A REMARK ON HOMOMORPHISMS FROM RIGHT-ANGLED ARTIN GROUPS TO MAPPING CLASS GROUPS

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Abstract. We study rigidity properties of certain homomorphisms from right-angled Artin groups to mapping class groups. As an application we show that if $\Gamma \subset \mathrm{Map}(S)$ is a subgroup that contains some power of every Dehn twist, then any injective homomorphism $\Gamma \to \mathrm{Map}(S)$ is a restriction of an automorphism of $\mathrm{Map}(S)$.

Résumé. Nous examinons la rigidité de certains homomorphismes entre groupes d'Artin rectangulaires et groupes modulaires. Nous démontrons que si $\Gamma \subset \mathrm{Map}(S)$ est un sous-groupe qui contient quelque puissance de tout twist de Dehn, alors tout homomorphisme injective $\Gamma \to \mathrm{Map}(S)$ est la restriction d'un automorphisme de $\mathrm{Map}(S)$.

Version française abrégée. Soit S une surface connexe et orientable, de genre g et avec n pointes. Nous supposerons que S est *non-exceptionelle*, c'est-à-dire que $3g + n \geq 5$ et $(g, n) \neq (1, 2)$. Le *groupe modulaire étendu* $\mathrm{Map}^*(S)$ est le groupe de difféomorphismes de S à isotopie près. Le *groupe modulaire* $\mathrm{Map}(S)$ est le sous-groupe d'indice 2 en $\mathrm{Map}^*(S)$ dont les éléments sont représentés par les difféomorphismes préservant l'orientation de S. Finalement, le *groupe modulaire pur* $P\,\mathrm{Map}(S) \subset \mathrm{Map}(S)$ est le sous-groupe des éléments de $\mathrm{Map}(S)$ qui fixent chaque pointe de S.

Le groupe d'Artin rectangulaire $\mathbb{A}(X)$ associé à un complexe simplicial X est le groupe engendré par l'ensemble $X^{(0)}$ des sommets de X, et tel que les éléments correspondant à deux sommets voisins commutent. On remarque que si $\Delta \subset X$ est un simplex, alors $\mathbb{A}(\Delta)$ est un sous-groupe de $\mathbb{A}(X)$ isomorphe à $\mathbb{Z}^{\dim(\Delta)+1}$.

Dans cet article on s'intéresse aux homomorphismes *faiblement injectifs* $\rho : \mathbb{A}(X) \to \mathrm{Map}(S)$, où X est un sous-complexe *rigide* [1] du complexe des courbes $\mathcal{C}(S)$ de S [14]. Ici, nous disons que $X \subset \mathcal{C}(S)$ est rigide si toute application injective et simpliciale $\omega : X \to \mathcal{C}(S)$ est la restriction d'un automorphisme de $\mathcal{C}(S)$. Un homomorphisme ρ est faiblement injectif si pour tous les simplexes $\Delta, \Delta' \subset X$ et pour tous les éléments $\gamma \in \mathbb{A}(\Delta), \gamma' \in \mathbb{A}(\Delta')$, si $\rho(\gamma) = \rho(\gamma')$ alors $\gamma = \gamma'$. Dénotant par δ_γ le twist de Dehn le long de $\gamma \in \mathcal{C}(S)$, on montre:

Théorème 1. *Soit S une surface connexe, orientable et non-exceptionnelle. Soit aussi $X \subset \mathcal{C}(S)$ un sous-complexe rigide avec $\dim(X) = \dim(\mathcal{C})$, et tel que chaque simplexe de X est l'intersection des simplexes de dimension maximale de X qui le contiennent. Pour tout homomorphisme faiblement injectif $\rho : \mathbb{A}(X) \to \mathrm{Map}(S)$ il existe $f \in \mathrm{Map}^*(S)$ et fonctions $a, b : X^{(0)} \to \mathbb{Z} \setminus \{0\}$ telles que $\rho(\gamma^{a(\gamma)}) = f \delta_\gamma^{b(\gamma)} f^{-1}$ pout tout $\gamma \in X^{(0)}$. De plus, f est unique si S n'est pas la surface fermeé de genre 2.*

