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MATHEMATIQUES
Equations différentielles

REMARKS ON THE EQUATIONS OF HEAVY GYROSTAT

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1. Non-integrability of the equations of heavy gyrostat. The motion of heavy gyrostat is governed by the equations

$$J\dot{\omega} = (J\omega+\lambda)\times\omega + \varepsilon e\times r, \quad \dot{e} = e\times\omega, \tag{1}$$

where $\omega=(\omega_1, \omega_2, \omega_3)$ is the angular velocity, $J\omega=(A\omega_1, B\omega_2, C\omega_3)$ is the kinetic momentum, $e=(e_1, e_2, e_3)$ is the unit vector along the direction of the gravitational field, $r=(x_0, y_0, z_0)$ is the centre of mass (the components of these vectors are referred to the fixed in the body frame, formed by the principal axes of inertia at the fixed point), ε is the mass of the body. A, B, C are the principal moments of inertia, and $\lambda=(\lambda_1, \lambda_2, \lambda_3)$ is the gyrostatic moment (see [8] for example). The system (1) admits an equivalent formulation as a Hamiltonian system with two degrees of freedom. Thus, for its Liouville complete integrability we need, besides the Hamiltonian

$$H=\frac{1}{2}\langle\omega, J\omega\rangle+\varepsilon\langle e, r\rangle, \tag{2}$$

and the two geometric first integrals

$$H_1=\langle J\omega+\lambda, e\rangle, \quad H_2=\langle e, e\rangle, \tag{3}$$

an additional fourth first integral. Such integral does exist in the following three cases

i) $x_0=y_0=z_0=0$, $H_4=\langle J\omega+\lambda, J\omega+\lambda\rangle$ (Zhukovskij [8]);

ii) $A=B$, $x_0=y_0=0$, $\lambda_1=\lambda_2=0$, $H_4=\omega_3$ (Lagrange [8]);

iii) $A=B=2C$, $y_0=z_0=0$, $\lambda_1=\lambda_2=0$,

$$H_4=(C(\omega_1^2-\omega_2^2)-\varepsilon x_0e_1)^2+(2C\omega_1\omega_2-\varepsilon x_0e_2)^2-4\varepsilon x_0\lambda_3\omega_1e_3 +2\lambda_3(\omega_1^2+\omega_2^2)(C\omega_3-\lambda_3)$$

(Yehia [1]).

Remark. Note that the integrable case $A=B=C$, $r\times\lambda=0$, is equivalent to ii) after suitable rotation of the inertial frame. For that reason we do not consider this case separately.

Suppose now that $\lambda=0$. The system (1) turns into the customary Euler-Poisson equations, and Zhukovskij and Yehia first integrals (see above) turn into the well known first integrals of Euler and Kowalevski, respectively [8, 9]. We recall here the following classical result

Theorem (Husson [2]). The Euler-Poisson equations possess an additional algebraic first integral only in the three cases of Euler, Lagrange, and Kowalevski.

Our first remark is the following generalization of the Husson's theorem.

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Theorem 1. The equations of heavy gyrostat (1) possess an additional algebraic first integral only in the three cases of Zhukovskij, Lagrange, and Yehia.

Remark. It was believed until recently, that the equations of heavy gyrostat (1) possessed an additional algebraic first integral only in the cases of Zhukovskij, Lagrange, and Kowalevski ($\lambda_1=\lambda_2=\lambda_3=0$ in the last case) [10]. Contrary to that assertion, Yehia [1] found a new fourth first integral in the case iii) above.

Sketch of the proof of Theorem 1. If the system of ordinary differential equations

$$dx_i/dt = F_i(x_1, x_2, \ldots, x_n), \quad i=1, 2, \ldots, n,$$

where F_i, $i=1, 2, \ldots, n$, are rational functions in $x_1, \ldots x_n$, possesses k algebraic, functionally independent first integrals, then it also possesses k rational, functionally independent first integrals. Thus, to prove Theorem 1 it is enough to consider only rational first integrals. Our proof consists of two steps. First we note that if the system (1) possesses an additional rational first integral, then the corresponding Euler-Poisson equations which are obtained from (1) after substituting $\lambda=0$, also possess an additional rational first integral (see [11] for details). According to Husson's theorem, we may restrict our attention only to the cases of Euler, Lagrange, and Kowalevski (there are no restrictions on λ_1, λ_2, and λ_3 at this step). However, as in the Euler case an additional rational first integral exists for any choise of $\lambda_1, \lambda_2, \lambda_3$, then we turn to Lagrange and Kowalevski cases.

