A note on the Legendre-Fenchel transform of convex composite functions.

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Dedicated to J.-J. MOREAU on the occasion of his 80th birthday.

Abstract. In this note we present a short and clear-cut proof of the formula giving the LEGENDRE-FENCHEL transform of a convex composite function $g \circ (f_1, \cdots, f_m)$ in terms of the transform of g and those of the $f_i's$.

Keywords. Convex functions, Composite functions, LEGENDRE-FENCHEL transform.

2000 Mathematics Subject Classification: 26B25, 52A41, 90C25.

1 Introduction.

The LEGENDRE-FENCHEL transform (or conjugate) of a function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is a function defined on the topological dual space $X^\star$ of X as

$$p \in X^\star \longmapsto \varphi^*(p) := \sup_{x \in X} \left[\langle p, x \rangle - \varphi(x) \right].$$

In Convex analysis, the transformation $\varphi \rightsquigarrow \varphi^*$ plays a role analogous to that of FOURIER's or LAPLACE's transform in other places in Analysis. In particular, one cannot avoid it in analysing a variational problem, more specifically the so-called dual version of it. That explains why the LEGENDRE-FENCHEL transform occupies a key place in any work on Convex analysis.

The purpose of the present note is to analyse the formula giving the LEGENDRE-FENCHEL transform of the convex composite function $g \circ (f_1, \cdots, f_m)$, with g and all the f_i convex, in terms of g^* and the ${f_i}^*$. Actually such a formula is not new, even if it does not appear explicitly in books but only in some specialized research papers. We intend here to derive such a result in a short and clear-cut way, using only a "pocket theorem" from Convex analysis; in particular, we shall not appeal to any result on convex mappings taking values in ordered vector spaces, as is usually done in the literature.

Before going further, some comments on the historical development of the convex analysis of $g \circ (f_1, \cdots, f_m)$ are in order. First of all, the setting: the ${f_i}'s$ are convex functions on some general vector space X and g is an increasing convex function on $\mathbb{R}^m$ (increasing means here that $g(y) \leq g(z)$ whenever $y_i \leq z_i$ for all i); the resulting composite function $g \circ (f_1, \cdots, f_m)$ is convex on X. Then, how things evolved:

- Expressing the subdifferential of $g \circ (f_1, \cdots, f_m)$ in terms of that of g and those of the ${f_i}'s$ was the first work carried out in the convex analysis of $g \circ (f_1, \cdots, f_m)$, as early as in the years $1965 - 1970$. The goal was made easier to achieve by the fact that one knew the formula aimed at (by extending to subdifferentials the so-called chain rule in Differential calculus). The objective of obtaining the subdifferential of the convex composite function $g \circ F$, with a vector-valued convex operator F, was pursued by several authors in various ways, see [1], [2], [8] and references therein for recent contributions.

- To our best knowledge, the first attempt to derive $[g \circ (f_1, \cdots, f_m)]^*$ in terms of g^* and the ${f_i}^{*\prime}s$ is due to KUTATELADZE in his note [4] and full-fledged paper [5]. The working context was that of convex operators taking values in ordered vector spaces, and this was also the case in most of the subsequent papers on the subject. Not only was the case of real-valued ${f_i}'s$ somehow hidden in the main results in these papers (Theorem 3.7.1 in [5], Proposition 4.11 (ii) in [1], Theorem 3.4 (ii) in [2], Theorem 2.8.10 in [8]), but more importantly, the theorems were derived after some heavy preparatory work : on subdifferential calculus rules for vector-valued mappings in [5], on perturbation functions in [2], on ε-subdifferentials in [8]. All these aspects are summarized in section 2.8 (especially bibliographical notes) of [8].

