particular case where $(X, \| \|)$ is a Hilbert space, $f = \frac{1}{2} \| \|^2$, and $h = i_S$ is the indicator function of a weakly closed subset of X, what (8) says is that the associated Asplund function $\Phi_S = \frac{1}{2} \left(\| \|^2 - d_S^2 \right)$ has a subdifferential expressed as

$$\partial \Phi_S(x^*) = \overline{\mathrm{co}} \left(P_S(x^*) \right), \tag{9}$$

where $P_S(x^*)$ is the projection set-valued mapping on S (i.e., $P_S(x^*)$ is the set of points in S at minimal distance of x^*). A formula like (9) is hopeless for general S, especially as $P_S(x^*)$ can be empty (while the continuous convex function Φ_S has a nonempty subdifferential everywhere on X). That is the reason why a different strategy had to be applied in order to express $\partial \Phi_S(x^*)$; this was done in [HULV, Section 3] by enlarging $P_S(x^*)$ to $P_S^\epsilon(x^*)$ for $\epsilon > 0$, and then filtering on the resulting closed convex set $\overline{\mathrm{co}} P_S^\epsilon(x^*)$. In a famous and rather hold result (1961), V. Klee proved that every weakly closed Tchebychev subset S of a Hilbert space X is necessary convex. In other words

$$(P_S(x^*) \text{ single-valued for all } x^* \in X) \Rightarrow (S \text{ convex}).$$

To know whether this implication holds true for general closed sets S is still an open question; see [HU] for an overview on this problem.

We now extend Klee's theorem to the set-valued proximal process introduced in this note :

Theorem 2. *Let X be reflexive, $f \in \Gamma(X)$, f cofinite, $h : X \to \mathbb{R} \cup \{+\infty\}$ weakly lower-semicontinuous and such that* $\mathrm{dom}\, f \cap \mathrm{dom}\, h = \emptyset$*. The following assertions are then equivalent :*

(i) $\begin{cases} h \textit{ is } f-\textit{Tchebychev, meaning that} \\ \mathrm{Prox}_h^f \textit{ is single-valued on } X^* \\ (\textit{that is to say}: \ \mathrm{Prox}_h^f(x^*) = \{\mathrm{prox}_h^f(x^*)\}, \ \forall x^* \in X^*) \end{cases}$

(ii) $f + h$ *is essentially strictly convex.*

In such a case, Φ_h^f is Gâteaux-differentiable on X^* and one has

$$\nabla \Phi_h^f(x^*) = \mathrm{prox}_h^f(x^*), \ \forall x^* \in X^*. \tag{10}$$

PROOF. According to the very definition of Prox_h^f (see (6)) and since J is cofinite, one clearly has

$$J \text{ adequate} \Leftrightarrow \mathrm{Prox}_h^f \text{ single-valued on } X^*.$$

Therefore, the equivalence between (i) and (ii) comes from Theorem 1. Formula (10) comes from (8).□

Theorem 2 says, among other things, that, even if the function h is not convex, the generalized Asplund function Φ_h^f is a primitive function of the mapping prox_h^f on X^*, whenever h is a weakly lower-semicontinuous $f-$Tchebychev function.

Remark 1. Taking for $(X, \| \|)$ a Hilbert space and $f = \frac{1}{2}\| \|^2$ in both Proposition 4 and Theorem 2, we extend Theorem 3.5 in [Wa] from Euclidean spaces to Hilbert spaces and answer Problem 8 posed in ([BMW] Section 8) by characterizing the class of Tchebychev functions in infinite dimensional spaces.