At the second step of the proof we apply the Painlevé property test [3] to the system (1) in Lagrange and Kowalevski cases. The conclusion is that this system may be of Painlevé type only if $\lambda_1=\lambda_2=0$, i. e. the cases ii) and iii) above are identified. If $\lambda_1^2+\lambda_2^2\neq 0$ the system under consideration is not of Painlevé type, as it possesses a five-parameter family of solutions with logarithmic branch points. Suppose now that the system (1) has an additional rational first integral H_4. Substituting the above five-parameter family of solutions into the first integrals H_1, H_2, H, and H_4, we obtain on each generic level set

$$\mathrm{A_C}=\{H_1=c_1,\ H_2=c_2,\ H=c_3,\ H_4=c_4\}\subset \mathbf{C}^6 \tag{4}$$

four algebraic relations for the five free parameters, and hence a union of algebraic curves. Denote this set by Γ^λ. Obviously Γ^λ depends upon the choice of the constants c_1, c_2, c_3, c_4, as well upon the gyrostatic moment λ (recall that the parameters $A, B, C, \varepsilon r$ are fixed according to either Lagrange or Kowalevski case). The next observation is that the algebraic relations defining Γ^λ (and in particular the algebraic relations defining $\Gamma^0=\Gamma^\lambda|_{\lambda=0}$) do not restrict the value of the free parameter α_1, corresponding to the Kowalevski's exponent $+1$. It means that Γ^λ is a union of genus zero curves (i. e. Riemann spheres). This fact is used earlier in [5]. In particular, taking the limit $\lambda\to 0$, we conclude that Γ^0 is also a union of genus zero curves. On the other hand the Kowalevski top (i. e. the system (1) in the case iii) under the assumption $\lambda=0$) is algebraically completely integrable [4]. The last means that $\mathrm{A_C}$ can be completed generically into an Abelian variety, after adjoining the set Γ^0, and it is a contradiction, as on an Abelian variety can not live a genus zero curve.

Let us turn now to the Lagrange case. One may check that if $(\omega_1, \omega_2, \omega_3, e_1, e_2, e_3)$ is the five-parameter family of solutions with logarithmic branch points, then after substituting $\lambda=0$, the variable $\omega_3=H_4$ is a linear function in α_1 which does not vanish identically. This implies that the four aglebraic relations (4) on the five free parameters restrict the values of α_1, which contradicts to the observation above. Thus Theorem 1 is proved. Complete proof of this theorem will appear elsewhere.

2. The gyrostat of Yehia as Clebsch geodesic motion on E(3). Recently Haine a. Horozov [6], using a transparent algebraic procedure, found a birational change of the variables, connecting the Kowalevski top and the so-called integrable Clebsch case of geodesic motion on the Euclidean motion Lie group $E(3)$. Our second remark is that the gyrostat of Yehia can be realised in a similar way as Clebsch geodesic motion on $E(3)$. This leads in particular to explicite formulae for its solutions in terms of genus two hyperelliptic theta functions [7]. We shall follow closely the notation of [6]. After substituting $A=B=2C=2$, $\varepsilon r=(1, 0, 0)$, $\lambda_1=\lambda_2=0$, in (1) and changing the variables as $x_1=\omega_1+i\omega_2$, $x_2=\omega_1-i\omega_2$, $x_3=\omega_3$, $y_3=e_3$,

$$y_1=x_1^2-(e_1+e_2), \quad y_2=x_2^2-(e_1-e_2),$$

the equations describing the gyrostat of Yehia take the form

$$\dot{x}_1=x_3x_1-y_3-\lambda_3x_1, \quad \dot{y}_1=2x_3y_1-2\lambda_3x_1^2; \quad \cdot=2i\frac{d}{dt},$$

$$\text{(5)} \qquad \dot{x}_2=-x_3x_2+y_3+\lambda_3x_2, \quad \dot{y}_2=-2x_3y_2+2\lambda_3x_2^2,$$

$$\dot{x}_3=x_2^2-x_1^2+y_1-y_2, \quad \dot{y}_3=x_1(x_2^2-y_2)-x_2(x_1^2-y_1).$$