In the setting we are considering in the present paper, the expected formula for the LEGENDRE-FENCHEL conjugate of $g \circ (f_1, \cdots, f_m)$ is as follows:

for all $p \in X^\star$,

$$[g \circ (f_1, \cdots, f_m)]^*(p) = \min_{\alpha_i \geq 0} \left[g^*(\alpha_1, \cdots, \alpha_m) + \left(\sum_{i=1}^{m} \alpha_i \, f_i \right)^* (p) \right]. \quad (1)$$

This formula was proved in a simple way when $m = 1$ in [3, Chapter X, Section 2.5]; we shall follow here the same approch as there, using only a standard result in Convex analysis, the one giving the LEGENDRE-FENCHEL conjugate of a sum of convex functions. However the formula (1) is not always informative, take for example $g(y_1, \cdots, y_m) := \sum_{i=1}^{m} y_i$, a situation in which (1) does not say anything new; we therefore shall go a step further in the expression of $[g \circ (f_1, \cdots, f_m)]^*(p)$ by developing $\left(\sum_{i=1}^{m} \alpha_i \, f_i \right)^* (p)$; hence the final formula (7) below is obtained.

We end with some illustrations enhancing the versatility of the proved formula.

2 The Legendre-Fenchel transform of $g \circ (f_1, \cdots, f_m)$.

We begin by recalling some basic notations and results from Convex analysis.

Let X be a real Banach space; by $X^\star$ we denote the topological dual space of X, and $(p, x) \in X^\star \times X \longmapsto \langle p, x \rangle$ stands for the duality pairing. The LEGENDRE-FENCHEL transform (or conjugate) of a function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is defined on $X^\star$ as

$$p \in X^\star \longmapsto \varphi^*(p) := \sup_{x \in X} \left[\langle p, x \rangle - \varphi(x) \right]. \quad (2)$$

Clearly only those x in $dom\, \varphi := \{x \in X \mid \varphi(x) < +\infty\}$ are relevant in the calculation of the supremum in (2).

As a particular example of φ, consider the indicator function of a nonempty set C in X, that is

$$i_C(x) := 0 \text{ if } x \in C, \quad +\infty \text{ if not} ; \quad (3)$$

then i_C^* is the so-called support function of C, that is

$$i_C^* =: \sigma_C : p \in X^\star \longmapsto \sigma_C(p) = \sup_{x \in C} \langle p, x \rangle. \tag{4}$$

When $\alpha > 0$, there is no ambiguity in defining $\alpha\varphi$: the resulting conjugacy result is : $(\alpha\varphi)^*(p) = \alpha\varphi^*(\frac{p}{\alpha})$. As for $\alpha = 0$, one should be more careful : we set $(0\varphi)(x) = 0$ if $x \in dom\,\varphi$, $+\infty$ if not; in other words $0\varphi = i_{dom\varphi}$. The corresponding conjugacy result is $(0\varphi)^* = \sigma_{dom\,\varphi}$, a fact coherent with the following result : $\alpha\varphi^*(\frac{p}{\alpha}) \underset{\alpha \to 0^+}{\longrightarrow} \sigma_{dom\,\varphi}(p)$ (at least for convex lower-semicontinuous φ).

We denote by $\Gamma_0(X)$ the set of functions $\varphi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ which are convex, lower-semicontinuous and not identically equal to $+\infty$ on X. The next theorem is the key result we shall rely on in our proofs; it is a classical one in Convex analysis (see [6, Théorème 6.5.8] for example).

Theorem 1. *Let $f_1, \cdots, f_k \in \Gamma_0(X)$. Suppose there is a point in $\bigcap_{i=1}^{k} dom\, f_i$ at which $f_1, \cdots, f_{k-1}$ are continuous. Then, for all $p \in X^\star$:*

$$(f_1 + \cdots + f_k)^*(p) = \min_{p_1 + \cdots + p_k = p} \left[f_1^*(p_1) + \cdots + f_k^*(p_k) \right]. \tag{5}$$

The context of our work is the following one :

- $f_1, \cdots, f_m \in \Gamma_0(X)$;

- $g \in \Gamma_0(\mathbb{R}^m)$ and is increasing on $\mathbb{R}^m$, i.e. $g(y) \leq g(z)$ whenever $y_i \leq z_i$ for all $i = 1, \cdots, m$.