Nous remarquons qu'il y a des constructions d'ensembles rigides *finis* de $\mathcal{C}(S)$ [1] qui satisfont les hypothèses du Théorème 1. Par exemple, soit $\mathbb{X}_n$ le complexe simplicial dont les simplexes de dimension k correspondent aux ensembles de $k + 1$ diagonales disjointes du polygone avec n sommets. Pour $n \geq 5$, $\mathbb{X}_n$ est un sous-complexe rigide du complexe de courbes de la sphère $S_{0,n}$ avec n pointes [1]. Donc, si $\rho_0 : \mathbb{A}(\mathbb{X}_n) \to P\,\mathrm{Map}(S_{0,n})$ est l'homomorphisme faiblement injectif donné par $\rho_0(\gamma) = \delta_\gamma$ pour tout γ, alors tout homomorphisme injectif $\rho : \mathbb{A}(\mathbb{X}_n) \to P\,\mathrm{Map}(S_{0,n})$ a la forme $\rho(\cdot) = f\,((\rho_{\mathbb{X}_n} \circ \tau)(\cdot))\,f^{-1}$, où $f \in \mathrm{Map}^*(S_{0,n})$ et $\tau : \mathbb{A}(\mathbb{X}_n) \to \mathbb{A}(\mathbb{X}_n)$ est un monomorphisme.

The second author has been partially supported by NSERC Discovery and Accelerator Supplement grants.

Par ailleurs, le complexe des courbes lui même est rigide [22]. En lui appliquant le Théorème 1, on montre:

Corollaire 2. *Soit S une surface connexe, orientable et non exceptionnelle, autre que la surface fermée de genre 2. Soit aussi $\Gamma \subset \mathrm{Map}(S)$ un sous-groupe tel que pour toute $\gamma \in \mathcal{C}(S)$ il y a $n(\gamma) \in \mathbb{N}$ avec $\delta_\gamma^{n(\gamma)} \in \Gamma$. Pour tout homomorphisme injectif $\sigma : \Gamma \to \mathrm{Map}(S)$ il existe un unique élément $f \in \mathrm{Map}^*(S)$ tel que $\sigma(g) = fgf^{-1}$ pour tout $g \in \Gamma$.*

Rappelons que si S a genre au moins 3, les noyaux des homomorphismes du groupe modulaire dans des groupes de Lie compactes - par exemple les représentations quantiques [21] - satisfont la condition du Corollaire 2 [3].

On observe aussi que le Corollaire 2 implique que le groupe Γ est cohopfien et que son commensurateur abstrait est isomorphe à $\mathrm{Map}^*(S)$. Ces résultats sont déjà connus pour le group de Torelli [11] et pour quelques autres sous-groupes normaux de $\mathrm{Map}(S)$ d'indice infini [8, 9]. Toutefois, la rigidité qu'on trouve dans le Corollaire 2 est bien plus forte que ce qu'on connaît dans ces cas là: ici on ne suppose pas que $\sigma(\Gamma) \subset \Gamma$, ni que Γ et $\sigma(\Gamma)$ soient commensurables.

1. Introduction

Let S be a connected, orientable surface of genus g with n punctures. Throughout, we will assume that S is *non-exceptional*, that is, $3g + n \geq 5$ and $(g, n) \neq (1, 2)$. Denote by $\mathrm{Map}^*(S)$ the *extended mapping class group*, that is the group of isotopy classes of self-diffeomorphisms of S. The *mapping class group* $\mathrm{Map}(S) \subset \mathrm{Map}^*(S)$ the index 2 subgroup consisting of isotopy classes of those diffeomorphisms that preserve the orientation of S; finally, the *pure mapping class group* $P\,\mathrm{Map}(S) \subset \mathrm{Map}(S)$ is the subgroup of those mapping classes fixing each puncture of S.