4. Adequate functions and Bregman-Tchebychev sets

Let $f \in \Gamma(X)$ be Gâteaux-differentiable on $\mathrm{int}(\mathrm{dom} f) \neq \emptyset$. The so-called Bregman "distance" associated with f (see e.g. [BBC]) is defined as

$$D^f(x, y) = f(x) - f(y) - \langle \nabla f(y), x - y\rangle,\ (x, y) \in \mathrm{dom}\, f \times \mathrm{int}(\mathrm{dom}\, f).$$

Given $S \subset X, S \cap \mathrm{dom}\, f \neq \emptyset$, and $y \in \mathrm{int}(\mathrm{dom}\, f)$, we set $D_S^f(y) = \inf_{u \in S} D^f(u, y)$ for the D^f-distance from y to S and $P_S^f(y) = \{x \in S : D^f(x, y) = D_S^f(y)\}$ for the D^f-projection of y on S. One says that S is D^f-Tchebychev whenever P_S^f is single-valued on $\mathrm{int}(\mathrm{dom}\, f)$. Taking X and f as in Remark 1 the D^f-Tchebychev sets boil down to the usual ones. We now introduce the additional assumption

$$\nabla f(\mathrm{int}(\mathrm{dom}\, f)) = X^*, \tag{11}$$

that holds, for instance, if f is a cofinite Legendre function. It is thus easy to check that

$$S \text{ is } D^f - \text{Tchebychev} \ \Leftrightarrow\ i_S \text{ is } f - \text{Tchebychev} \ \Leftrightarrow\ f + i_S \text{ is adequate.} \tag{12}$$

Proposition 5. *Let X be reflexive and $f \in \Gamma(X)$ satisfying (11). For any weakly closed $S \subset X$ such that $S \cap \mathrm{dom}\, f \neq \emptyset$, the following assertions are equivalent :*

(i) S is D^f-Tchebychev

(ii) $f + i_S$ is essentially strictly convex.

So, if S is D^f-Tchebychev, then $S \cap \mathrm{dom}\, f$ is convex. In particular, any weakly closed D^f-Tchebychev set included in $\mathrm{dom}\, f$ *is convex.*

Proof. Due to (12), Theorem 1 gives the equivalence between (i) and (ii). The other assertions are then trivial.□

Corollary 1. *Assume f is a cofinite Legendre function and S is a weakly closed set included in* $\mathrm{int}(\mathrm{dom}\, f)$*. Then S is D^f-Tchebychev if and only if S is convex.*

Proof. Sufficiency comes from [BBC, Corollary 7.9]. Necessity is due to the second part of Proposition 5.□

Remark 2. Since there are Legendre functions that are cofinite but not 1-coercive ([BBC, Example 7.5]), Corollary 1 above improves a bit [LSY, Theorem 4.1].

5. The finite dimensional case

In this context, Theorem B reads as follows :

Theorem B'. *Let $J : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower-semicontinuous. Then, for any $x^* \in \mathrm{int}(\mathrm{dom}\, h^*)$, one has :*

$$\partial J^*(x^*) = \mathrm{co}MJ(x^*). \tag{13}$$

Proof. Since $x^* \in \mathrm{int}(\mathrm{dom}\, J^*)$ and J is lower-semicontinuous, the set $MJ(x^*)$ is compact and $\mathrm{co}MJ(x^*)$ is compact too. Thus (13) is just (5).□

We now give another proof of Theorem B' that does not use Theorem B. It is based on a formula given in [BHU] which is limited to finite dimensional spaces. Since $x^* \in \mathrm{int}(\mathrm{dom}\, h^*)$ we know that there exist $\alpha > 0$ and $r \in \mathbb{R}$ such that

$$J - \langle x^*, .\rangle \geq \alpha \|.\| + r.$$

Thus the asymptotic function $(J - \langle x^*, .\rangle)'_\infty$ of $J - \langle x^*, .\rangle$, that is $J'_\infty - \langle x^*, .\rangle$ where, for any $x \in \mathbb{R}^n$, $J'_\infty(x) = \liminf_{(t,u)\to(0^+,x)} tJ(u/t)$, is nonnegative and vanishes at the origin of $\mathbb{R}^n$ only. We then have (see [HULV] Theorem 6 or [BHU] p 1672) :

$$\partial J^*(x^*) = \mathrm{argmin}(J - \langle x^*, .\rangle)^{**} = \mathrm{co}\ \mathrm{argmin}(J - \langle x^*, .\rangle) = \mathrm{co}JM(x^*).$$

Remark 3. According to Theorem B' one can replace $\overline{\mathrm{co}}$ by co in the formulas (8) and (9) when X is finite dimensional.

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