The composite function $g \circ (f_1, \cdots, f_m)$ is defined on X as follows :

$$[g \circ (f_1, \cdots, f_m)](x) = \begin{cases} g[f_1(x), \cdots, f_m(x)] & \text{if } f_i(x) < +\infty \text{ for all } i, \\ +\infty & \text{if not}. \end{cases}$$

The resulting function $g \circ (f_1, \cdots, f_m)$ is now convex on X. The minimal assumption to secure that $g \circ (f_1, \cdots, f_m)$ does not identically equal $+\infty$ is : there is a point $x_0 \in \bigcap_{i=1}^{m} dom\, f_i$ such that $(f_1(x_0), \cdots, f_m(x_0)) \in dom\, g$. We shall actually assume a little more to derive our main result below.

Theorem 2. *With the assumptions listed above on g and the $f_i's$, we suppose :*

$$(\mathcal{H}) \begin{cases} \text{There is a point } x_0 \in \bigcap_{i=1}^{m} dom\, f_i \text{ such that} \\ (f_1(x_0), \cdots, f_m(x_0)) \text{ lies in the interior of } dom\, g. \end{cases}$$

Then : for all $p \in X^\star$,

$$[g \circ (f_1, \cdots, f_m)]^*(p) = \min_{\alpha_1 \geq 0, \cdots, \alpha_m \geq 0} \left[g^*(\alpha_1, \cdots, \alpha_m) + \left(\sum_{i=1}^{m} \alpha_i\, f_i \right)^* (p) \right]. \tag{6}$$

(with $0\, f_i = i_{dom\, f_i}$).

If moreover there is a point in $\bigcap_{i=1}^{m} dom\, f_i$ at which $f_1, \cdots, f_{m-1}$ are continuous, then for all $p \in X^\star$,

$$[g \circ (f_1, \cdots, f_m)]^*(p) = \min_{\substack{\alpha_1 \geq 0, \cdots, \alpha_m \geq 0 \\ p_1 + \cdots + p_m = p}} \left[g^*(\alpha_1, \cdots, \alpha_m) + \sum_{i=1}^{m} \alpha_i\, f_i^*(\frac{p_i}{\alpha_i}) \right] \tag{7}$$

(where we interpret $0\, f_i^*(\frac{p_i}{0}) = \sigma_{dom\, f_i}(p_i)$).

Proof. By definition, given $p \in X^\star$,

$$-[g \circ (f_1, \cdots, f_m)]^*(p) = \inf_{x \in X} \{[g \circ (f_1, \cdots, f_m)](x) - \langle p, x \rangle\}$$

$$= \inf_{\substack{f_i(x) < +\infty \\ \text{for all } i}} \{g[f_1(x), \cdots, f_m(x)] - \langle p, x \rangle\}$$

$$= \inf_{(y_1, \cdots, y_m) \in \mathbb{R}^m} [g(y_1, \cdots, y_m) \mid f_i(x) \leq y_i \text{ for all } i] \tag{8}$$

because g is assumed increasing on $\mathbb{R}^m$.

Let now F_1 and F_2 be defined on $X \times \mathbb{R}^m$ as follows :

For $(x, y_1, \cdots, y_m) \in X \times \mathbb{R}^m$,

$$F_1(x, y_1, \cdots, y_m) := -\langle p, x \rangle + g(y_1, \cdots, y_m),$$

$$F_2(x, y_1, \cdots, y_m) := i_{epi\, f_1}(x, y_1) + \cdots + i_{epi\, f_m}(x, y_m),$$

where *epi* f_i denotes the epigraph of f_i, that is the set of $(x, y_i) \in X \times \mathbb{R}$ such that $f_i(x) \leq y_i$.

Thus, (8) can be written as

$$-[g \circ (f_1, \cdots, f_m)]^*(p) = \inf_{(x, y_1, \cdots, y_m) \in X \times \mathbb{R}^m} [F_1(x, y_1, \cdots, y_m) + F_2(x, y_1, \cdots, y_m)] . \tag{9}$$

We then have to compute the conjugate (at 0) of a sum of functions, however in a favorable context since :

$F_1 \in \Gamma_0(X \times \mathbb{R})$, $F_2 \in \Gamma_0(X \times \mathbb{R})$, $dom\, F_1 = X \times dom\, g$.