Given a simplicial complex X, the *right-angled Artin group* $\mathbb{A}(X)$ associated to X is the group generated by the set $X^{(0)}$ of vertices of X, subject to the relation that $\gamma_i, \gamma_j \in X^{(0)}$ commute if and only if they are adjacent in X. Note that every simplex Δ of X determines an abelian subgroup $\mathbb{A}(\Delta)$ of $\mathbb{A}(X)$, isomorphic to $\mathbb{Z}^{\dim(\Delta)+1}$.

In this note we are interested in homomorphisms from right-angled Artin groups to mapping class groups. We remark that there are numerous examples of such homomorphisms: for instance, if $X \neq \emptyset$ then $\mathbb{A}(X)$ surjects onto $\mathbb{Z}$, and hence we obtain infinitely many homomorphisms $\mathbb{A}(X) \to \mathrm{Map}(S)$. In addition, so long as X has at least two non-adjacent vertices, $\mathbb{A}(X)$ surjects onto the non-abelian free group of rank 2 and thus we obtain still more homomorphisms $\mathbb{A}(X) \to \mathrm{Map}(S)$. Observe, however, that the homomorphisms just described fail to be injective. On the other hand, Koberda [18] and Clay-Leininger-Mangahas [10] showed that *every* finitely generated right-angled Artin group embeds as a subgroup of some mapping class group.

Below, we will prove a rigidity result for a certain class of homomorphisms $\mathbb{A}(X) \to \mathrm{Map}(S)$, called *weakly injective*, in the case when X is a *rigid* subset of the curve complex $\mathcal{C}(S)$. We need a couple of definitions before stating our main result:

Definition (Weak injectivity)**.** *Let X be a simplicial complex, and G a group. A homomorphism $\rho : \mathbb{A}(X) \to G$ is* weakly injective *if the following holds: for all simplices $\Delta, \Delta' \subset X$, and for all $\gamma \in \mathbb{A}(\Delta), \gamma' \in \mathbb{A}(\Delta')$, if $\rho(\gamma) = \rho(\gamma')$ then $\gamma = \gamma'$.*

Recall that the curve complex $\mathcal{C}(S)$ is the simplicial complex whose k-simplices correspond to sets of $k+1$ distinct free isotopy classes of essential simple closed curves on S with pairwise disjoint representatives [14]. Denote by δ_γ the right Dehn twist along the simple closed curve $\gamma \in \mathcal{C}(S)$. If $X \subset \mathcal{C}(S)$ is an arbitrary subcomplex, then the homomorphism

$$\rho_0 : \mathbb{A}(X) \to \mathrm{Map}(S), \quad \rho_0(\gamma) = \delta_\gamma \text{ for every vertex } \gamma \in X^{(0)} \tag{1}$$

is weakly injective; see [12], in particular Section 3.3, for basic facts about Dehn twists. Note, however, that the map ρ_0 is not injective in general; compare with [13]. As mentioned earlier, we will be interested in subcomplexes of $\mathcal{C}(S)$ that are *rigid*:

Definition (Rigid subcomplex)**.** *A simplicial subcomplex X of $\mathcal{C}(S)$ is* rigid *if for every injective simplicial map $\omega : X \to \mathcal{C}(S)$ there is an automorphism $\phi \in \mathrm{Aut}(\mathcal{C}(S))$ of $\mathcal{C}(S)$ with $\omega(\gamma) = \phi(\gamma)$ for all $\gamma \in X^{(0)}$.*

Since S is assumed to be non-exceptional, the combination of results of Ivanov [17], Korkmaz [19], and Luo [20], implies that every automorphism of $\mathcal{C}(S)$ is induced by an element of $\mathrm{Map}^*(S)$. Furthermore, if S is not the closed surface of genus 2, then the said element is unique – see [20].

