Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 260 (2016) 3963–3990
Journal of Differential Equations
www.elsevier.com/locate/jde

Cubic perturbations of elliptic Hamiltonian vector fields of degree three

Lubomir Gavrilov [a,*], Iliya D. Iliev [b]

[a] *Institut de Mathématiques de Toulouse, Université de Toulouse, F-31062 Toulouse, France*
[b] *Institute of Mathematics, Bulgarian Academy of Sciences, Bl. 8, 1113 Sofia, Bulgaria*

Received 27 March 2015; revised 24 August 2015

Available online 18 November 2015

Abstract

The purpose of the present paper is to study the limit cycles of one-parameter perturbed plane Hamiltonian vector field X_ε

$$X_\varepsilon : \begin{cases} \dot{x} = \ \ H_y + \varepsilon f(x,y) \\ \dot{y} = -H_x + \varepsilon g(x,y), \end{cases} \qquad H = \frac{1}{2}y^2 + U(x)$$

which bifurcate from the period annuli of X_0 for sufficiently small ε. Here U is a univariate polynomial of degree four without symmetry, and f, g are *arbitrary* cubic polynomials in two variables.

We take a period annulus and parameterize the related displacement map $d(h,\varepsilon)$ by the Hamiltonian value h and by the small parameter ε. Let $M_k(h)$ be the k-th coefficient in its expansion with respect to ε. We establish the general form of M_k and study its zeroes. We deduce that the period annuli of X_0 can produce for sufficiently small ε, at most 5, 7 or 8 zeroes in the interior eight-loop case, the saddle-loop case, and the exterior eight-loop case respectively. In the interior eight-loop case the bound is exact, while in the saddle-loop case we provide examples of Hamiltonian fields which produce 6 small-amplitude limit cycles. Polynomial perturbations of X_0 of higher degrees are also studied.

* Corresponding author.
E-mail address: lubomir.gavrilov@math.univ-toulouse.fr (L. Gavrilov).

http://dx.doi.org/10.1016/j.jde.2015.10.052
0022-0396/

Contents

1. Introduction

We consider cubic systems in the plane which are small perturbations of Hamiltonian systems with a center. Our goal is to estimate the number of limit cycles produced by the perturbation. The Hamiltonians we consider have the form $H = y^2 + U(x)$ where U is a polynomial of degree 4. In this paper we exclude from consideration the four symmetric Hamiltonians $H = y^2 + x^2 \pm x^4$, $H = y^2 - x^2 + x^4$ and $H = y^2 + x^4$ because they require a special treatment, see [6]. Therefore, one can use the following normal form of the Hamiltonian

$$H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{a}{4}x^4, \quad a \neq 0, \frac{8}{9}. \tag{1}$$

An easy observation shows that the following four topologically different cases occur:

$a < 0$	saddle-loop,
$0 < a < 1$	eight loop,
$a = 1$	cuspidal loop,
$a > 1$	global center.

There is one period annulus in the saddle-loop and the global center cases, two annuli in the cuspidal loop case, and three annuli in the eight loop case. Note that $a = \frac{8}{9}$ is the symmetric eight loop case which we are not going to deal with. Take small $\varepsilon > 0$ and consider the following one-parameter perturbation of the Hamiltonian vector field associated to H:

$$\begin{aligned} \dot{x} &= H_y + \varepsilon f(x, y), \\ \dot{y} &= -H_x + \varepsilon g(x, y), \end{aligned} \tag{2}$$

where f and g are arbitrary cubic polynomials with coefficients a_{ij} and b_{ij} at $x^i y^j$, respectively. As well known, if we parameterize the displacement map by the Hamiltonian level h, then the following expansion formula holds

$$d(h,\varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \varepsilon^3 M_3(h) + \dots, \quad h \in \Sigma \tag{3}$$

where Σ is an open interval depending on the case and the period annulus we consider. There is a lot of papers investigating system (2), but most of them deal with $M_1(h)$ only or consider perturbations like $f(x,y)=0$, $g(x,y)=(\alpha_0+\alpha_1 x+\alpha_2 x^2)y$. See e.g. the book by Colin Christopher and Chengzhi Li [1] for more comments and references. In what follows we consider for a first time the full 20-parameter cubic deformation (2) of the Hamiltonian system associated to H. We suppose, however, that the arbitrary cubic polynomials f, g do not depend on the small parameter ε. To study the full neighborhood of the Hamiltonian system associated to H, it is also necessary to allow that f, g depend analytically on ε.