According to the assumption $(\mathcal{H})$ we made, there is a point $(x_0, f_1(x_0), \cdots, f_m(x_0)) \in dom\, F_2$ at which F_1 is continuous (indeed g is continuous at $(f_1(x_0), \cdots, f_m(x_0))$ whenever $(f_1(x_0), \cdots, f_m(x_0))$ lies in the interior of $dom\, g$). We therefore are in a situation where Theorem 1 applies :

$$[g \circ (f_1, \cdots, f_m)]^*(p) = (F_1 + F_2)^*(0) \quad [\text{from } (9)]$$

$$= \min_{\substack{s \in X^\star \\ (\alpha_1, \cdots, \alpha_m) \in \mathbb{R}^m}} [F_1^*(-s, \alpha_1, \cdots, \alpha_m) + F_2^*(s, -\alpha_1, \cdots, -\alpha_m)] . \tag{10}$$

The computation of the above two conjugate functions (F_1^* and F_2^*) is easy and gives :

$$F_1^*(-s, \alpha_1, \cdots, \alpha_m) = \begin{cases} g^*(\alpha_1, \cdots, \alpha_m) \text{ if } s = p, \\ +\infty \text{ if not ;} \end{cases}$$

$$F_2^*(s, \beta_1, \cdots, \beta_m) = \sup_{\substack{f_i(x) \leq y_i \\ \text{for all } i}} (\langle s, x \rangle + \sum_{i=1}^m \beta_i y_i)$$

$$= +\infty \text{ if at least one } \beta_i \text{ is } > 0,$$

$$= \sup_{\substack{x \in dom\, f_i \\ \text{for all } i}} [\langle s, x\rangle + \sum_{i=1}^{m} \beta_i\, f_i(x)] \text{ if } \beta_i \leq 0 \text{ for all } i.$$

Consequently, the minimum in (10) is taken over $s = p$ and those $(\alpha_1, \cdots, \alpha_m) \in \mathbb{R}^m$ whose components are all nonnegative. Whence the formula (6) follows.

If there is a point $\tilde{x} \in \bigcap_{i=1}^{m} dom\, f_i$ at which $f_1, \cdots, f_{m-1}$ are continuous, then $\tilde{x} \in \bigcap_{i=1}^{m} dom(\alpha_i\, f_i)$ and $\alpha_1\, f_1, \cdots, \alpha_{m-1}\, f_{m-1}$ are continuous at $\tilde{x}$ (if $\alpha_{i_0} = 0$, $\alpha_{i_0}\, f_{i_0} = i_{dom\, f_{i_0}}$ is indeed continuous at $\tilde{x}$, since $\tilde{x}$ necessarily lies in the interior of $dom\, f_{i_0}$). We apply Theorem 1 again :

$$(\sum_{i=1}^{m} \alpha_i\, f_i)^*(p) = \min_{p_1+\cdots+p_m=p} \left[\sum_{i=1}^{m} \alpha_i\, f_i{}^*(\frac{p}{\alpha_i})\right].$$

Plugging this into (6) yields the formula (7). ■

3 By way of illustrations.

Since g is assumed increasing in our setting, it is easy to prove that $dom\, g^*$ is indeed included in $(\mathbb{R}_+)^m$, thus the restriction in (6) or (7) to those α_i which are positive is not surprising. However one may wonder if the optimal $\alpha_i's$ in these formulas are strictly positive or not. The answer is no, some of the $\alpha_i's$ may be null, possibly all of them. As a general rule, one can say more about the optimal $\alpha_i's$ only in particular instances.