We are finally ready to state our main result:

Theorem 1. *Let S be a connected, orientable and non-exceptional surface. Suppose that X is a rigid subcomplex of $\mathcal{C}(S)$, with $\dim(X) = \dim(\mathcal{C}(S))$, and such that every simplex of X is equal to the intersection of all maximal dimensional simplices of X that contain it. For every weakly injective homomorphism $\rho : \mathbb{A}(X) \to \mathrm{Map}(S)$ there are $f \in \mathrm{Map}^*(S)$ and functions $a, b : X^{(0)} \to \mathbb{Z} \setminus \{0\}$ with $\rho(\gamma^{a(\gamma)}) = f\delta_\gamma^{b(\gamma)} f^{-1}$, for every $\gamma \in X^{(0)}$. Moreover, f is unique unless S is a closed surface of genus 2.*

The equality $\rho(\gamma^{a(\gamma)}) = f\delta_\gamma^{b(\gamma)} f^{-1}$ asserts that $\rho(\gamma)$ is a root of a power of the Dehn twist along $f(\gamma)$. In the absence of roots – for instance if ρ takes values is the pure mapping class group $P\,\mathrm{Map}(S_{0,n})$ of the n-punctured sphere – we deduce that $\rho(\gamma)$ is in fact a power of a Dehn twist; with the notation of Theorem 1 this means that $a(\gamma) = 1$.

Concrete examples of finite rigid subsets of $\mathcal{C}(S_{0,n})$ were given in [1]. Indeed, the simplicial complex $\mathbb{X}_n$ whose k-simplices correspond to sets of $k+1$ pairwise disjoint diagonals of the polygon with n vertices is a rigid subcomplex of $\mathcal{C}(S_{0,n})$ for $n \geq 5$. The complex $\mathbb{X}_n$ is the dual polytope to the associahedron, and hence every simplex is equal to the intersection of all maximal dimensional simplices containing it. We thus deduce from Theorem 1 that, for $n \geq 5$, every weakly injective homomorphism $\rho : \mathbb{A}(\mathbb{X}_n) \to P\,\mathrm{Map}(S_{0,n})$ is of the form $\rho(\cdot) = f\left((\rho_0 \circ \tau)(\cdot)\right) f^{-1}$, where $f \in \mathrm{Map}^*(S_{0,n})$, ρ_0 is as in (1), and $\tau : \mathbb{A}(\mathbb{X}_n) \to \mathbb{A}(\mathbb{X}_n)$ is the injective homomorphism determined by $\tau(\gamma) = \gamma^{b(\gamma)}$ for $\gamma \in \mathbb{X}_n^{(0)}$.

We stress that in Theorem 1 we are *not* assuming that the subcomplex X be finite. In particular, applying the theorem to $X = \mathcal{C}(S)$, which is itself rigid by the work of Shackleton [22], we prove:

Corollary 2. *Let S be a connected, orientable and non-exceptional surface, other than the closed surface of genus 2. Let $\Gamma \subset \mathrm{Map}(S)$ be a subgroup such that for every $\gamma \in \mathcal{C}(S)$ there is $n(\gamma) \in \mathbb{N}$ with $\delta_\gamma^{n(\gamma)} \in \Gamma$. For every injective homomorphism $\sigma : \Gamma \to \mathrm{Map}(S)$ there is a unique $f \in \mathrm{Map}^*(S)$ such that $\sigma(g) = fgf^{-1}$, for all $g \in \Gamma$.*

Since any finite index subgroup $\Gamma \subset \mathrm{Map}(S)$ automatically satisfies the hypothesis above, Corollary 2 implies the results in [6, 7, 15, 16, 22] about injections of finite index subgroups of mapping class groups.

In addition, there are numerous subgroups $\Gamma \subset \mathrm{Map}(S)$ of infinite index that satisfy the condition of Corollary 2, for example the kernel of any representation of $\mathrm{Map}(S)$ to a compact Lie group, provided that S has genus at least 3 – see Corollary 2.6 of [3]. This applies for instance to the so-called *quantum representations* [21] of $\mathrm{Map}(S)$, many of which have infinite image. Note also that Corollary 2 implies that the subgroup Γ is co-Hopfian, and that its abstract commensurator is isomorphic to $\mathrm{Map}^*(S)$. Such results were already known for the Torelli group [11], as well as for other infinite index normal subgroups of $\mathrm{Map}(S)$ [8, 9]. However, we remark that the rigidity statement in Corollary 2 is more powerful than any of the existing ones, since we do not assume that $\sigma(\Gamma) \subset \Gamma$, or that Γ and $\sigma(\Gamma)$ are commensurable.