Our first goal will be to calculate explicitly the first several coefficients M_1, M_2, etc., in (3) and then determine the least integer m such that system (2) becomes integrable provided that the first m coefficients in (3) do vanish.

Let us rewrite system (2) in a Pfaffian form

$$dH = \varepsilon\omega, \quad \omega = g(x,y)dx - f(x,y)dy. \tag{4}$$

We first establish that if $M_1(h) \equiv 0$, then one can express the cubic one-form ω in the perturbation as

$$\omega = d[Q(x,y) - (\tfrac{a}{5}\lambda - \tfrac{2}{5}\mu)x^5 - \tfrac{a}{6}\mu x^6] + (\lambda x + \mu x^2)dH \tag{5}$$

where $Q(x,y) = \sum_{1\le i+j\le 4} q_{ij}x^i y^j$ and λ, μ are parameters. Obviously, there are simple explicit linear formulas connecting q_{ij}, λ and μ to the coefficients of f and g, see the Appendix. Below, we shall consider q_{ij}, λ and μ as the parameters of the perturbation.

Theorem 1. *The perturbation (4)–(5) is integrable if and only if either of the two conditions holds:*

1) $\lambda = \mu = 0$;
2) $q_{01} = q_{11} = q_{21} = q_{31} = q_{03} = q_{13} = 0$.

In the first case system (4) becomes Hamiltonian and in the second one it becomes time-reversible.

If $M_1(h) = M_2(h) = M_3(h) = M_4(h) \equiv 0$, then the perturbation is integrable.

Corollary 1. *The perturbation (2) is integrable if and only if either of the two conditions holds:*

1) *The divergence $f_x + g_y$ is zero.*
2) *The polynomials f and g are respectively odd and even in y.*

If $M_1(h) = M_2(h) = M_3(h) = M_4(h) \equiv 0$, then the perturbation is integrable.

When the perturbation is integrable, all coefficients $M_k(h)$ do vanish in the respective period annulus and the Poincaré map is the identity. When the perturbation is not integrable (that is neither of the conditions in Theorem 1 holds), one can prove the following result. Take an oval $\delta(h)$ contained in the level set $H = h$, $h \in \Sigma$ and define the integrals

$$I_k(h) = \oint\limits_{\delta(h)} x^k y dx, k = 0, 1, 2, \ldots$$

Theorem 2. *The first four coefficients $M_k(h)$, $1 \le k \le 4$ have the form*

$$M_k(h) = \alpha_k(h) I_0(h) + \beta_k(h) I_1(h) + \gamma_k(h) I_2(h)$$

where $\alpha_k(h)$, $\beta_k(h)$, $\gamma_k(h)$ are polynomials of degree at most one. The second coefficient $M_2(h)$ has the maximum possible number of zeroes in Σ among $M_k(h)$.

We use the above results in deriving upper bounds for the number of limit cycles bifurcating from the open period annuli in the cases when the Hamiltonian has three real and different critical values. For this, we take a perturbation with $M_1(h) \equiv 0$ and $M_2(h) \not\equiv 0$, with all six coefficients independently free.

Theorem 3.

(i) *In the interior eight-loop case, at most five limit cycles bifurcate from each one of the annuli inside the loop.*
(ii) *In the exterior eight-loop case, at most eight limit cycles bifurcate from the annulus outside the loop.*
(iii) *In the saddle-loop case, at most seven limit cycles bifurcate from the unique period annulus.*

The proof is based on a refinement of Petrov's method which we apply to the much more general case when the coefficients in $M_k(h)$ are polynomials of arbitrary degree n, thus $M_k(h)$ being an element of a module of dimension $3n + 3$.