Consider for example $p \in X^\star$ such that $[g \circ (f_1, \cdots, f_m)]^*(p) < +\infty$; then, if $\sigma_{\bigcap_{i=1}^{m} dom\, f_i}(p) = +\infty$ or if $g^*(0, \cdots, 0) = +\infty$, one is sure that some of the optimal $\alpha_i's$ in (6) or (7) are strictly positive (in either case, having all the α_i null leads to impossible equalities (6) and (7)). In some other situations however, one is sure that all the optimal $\alpha_i's$ are strictly positive; see some examples below.

3.1 The sum of the k largest values.

Suppose that the increasing $g \in \Gamma_0(\mathbb{R}^m)$ is positively homogeneous; it is then the support function of a closed convex set $C \subset (\mathbb{R}_+)^m$ (namely the subdifferential of g at the origin). The LEGENDRE-FENCHEL transform g^* of g is the indicator function of C so that (6) and (7) are simplified into :

$$[g \circ (f_1, \cdots, f_m)]^*(p) = \min_{\alpha_1,\cdots,\alpha_m \in C} \left[(\sum_{i=1}^{m} \alpha_i\, f_i)^*(p) \right] ; \tag{11}$$

$$= \min_{\substack{\alpha_1, \cdots, \alpha_m \in C \\ p_1 + \cdots + p_m = p}} \left[\sum_{i=1}^{m} \alpha_i\, f_i^*(\frac{p_i}{\alpha_i}) \right] . \tag{12}$$

A first example, with a bounded C, is $g(y_1, \cdots, y_m) = \sum_{i=1}^{m} y_i^+$ (where y_i^+ stands for the positive part of y_i). Here g is the support function of $C = [0,1]^m$ and (11)-(12) give an expression of $\left(\sum_{i=1}^{m} f_i^+ \right)^* (p)$.

A more interesting example of g as the support function of C, still with a bounded C, is the following one : for an integer $k \in \{1, \cdots, m\}$, let

$$g_k(y_1, \cdots, y_m) := \text{ the sum of the } k \text{ largest values among the } \alpha_i's.$$

It is not difficult to realize that g_k is the support function of the compact convex polyhedron

$$C_k := \{(\alpha_1, \cdots, \alpha_m) \in [0,1]^m \mid \alpha_1 + \cdots + \alpha_m = k\} . \tag{13}$$

Since $dom\, g_k = \mathbb{R}^m$, the corollary below readily follows from Theorem 2.

Corollary 3. *Let $f_1, \cdots, f_m \in \Gamma_0(X)$, let φ_k be defined on X as*

$$\varphi_k(x) := \text{ the sum of the } k \text{ largest values among the } f_i(x)'s.$$

We suppose there is a point in $\bigcap_{i=1}^{m} dom\, f_i$ at which $f_1, \cdots, f_{m-1}$ are continuous. Then : for all $p \in X^\star$

$$\varphi_k^*(p) = \min_{\substack{\alpha_1, \cdots, \alpha_m \in C_k \\ p_1 + \cdots + p_m = p}} \left[\sum_{i=1}^{m} \alpha_i\, f_i^*(\frac{p_i}{\alpha_i}) \right] . \tag{14}$$

The two "extreme" cases, $k=1$ or $k=m$, are interesting to consider.

If $k=1$, $\varphi_1(x) = \max\limits_{i=1,\cdots,m} f_i(x)$, C_1 is the so-called unit-simplex in $\mathbb{R}^m$, and (14) reduces to a well-known formula on $(\max\limits_i f_i)^*(p)$:

$$(\max_i f_i)^*(p) = \min_{\substack{\alpha_1 \geq 0, \cdots, \alpha_m \geq 0 \\ \alpha_1 + \cdots + \alpha_m = 1 \\ p_1 + \cdots + p_m = p}} \left[\sum_{i=1}^{m} \alpha_i \, f_i^*(\frac{p_i}{\alpha_i}) \right]. \tag{15}$$

If $k=m$, $\varphi_m(x) = \sum_{i=1}^{m} f_i(x)$, C_m is the singleton $\{(1,\cdots,1)\}$, and Corollary 3 takes us back to Theorem 1.