The first integrals of this system read

$$(x_1+x_2)^2+x_3^2-y_1-y_2=A$$

$$\text{(6)} \qquad x_1x_2(x_1+x_2)-y_1x_2-y_2x_1+(x_3+\lambda_3)y_3=B$$

$$x_1^2x_2^2+y_3^2-y_1x_2^2-y_2x_1^2+y_1y_2=C+D^2$$

$$y_1y_2-2\lambda_3((x_1+x_2)y_3-x_1x_2(x_3-\lambda_3))=D^2.$$

One may prove that the system (5) is algebraically completely integrable. Thus each generic level set A_C (4) can be completed into an Abelian variety T. The natural involution $\sigma\colon (x_1, x_2, x_3, y_1, y_2, y_3)\to(x_2, x_1, x_3, y_2, y_1, y_3)$ flips the sign of the vector field (5) and represents the reflection on T (about some of the 16 fixed points of σ on T). The zero locus of the Abelian function x_1-x_2 on T is an odd divisor K, linearly equivalent to the pole divisor Γ^0 (the analytic set Γ^0 is defined in section 1). The space $L(K)=\{f$ meromorphic on T, $(f)\geq -K\}$ splits into an odd and an even piece $L(K)=L_-(K)\oplus L_+(K)$ with

$$L_-(K)=\Big\{p_1=\frac{1+x_1x_2}{x_1-x_2}i, \; p_2=\frac{x_1+x_2}{x_1-x_2},$$

$$p_3=\frac{1-x_1x_2}{x_1-x_2}, \; l_2=\frac{y_3+\lambda_3(x_1+x_2)}{x_1-x_2}$$

$$\text{(7)} \qquad l_1=\frac{(x_3-\lambda_3)(1-x_1x_2)+(x_1+x_2)y_3+2\lambda_3(1+x_1x_2)}{2(x_1-x_2)}i,$$

$$l_3=\frac{(x_3-\lambda_3)(1+x_1x_2)-(x_1+x_2)y_3+2\lambda_3(1-x_1x_2)}{2(x_1-x_2)}\Big\}, \quad i=\sqrt{-1},$$

and $L_+(K)=\Big\{1, \frac{(x_3-\lambda_3)\dot{y}_3-y_3\dot{x}_3}{x_1-x_2}\Big\}$. The embedding

$$A_C=T\setminus K\to \mathbf{C}^6\colon (x_1, x_2, x_3, y_1, y_2, y_3)\to(p_1, p_2, p_3, l_1, l_2, l_3)$$

maps the vector field (5) into the system

$$\dot{p}=\nabla_l\tilde{H}\times p \qquad , \cdot=\frac{d}{dt},$$

$$\text{(8)} \qquad \dot{l}=\nabla_p\tilde{H}\times p+\nabla_l\tilde{H}\times l,$$

$$\tilde{H}=\frac{1}{2}(|l|^2+\langle Qp, p\rangle)$$

$$Q=\frac{1}{4}\begin{pmatrix} C-1 & -iB & i(1+C) \\ -iB & -A+\lambda_3^2 & B \\ i(1+C) & B & 1-C \end{pmatrix},$$

which describes integrable Clebsch geodesic motion on $E(3)$ for the right invariant metric $\begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix}$.

The system (8) possesses two geometric first integrals $\widetilde{H}_1=\langle p,p\rangle$, $\widetilde{H}_2=\langle l, p\rangle$, and the additional fourth first integral reads $\widetilde{H}_4=-\langle Ql, l\rangle+\det Q\langle Q^{-1}p, p\rangle$.

At last we note that in (l, p) coordinates the level surface (6) takes the form

$$\{\widetilde{H}_1=1,\ \widetilde{H}_2=\lambda_3,\ \widetilde{H}=(5\lambda_3^2-A)/8,\ \widetilde{H}_4=(\lambda_3^2(A-\lambda_3^2)-D^2)/4\}\subset \mathbb{C}^6,$$

The system (8) is integrated first by Kötter [7]. It coincides, up to linear change of the variables, with the Euler-Manakov equations on so(4). In this context the problem is integrated by Dubrovin [12].

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