Theorem 4. *Let the coefficients $\alpha_k(h)$, $\beta_k(h)$ and $\gamma_k(h)$ in the expression of $M_k(h)$ be polynomials of degree n with real coefficients. Then $M_k(h)$ has in the respective interval Σ at most $3n + 2$ zeroes in the interior eight-loop case, at most $4n + 4$ in the exterior eight-loop case, and at most $4n + 3$ zeroes in the saddle-loop case.*

In order to demonstrate that Chebyshev's property (no more zeroes than the dimension minus one) would not also hold in the saddle-loop case, we provide an estimate from below for the number of bifurcating small-amplitude limit cycles around the center at the origin which concerns all Hamiltonian parameters $a \ne 0, \frac{8}{9}$.

Theorem 5. *For a close to $-\frac{8}{3}$, function $M_1(h)$ can produce four small limit cycles around the origin. For a close to $-\frac{8}{9}$, function $M_2(h)$ can produce six such limit cycles. For all other values of $a \in \mathbb{R}$, the number of small limit cycles produced by the function $M_k(h)$ equals its dimension minus one.*

The limit cycle in addition in the saddle-loop case is obtained by moving slightly the Hamiltonian parameter a in appropriate direction from the respective fraction.

The paper is organized as follows. At the beginning, we compute explicitly the coefficients M_k for $k = 1, 2, 3, 4$. It is easily seen from their explicit expressions that for each k they form a set which is

- a vector space of dimension four, for $k = 1$
- a vector space of dimension six, for $k = 2$
- a union of three distinct five-dimensional vector spaces, for $k = 3$
- a union of three distinct straight lines, for $k = 4$,

and when $M_1 = M_2 = M_3 = M_4 = 0$, then the perturbation becomes integrable. The function M_2 takes therefore the form

$$M_2(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_2(h) \tag{6}$$

where α, β, γ are arbitrary linear functions in h.

Next, considering the generalized situation when M_2 is a function of the form (6) in which α, β, γ are arbitrary degree n polynomials in h, we establish that M_2 would have at most $3n + 2$ zeroes in the interior eight-loop case, $4n + 4$ zeroes in the exterior eight-loop case, $4n + 3$ zeroes in the saddle-loop case. We apply these results to our problem by taking $n = 1$. Finally, we provide examples of Hamiltonian fields in the saddle-loop case which produce 4 and 6 small-amplitude limit cycles, respectively when $M_1 \not\equiv 0$, and $M_1 \equiv 0$ but $M_2 \not\equiv 0$. For all other cases, the number of such small-amplitude limit cycles is less than the respective dimension.

2. Calculation of the coefficients $M_k(h)$

In this section we are going to calculate the first four coefficients in (3). We use the recursive procedure proposed by Françoise [2], see also [7,8].

2.1. The coefficient $M_1(h)$

We begin with the easy calculation of $M_1(h)$.

Proposition 1.

(i) *The function $M_1(h)$ has the form*

$$M_1(h) = \alpha_1 I_0(h) + \beta_1 I_1(h) + \gamma_1 I_2(h), \tag{7}$$

where α_1 is a first-degree polynomial in h and β_1, γ_1 are constants, depending on the perturbation.

(ii) *If $M_1(h) \equiv 0$, then one can rewrite the one-form ω as (5) where Q is a polynomial of degree four without constant term and λ, μ are constant parameters.*

Proof. By a simple calculation, one can rewrite ω in the form $\omega = dQ(x, y) + yq(x, y)dx$ with Q and q certain polynomials of degree 4 and 2, respectively. Denote for a moment by c_{ij} the coefficient in q at $x^i y^j$. Then

$$yq(x,y)dx = (c_{01}+c_{11}x)y^2dx + (c_{00}+c_{10}x+c_{20}x^2)ydx + c_{02}y^3dx.$$

Next, $y^3dx = (2H - x^2 + \frac{4}{3}x^3 - \frac{a}{2}x^4)ydx = (2H - x^2)ydx + yd(\frac{1}{3}x^4 - \frac{a}{10}x^5)$. Using the identity $\frac{1}{3}x^4 - \frac{a}{10}x^5 = \frac{4}{15a}H - \frac{2}{5}xH + (\frac{8}{45a} + \frac{1}{5})x^3 - \frac{2}{15a}x^2 - \frac{2}{15a}y^2 + \frac{1}{5}xy^2$ we derive the equation

$$y^3dx = d(\tfrac{1}{7}xy^3 - \tfrac{2}{21a}y^3) + (\tfrac{2}{7a} - \tfrac{3}{7}x)ydH + [\tfrac{12}{7}H - \tfrac{2}{7a}x + (\tfrac{4}{7a} - \tfrac{3}{7})x^2]ydx. \tag{8}$$