3.2 Smoothing the max function.

A way of smoothing the nondifferentiable function $\max\limits_i(y_1,\cdots,y_m)$ is via the so-called log-exponential function. Given $\varepsilon > 0$, let $\Theta_\varepsilon : \mathbb{R}^m \longrightarrow \mathbb{R}$ be defined as following:

$$(y_1,\cdots,y_m) \in \mathbb{R}^m \longmapsto \Theta_\varepsilon(y_1,\cdots,y_m) := \varepsilon \, log \, (e^{\frac{y_1}{\varepsilon}} + \cdots + e^{\frac{y_m}{\varepsilon}}).$$

Such a function is studied in full detail in [7] for example : Θ_ε is a (finite-valued) increasing convex function on $\mathbb{R}^m$, whose LEGENDRE-FENCHEL transform is given below ([7, p. 482]):

$$\Theta_\varepsilon^*(\alpha_1,\cdots,\alpha_m) = \begin{cases} \varepsilon \displaystyle\sum_{i=1}^{m} \alpha_i \, log \, \alpha_i & \text{if } \alpha_1 \geq 0, \cdots, \alpha_m \geq 0 \text{ and } \alpha_1 + \cdots + \alpha_m = 1, \\ +\infty & \text{if not} \end{cases} \tag{16}$$

(the entropy function multiplied by ε).

When $f_1,\cdots,f_m \in \Gamma_0(X)$, the nonsmooth function $\max\limits_{i=1,\cdots,m} f_i$ can be approximated by the function $\Theta_\varepsilon \circ (f_1,\cdots,f_m)$, smooth whenever all the $f_i's$ are smooth.

Here again $dom\,\Theta_\varepsilon = \mathbb{R}^m$, so that the next corollary is an immediate application of Theorem 2.

Corollary 4. *Let $f_1, \cdots, f_m \in \Gamma_0(X)$; we assume there is a point in $\bigcap_{i=1}^{m} dom\, f_i$ at which $f_1, \cdots, f_{m-1}$ are continuous.*

Then : for all $p \in X^\star$,

$$[\Theta_\varepsilon \circ (f_1, \cdots, f_m)]^*(p) = \min_{\substack{\alpha_1 \geq 0, \cdots, \alpha_m \geq 0 \\ \alpha_1 + \cdots + \alpha_m = 1 \\ p_1 + \cdots + p_m = p}} \left[\varepsilon \sum_{i=1}^{m} \alpha_i \log \alpha_i + \sum_{i=1}^{m} \alpha_i\, f_i^*(\frac{p_i}{\alpha_i}) \right]. \tag{17}$$

Compare (17) with (15): since the entropy function $\varepsilon \sum_{i=1}^{m} \alpha_i \log \alpha_i$ is negative and bounded from below by $-\varepsilon \log m$ (achieved at $\alpha_1 = \cdots = \alpha_m = \frac{1}{m}$), we have :

$$(\max_i f_i)^* - \varepsilon \log m \leq [\Theta_\varepsilon \circ (f_1, \cdots, f_m)]^* \leq (\max_i f_i)^*. \tag{18}$$

3.3 Optimality conditions in Convex minimization.

Let K be a closed convex cone in $\mathbb{R}^m$, let $f_1, \cdots, f_m$ be convex functions on X and

$$S := \{x \in X \mid (f_1(x), \cdots, f_m(x)) \in K\}. \tag{19}$$

As an example, suppose $K = (\mathbb{R}_-)^m$: S is then a constraint set represented by inequalities in Convex minimization.

The indicator function i_S of S is nothing else than $i_K \circ (f_1, \cdots, f_m)$. We are again in the context considered in section 3.1 with $g = i_K$ the support function of the polar cone K° of K. We can thus express the support function of S, that is the LEGENDRE-FENCHEL transform of the composite function $i_K \circ (f_1, \cdots, f_m)$ in terms of the $f_i^{*\prime}s$.

