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NOTES

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Pythagoras' Theorem for Areas

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The theorem of Pythagoras relating the squares of the lengths of the sides of a right triangle is one of the key results in elementary geometry. At a more advanced level, one learns that the Pythagorean theorem extends to prehilbert spaces, but it still expresses a relation among *lengths* of vectors.

We incidentally came across a result relating the areas of the faces of a right tetrahedron whose analogy with the Pythagorean theorem is striking. Here it is.

Theorem 1. *Let $OABC$ be a tetrahedron with three perpendicular triangular faces OAB, OAC, OBC, and "hypotenuse-face" ABC; see Figure 1. Let S_1, S_2, S_3 denote the areas of the perpendicular faces and let S denote the area of the hypotenuse-face. Then*

$$S^2 = S_1^2 + S_2^2 + S_3^2. \tag{1}$$

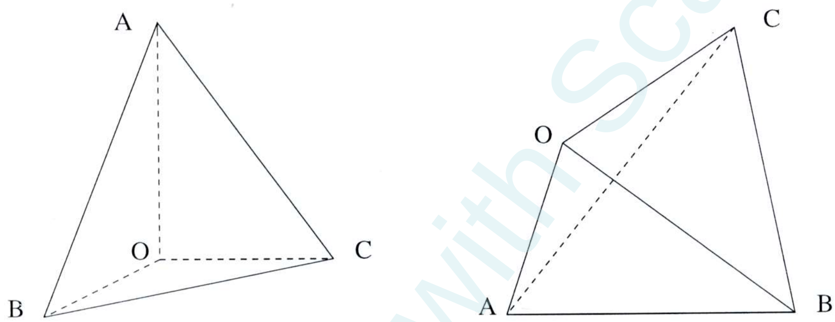

Figure 1. The right tetrahedron OABC

Theorem 1 is very easy to prove; we derive a generalized version in the n-dimensional context.

When we came across the result of Theorem 1, which we propose to call *the Pythagorean theorem for areas*, we found it noteworthy and asked colleagues: is it original (of course we thought no ...)? is it well-known (maybe ...)? does it bear a name (who knows ...)? After several unsuccessful attempts, we finally found mention of this result; here is a brief historical account.

The result of Theorem 1 indeed appears in some old books of geometry, in chapters devoted to "Applications of vector calculus to analytical geometry in 3-dimensional space"; the areas are there calculated via vector products. A sample reference is [**8**, pp. 121–123]. According to [**4**, p. 98], it was very likely known to R. Descartes. Theorem 1 also appears in [**7**] (a "bible" for results in elementary geometry) where (p. 911–912) it is attributed to J.-P. Gua de Malves (published in his memoirs of 1783). However, the analogy with the Pythagorean theorem for lengths is nowhere pointed out. Later on, in his famous work [**3**], L.N.M. Carnot states the result of Theorem 1 and refers to it as a "known result" [**3**, section 263, p. 311]; he also provides an extension to any convex polyhedron ([**3**, section 268, p. 313]), which is a generalization of the cosine theorem for triangles.

We now propose a generalization of Theorem 1 to n-dimensional spaces. Let the standard Euclidean affine space $\mathbb{R}^n$ be marked off with $(O; \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$, where $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is an orthonormal basis of the vector space $\mathbb{R}^n$. We consider there the compact convex polyhedron (or polytope, or n-simplex) Ω_n described as follows:

$$\Omega_n := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{x_i}{a_i} \leq 1, x_i \geq 0, \text{ for all } i = 1, \dots, n \right\}, \tag{2}$$

where $a_i > 0$ for all $i = 1, \dots, n$.

The "multi-orthogonal" Ω_n is just a generalized version of the right tetrahedron of Theorem 1, and its structure from the convexity viewpoint is well-known ([**1**, p. 84], [**2**, p. 77–79], [**6**, p. 53]). Indeed, Ω_n has

- $n+1$ vertices: the origin O and the n points A_i defined as $\overrightarrow{OA_i} = a_i \vec{e}_i$ for all $i = 1, 2, \dots, n$.
- $n+1$ $(n-1)$-dimensional faces (also called *facets*): n of these facets contain the origin (obtained as convex hulls of O and $n-1$ points among the A_i's); we call them *facets issuing from the origin*. One facet does not contain the origin (obtained as the convex hull of the A_i's); we call it *hypotenuse facet*.

The $(n-1)$-dimensional content of the facets of Ω_n is called *area*, in analogy with the usual case with $n = 3$. The n-dimensional content of Ω_n, called the "volume" V of Ω_n, can be computed from the areas of facets by the following formula [**1**, p. 87]:

$$V = \frac{1}{n} \text{(area of a facet)} \times \text{(height from the vertex outside the considered facet)}. \tag{3}$$

Theorem 2. (an n-Dimensional Version of the Pythagorean Theorem for Areas) *For the compact convex polyhedron Ω_n in (2), the square of the area of the hypotenuse facet is equal to the sum of the squares of the areas of the n facets issuing from the origin.*

For example, with $n = 4$, Theorem 2 gives a relation among volumes (in the usual sense of the word) of the four 3-dimensional faces: $V^2 = V_1^2 + V_2^2 + V_3^2 + V_4^2$.

Proof. To follow the proof, the reader is invited to take $n = 3$ and keep in mind Figure 1. According to (3), if S_i denotes the area of the facet issuing from the origin "opposed" to the vertex A_i,

$$V = \frac{1}{n} S_i \times \|\overrightarrow{OA_i}\| = \frac{1}{n} S_i a_i. \tag{4}$$

The height from the origin O to the hypotenuse facet is the distance from O to the affine hyperplane whose equation is $\sum_{i=1}^{n} x_i/a_i = 1$ (the one containing all the vertices $A_1, A_2, \dots, A_n$); it equals $(\sum_{i=1}^{n} a_i^{-2})^{-1/2}$. Hence, if S denotes the area of the hypotenuse facet of Ω_n, we infer from (3) that

$$V = \frac{1}{n} S \times \left(\sum_{i=1}^{n} \frac{1}{a_i^2} \right)^{-1/2}. \tag{5}$$

We infer from (4)

$$S_i^2 = n^2 V^2 \frac{1}{a_i^2} \qquad \text{for all } i = 1, \ldots, n$$

and from (5)

$$S^2 = n^2 V^2 \left(\sum_{i=1}^{n} \frac{1}{a_i^2} \right). \qquad \blacksquare$$

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