Replacing in the formula above and taking into account that $M_1(h) = \oint_{\delta(h)} \omega = \oint_{\delta(h)} yq(x,y)dx$, one obtains formula (7) with

$$\alpha_1 = c_{00} + \tfrac{12}{7}c_{02}h, \quad \beta_1 = c_{10} - \tfrac{2}{7a}c_{02}, \quad \gamma_1 = c_{20} + (\tfrac{4}{7a} - \tfrac{3}{7})c_{02}.$$

Now, $M_1(h) \equiv 0$ is equivalent to $c_{00} = c_{10} = c_{20} = c_{02} = 0$ (see Corollary 2 below) and ω becomes $\omega = dQ - \frac{1}{2}y^2(\lambda + 2\mu x)dx$ where $\lambda = -2c_{01}$, $\mu = -c_{11}$. On the other hand (modulo terms dQ)

$$\begin{aligned}
&-\tfrac{1}{2}y^2(\lambda + 2\mu x)dx = (\lambda x + \mu x^2)d(H - \tfrac{1}{2}x^2 + \tfrac{2}{3}x^3 - \tfrac{a}{4}x^4)\\
&= (\lambda x + \mu x^2)dH + (\lambda x + \mu x^2)(-x + 2x^2 - ax^3)dx\\
&= (\lambda x + \mu x^2)dH + d(-\tfrac{a}{5}\lambda x^5 + \tfrac{2}{5}\mu x^5 - \tfrac{a}{6}\mu x^6).
\end{aligned}$$

Proposition 1 is proved. □

2.2. The coefficient $M_2(h)$

By (5), if $\lambda = \mu = 0$, then the perturbation is Hamiltonian and all coefficients M_k do vanish. We will assume below that λ and μ are not both zero. Then the calculation of $M_2(h)$ makes sense. Denote by q_{ij} the coefficient at x^iy^j in Q. Below, we split Q into an odd and even part $Q = Q_1 + Q_2$ with respect to y.

Proposition 2.

(i) *If $M_1(h) \equiv 0$, then the function $M_2(h)$ has the form*

$$M_2(h) = \alpha_2 I_0(h) + \beta_2 I_1(h) + \gamma_2 I_2(h), \tag{9}$$

where α_2, β_2 and γ_2 are first-degree polynomials in h with coefficients depending on the perturbation.

(ii) *If $M_1(h) = M_2(h) \equiv 0$, then the odd part of $Q(x,y)$ becomes:*
   (a) *$Q_1 = q_{11}(x - 2x^2 + ax^3)y$, if $\mu = 0$;*
   (b) *$Q_1 = -\frac{1}{2}q_{11}(1 - 2x + ax^2)y$, if $\lambda = 0$;*
   (c) *$Q_1 = q_{11}(x + \frac{a\lambda}{2\mu}x^2)y$, if $a \le 1$ and $a\lambda^2 + 4\lambda\mu + 4\mu^2 = 0$;*
   (d) *$Q_1 = 0$, if $\lambda\mu \ne 0$ and $a\lambda^2 + 4\lambda\mu + 4\mu^2 \ne 0$.*

Proof. As well known, the second coefficient in (3) is obtained by integrating the one-form $\omega_2 = (\lambda x + \mu x^2)\omega$, that is

$$M_2(h) = \oint_{\delta(h)} \omega_2 = \oint_{\delta(h)} (\lambda x + \mu x^2) dQ(x, y) = -\oint_{\delta(h)} (\lambda + 2\mu x) Q_1(x, y) dx$$
$$= -\oint_{\delta(h)} (\lambda + 2\mu x)[(q_{01} + q_{11}x + q_{21}x^2 + q_{31}x^3)y + (q_{03} + q_{13}x)y^3]dx.$$