2. Abelian subgroups of the mapping class group.

We recall a few standard facts about abelian subgroups of the mapping class group. See [12] for basic facts on the mapping class group and [5] for details on its abelian subgroups.

Let S be a connected orientable surface, and A an abelian subgroup of $\mathrm{Map}(S)$. By the *rank* of A we understand the dimension of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-vector space. A *reducing system* for A is an A-invariant multicurve $\lambda \subset S$. If there is no reducing system for A, then A contains a pseudo-Anosov and hence $\mathrm{rank}(A) = 1$. Thus, every abelian subgroup A of $\mathrm{Map}(S)$ with $\mathrm{rank}(A) \geq 2$ is reducible. Given any reducing system λ for A we have the exact sequence:

$$1 \to A \cap \mathbb{T}_\lambda \to A \to \mathrm{Map}(S \setminus \lambda), \tag{2}$$

where $\mathbb{T}_\lambda$ is the group generated by the Dehn twists along the components of λ (or half-twists in the case when the given component bounds a twice-punctured disk or a once-punctured torus). We say that λ is a *complete reducing system* for A if, for every component W of $S \setminus \lambda$, either:

(a) there are $d > 0$ and f in the image of the third homomorphism in (2), such that $f^d(W) = W$ and $f^d|_W$ is pseudo-Anosov, or
(b) there is $d > 0$ with $f^d|_W = \mathrm{Id}$ for every f in the image of the third homomorphism in (2).

There is a unique complete reducing system $\lambda(A)$ for A, the *canonical reducing system* [5], contained in every other complete reducing system for A. The *active surface* $S(A)$ of A is the union of those components of $S \setminus \lambda(A)$ for which (a) above is satisfied. Noting that no component of $S(A)$ is homeomorphic to a three-times punctured sphere, the pigeonhole principle and (2) together imply:

Fact 1. Suppose that S has genus g and n punctures. Every abelian subgroup A of $\mathrm{Map}(S)$ satisfies $\mathrm{rank}(A) \leq 3g + n - 3 = \dim(\mathcal{C}(S)) + 1$. Moreover, in the equality case one has:

(1) $3g + n - 3 = l(A) + s(A)$, where $l(A)$ and $s(A)$ are, respectively, the number of components of $\lambda(A)$ and $S(A)$;
(2) Every component of $S(A)$ is homeomorphic to either a once-punctured torus or a four-times punctured sphere;
(3) The group A does not permute the components of $\lambda(A)$ (resp. $S(A)$).

Suppose now that $f \in \mathrm{Map}(S)$ has infinite order and is contained in some abelian subgroup $A < \mathrm{Map}(S)$ of maximal rank. In particular, f is not pseudo-Anosov and thus $\lambda(\langle f \rangle) \neq \emptyset$. Moreover, observe that $\lambda(\langle f \rangle) \subset \lambda(A)$, and that $S(\langle f \rangle)$ is a union of connected components of $S(A)$. Moreover, both $\lambda(\langle f \rangle)$ and $S(\langle f \rangle)$ are preserved by the centralizer $\mathcal{Z}_{\mathrm{Map}(S)}(f)$ of f. In fact, the subgroup of $\mathcal{Z}_{\mathrm{Map}(S)}(f)$ that preserves each component of $\lambda(f)$ and each component of $S \setminus \lambda(f)$ has finite index in $\mathcal{Z}_{\mathrm{Map}(S)}(f)$ and contains $\langle f, \mathbb{T}_{\lambda(f)} \rangle$ in its center. Notice that $\mathrm{rank}(\langle f, \mathbb{T}_{\lambda(f)} \rangle) \geq 2$ unless $S(\langle f \rangle) = \emptyset$ and $\lambda(f)$ has a single component. Altogether we have:

Fact 2. Let S be a connected, orientable and non-exceptional surface. Suppose that $f \in \mathrm{Map}(S)$ has infinite order and is contained in an abelian group of maximal rank. Then, either f is a root of a power of a Dehn twist, or the centralizer of f in $\mathrm{Map}(S)$ has a finite index subgroup G whose center $\mathcal{Z}(G)$ satisfies $\mathrm{rank}(\mathcal{Z}(G)) \geq 2$.