Corollary 5. *We assume the following on the* $f_i's$ *and* K *:*

• *All the* $f_i : X \longrightarrow \mathbb{R}$ *are convex and continuous on* X *(as it is usually the case in applications);*

• *There is a point* $x_0 \in X$ *such that* $(f_1(x_0), \cdots, f_m(x_0))$ *lies in the interior of* K *(this is SLATER's constraint qualification condition).*

• K *is a closed convex cone of* $\mathbb{R}^m$ *containing* $(\mathbb{R}_-)^m$*.*

Then : for all $0 \neq p \in X^\star$ *such that* $\sigma_S(p) < +\infty$*,*

$$\sigma_S(p) = \min_{\substack{0 \neq (\alpha_1, \cdots, \alpha_m) \in K^\circ \\ p_1 + \cdots + p_m = p}} \left[\sum_{i=1}^{m} \alpha_i \, f_i^*(\frac{p_i}{\alpha_i}) \right]. \tag{20}$$

Proof. The function $i_K \in \Gamma_0(X)$ is increasing because of the assumption $(\mathbb{R}_-)^m \subset K$. We have assumed there is a point $x_0 \in X$ such that $(f_1(x_0), \cdots, f_m(x_0))$ lies in the interior of the domain of i_K. Thus, by applying Theorem 2, we obtain :

$$\sigma_S(p) = [i_K \circ (f_1, \cdots, f_m)]^*(p) = \min_{\substack{(\alpha_1, \cdots, \alpha_m) \in K^\circ \\ p_1 + \cdots + p_m = p}} \left[\sum_{i=1}^{m} \alpha_i \, f_i^*(\frac{p_i}{\alpha_i}) \right].$$

Note that $K^\circ \subset (\mathbb{R}_+)^m$ but, since $\sigma_{dom\, f_i}(p_i) = +\infty$ for $p_i \neq 0$, all the optimal α_i are not null simultaneously. ■

To pursue our illustration further, consider the following minimization problem :

$$(\mathcal{P}) \quad \text{Minimize } f_0(x) \text{ over } S,$$

where S is described as in (19).

We suppose:

- $f_0 \in \Gamma_0(X)$ is continuous at some point of S;
- $v_{opt} := \inf_S f_0 > -\infty$ (f_0 is bounded from below on S);
- $v_{opt} > \inf_X f_0$ (($\mathcal{P}$) is genuinely a constrained problem);
- Assumptions on the $f_i's$ and K made in Corollary 5.

Corollary 6. *Under the assumptions listed above, we have :*

$$-v_{opt} = \min_{\substack{0 \neq (\alpha_1, \cdots, \alpha_m) \in K^\circ \\ p_1 + \cdots + p_m = p \neq 0}} \left[f_0^*(-p) + \sum_{i=1}^{m} \alpha_i \, f_i^*(\frac{p_i}{\alpha_i}) \right]. \tag{21}$$

This result is an alternate formulation, in the dual form (in the spirit of [6, Chapter VII]), of the existence of LAGRANGE-KARUSH-KUHN-TUCKER multipliers in $(\mathcal{P})$: there exist positive $\alpha_1, \cdots, \alpha_m$ such that

$$v_{opt} = \min_X \left[f_0(x) + \sum_{i=1}^{m} \alpha_i \, f_i(x) \right].$$

Proof of Corollary 6. By definition,

$$v_{opt} = -(f_0 + i_S)^*(0).$$

Applying Theorem 1, we transform the above into :

$$-v_{opt} = \min_{p \in X^\star} \left[f_0^*(-p) + \sigma_S(p) \right]. \tag{22}$$

The optimal p cannot be null, as otherwise we would have $-v_{opt} = f_0^*(0) = -\inf_X f_0$, which is excluded by assumption.

It then remains to apply the result of Corollary 5 to develop $\sigma_S(p)$ in (22). ■

Acknowledgment. We would like to thank Professors S. ROBINSON (University of Wisconsin at Madison) and L. THIBAULT (Université des sciences et techniques du Languedoc à Montpellier) for their remarks on the first version of this paper.

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