We can now prove:

Lemma 3. *Let S be a connected, orientable and non-exceptional surface. If $\{A_i\}_{i \in I}$ is a collection of maximal rank abelian subgroups of* $\mathrm{Map}(S)$ *such that* $\mathrm{rank}(\cap A_i) = 1$*, then $\cap A_i$ is a cyclic group generated by a root of a power of a Dehn twist.*

Proof. First, it follows from Fact 1 and the work of Birman-Hilden [4] (see Theorem 2.8 of [2] for an explicit statement) that the centralizer of a non-trivial finite order element of $\mathrm{Map}(S)$ does not contain abelian groups of maximal rank. Therefore $\cap A_i$ is torsion free, and hence cyclic.

Let G be a finite index subgroup of the centralizer of $\cap A_i$. Maximality of the rank of A_i implies that $\mathcal{Z}(G) \cap A_i$ has finite index in $\mathcal{Z}(G)$ for all i. In particular, also $\mathcal{Z}(G) \cap (\cap A_i)$ has finite index in $\mathcal{Z}(G)$; this proves that $\mathcal{Z}(G)$ has rank 1. By Fact 2, $\cap A_i$ is generated by a root of a power of a Dehn twist, as claimed. □

A remark on roots. We remark that if f is a half-twist along a curve that bounds a twice-punctured disk or a once-punctured torus in S, then f is indeed contained in a maximal rank abelian subgroup of $\mathrm{Map}(S)$. In fact, it is not difficult, albeit not so interesting and slightly cumbersome, to prove that these are the only roots which can appear in Lemma 3 as long as $(g, n) \neq (2, 0)$. It follows that Theorem 1 can be marginally improved to assert that $a(\gamma) \in \{1, 2\}$.

3. Proofs

Before proving the results announced in the introduction, we need a preparatory observation:

Lemma 4. *Let S be connected, orientable and non-exceptional surface. Suppose that X is a simplicial complex with $\dim(X) = \dim(\mathcal{C}(S))$, and whose every simplex is equal to the intersection of the maximal dimensional simplices of X that contain it. Then every weakly injective homomorphism $\rho : \mathbb{A}(X) \to \mathrm{Map}(S)$ maps each standard generator of $\mathbb{A}(X)$ to a root of a power of a Dehn twist along a single curve.*

Proof. Let $\gamma \in X^{(0)}$ be a vertex, and consider the collection $\{\Delta_i\}_{i \in I}$ of maximal dimensional simplices of X that contain γ. Our assumption implies that the cyclic group $\langle \gamma \rangle$ is equal to $\cap_i \mathbb{A}(\Delta_i)$. Since ρ is weakly injective, $\rho(\langle \gamma \rangle)$ is also an infinite cyclic subgroup of $\mathrm{Map}(S)$, which moreover satisfies

$$\rho(\langle \gamma \rangle) = \cap_i \rho(\mathbb{A}(\Delta_i)) \subset \mathrm{Map}(S).$$

Now, $\mathrm{rank}(\rho(\mathbb{A}(\Delta_i))) = \mathrm{rank}(\mathbb{A}(\Delta_i)) = \dim(X) + 1 = \dim(\mathcal{C}(S)) + 1$. We can hence apply Lemma 3 to $\{\rho(\Delta_i)\}_{i \in I}$, thus deducing that $\rho(\langle \gamma \rangle)$ is generated by a root of a power of a Dehn twist, as we needed to prove. □

We are now ready to prove Theorem 1:

Proof of Theorem 1. By Lemma 4, $\rho(\gamma)$ is a root of a power of a Dehn twist along a single curve, for every $\gamma \in X^{(0)}$. In other words, there are $\rho_*(\gamma) \in \mathcal{C}(S)$ and $a(\gamma), b(\gamma) \in \mathbb{Z} \setminus \{0\}$ with

$$\rho(\gamma^{a(\gamma)}) = \delta_{\rho_*(\gamma)}^{b(\gamma)}.$$

Since the elements of $\mathbb{A}(X)$ corresponding to adjacent vertices $\gamma, \eta \in X^{(0)}$ commute, $\rho_*(\gamma)$ and $\rho_*(\eta)$ do not intersect. Moreover, if γ, η are arbitrary distinct vertices of X, then $\rho_*(\gamma) \neq \rho_*(\eta)$ because ρ is weakly injective. Therefore, we deduce that the map $\rho_* : X \to \mathcal{C}(S)$ is an injective simplicial map. Since $X \subset \mathcal{C}(S)$ is assumed to be rigid, there is $\phi \in \mathrm{Aut}(\mathcal{C}(S))$ with $\rho_*(\gamma) = \phi(\gamma)$ for all $\gamma \in X$. As S is not exceptional, the aforementioned results of Ivanov [17], Korkmaz [19] and Luo [20] together imply that there is $f \in \mathrm{Map}^*(S)$ with $\phi(\gamma) = f(\gamma)$ for all $\gamma \in \mathcal{C}(S)$; moreover, f is unique unless S is a closed surface of genus 2. Therefore, we obtain

$$\rho(\gamma^{a(\gamma)}) = \delta_{\rho_*(\gamma)}^{b(\gamma)} = \delta_{f(\gamma)}^{b(\gamma)} = f \delta_{\gamma}^{b(\gamma)} f^{-1}$$

for all $\gamma \in X^{(0)}$, as desired. □

Finally, we prove Corollary 2:

Proof of Corollary 2. Let $\sigma : \Gamma \to \mathrm{Map}(S)$ be an injective homomorphism, and $\rho : \mathbb{A}(\mathcal{C}(S)) \to \mathrm{Map}(S)$ the homomorphism $\rho(\gamma) = \delta_\gamma^{n(\gamma)}$, noting that its image is contained in Γ. Hence, we can also consider the homomorphism

$$\rho' = \sigma \circ \rho : \mathbb{A}(\mathcal{C}(S)) \to \mathrm{Map}(S).$$

As S is assumed to be non-exceptional, $\mathcal{C}(S)$ is rigid [22] and thus Theorem 1 implies that there are $f \in \mathrm{Map}^*(S)$ and functions $a, b : \mathcal{C}(S) \to \mathbb{Z} \setminus \{0\}$ with $\rho'(\gamma^{a(\gamma)}) = f\delta_\gamma^{b(\gamma)} f^{-1}$ for every $\gamma \in \mathcal{C}(S)$. Moreover, f is unique since S is not the closed surface of genus 2. Conjugating σ by f^{-1}, we may in fact assume that

$$\rho'(\gamma^{a(\gamma)}) = \delta_\gamma^{b(\gamma)}$$

for every vertex $\gamma \in \mathcal{C}(S)$. After this normalization, σ maps roots of powers of Dehn twists along a curve to roots of powers of Dehn twists along the same curve. We claim that $\sigma(h) = h$ for every $h \in \Gamma$. Indeed, note that for every $h \in \Gamma$ and $\gamma \in \mathcal{C}(S)$ there are $a, b, c \in \mathbb{Z}$ such that

$$\delta^a_{h(\gamma)} = \sigma(\delta^b_{h(\gamma)}) = \sigma(h\delta^b_\gamma h^{-1}) = \sigma(h)\sigma(\delta^b_\gamma)\sigma(h)^{-1} = \sigma(h)\delta^c_\gamma\sigma(h)^{-1} = \delta^c_{\sigma(h)(\gamma)}$$

This proves, in particular, that $h(\gamma) = \sigma(h)(\gamma)$. Since $\gamma \in \mathcal{C}(S)$ was arbitrary and S is not the closed surface of genus 2, it follows that $\sigma(h) = h$, as we needed to prove. □

Acknowledgements. This paper was written during the program "Automorphisms of Free Groups: Algorithms, Geometry and Dynamics" at the CRM, Barcelona. We would like to thank the organizers of the program, as well as to express our gratitude to the CRM for its hospitality. Se lo dedicamos a los hijos de la Petra.

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