Next, multiplying (8) by x and expressing the first term on the right-hand side in a proper form, we obtain identity

$$\begin{aligned} xy^3dx &= d(\tfrac{1}{8}x^2y^3 - \tfrac{1}{14a}xy^3 - \tfrac{1}{126a^2}y^3) + (\tfrac{1}{42a^2} + \tfrac{3}{14a}x - \tfrac{3}{8}x^2)ydH \\ &+ [(\tfrac{1}{7a} + \tfrac{3}{2}x)H - \tfrac{1}{42a^2}x + (\tfrac{1}{21a^2} - \tfrac{2}{7a})x^2 + (\tfrac{1}{2a} - \tfrac{3}{8})x^3]ydx. \end{aligned} \tag{10}$$

In a similar way, multiplying (10) by x, we get

$$\begin{aligned} x^2y^3dx &= d(\tfrac{1}{9}x^3y^3 - \tfrac{1}{18a}x^2y^3 - \tfrac{2}{189a^2}xy^3 - \tfrac{2}{1701a^3}y^3) \\ &+ (\tfrac{2}{567a^3} + \tfrac{2}{63a^2}x + \tfrac{1}{6a}x^2 - \tfrac{1}{3}x^3)ydH + [(\tfrac{4}{189a^2} + \tfrac{2}{9a}x + \tfrac{4}{3}x^2)H \\ &- \tfrac{2}{567a^3}x + (\tfrac{4}{567a^3} - \tfrac{8}{189a^2})x^2 + (\tfrac{2}{27a^2} - \tfrac{5}{18a})x^3 + (\tfrac{4}{9a} - \tfrac{1}{3})x^4]ydx. \end{aligned} \tag{11}$$

Replacing the values from (8), (10) and (11) in the above formula of $M_2(h)$, we obtain

$$M_2(h) = -[q_0I_0(h) + q_1I_1(h) + q_2I_2(h) + q_3I_3(h) + q_4I_4(h)]$$

where

$$\begin{aligned} q_0 &= \lambda q_{01} + [\tfrac{12}{7}\lambda q_{03} + \tfrac{1}{7a}(\lambda q_{13} + 2\mu q_{03}) + \tfrac{8}{189a^2}\mu q_{13}]h, \\ q_1 &= \lambda q_{11} + 2\mu q_{01} - \tfrac{2}{7a}\lambda q_{03} - \tfrac{1}{42a^2}(\lambda q_{13} + 2\mu q_{03}) - \tfrac{4}{567a^3}\mu q_{13} \\ &\quad + [\tfrac{3}{2}\lambda q_{13} + 3\mu q_{03} + \tfrac{4}{9a}\mu q_{13}]h, \\ q_2 &= \lambda q_{21} + 2\mu q_{11} + (\tfrac{4}{7a} - \tfrac{3}{7})\lambda q_{03} + (\tfrac{1}{21a^2} - \tfrac{2}{7a})(\lambda q_{13} + 2\mu q_{03}) \\ &\quad + (\tfrac{8}{567a^3} - \tfrac{16}{189a^2})\mu q_{13} + \tfrac{8}{3}\mu q_{13}h, \\ q_3 &= \lambda q_{31} + 2\mu q_{21} + (\tfrac{1}{2a} - \tfrac{3}{8})(\lambda q_{13} + 2\mu q_{03}) + (\tfrac{4}{27a^2} - \tfrac{5}{9a})\mu q_{13}, \\ q_4 &= 2\mu q_{31} + (\tfrac{8}{9a} - \tfrac{2}{3})\mu q_{13}. \end{aligned}$$

In order to remove integrals I_3, I_4, we use the identity

$$\oint_{\delta(h)} (x^kU' + \tfrac{2}{3}kx^{k-1}U)ydx = 0, \quad U = h - \tfrac{1}{2}x^2 + \tfrac{2}{3}x^3 - \tfrac{1}{4}ax^4$$

which is equivalent to

$$\tfrac{k+6}{6} a I_{k+3} = \tfrac{4k+18}{9} I_{k+2} - \tfrac{k+3}{3} I_{k+1} + \tfrac{2k}{3} h I_{k-1}. \tag{12}$$

Used with $k = 0, 1, 2$, this relation yields

$$\begin{aligned} I_3 &= \tfrac{2}{a} I_2 - \tfrac{1}{a} I_1, \\ I_4 &= (\tfrac{88}{21a^2} - \tfrac{8}{7a}) I_2 - \tfrac{44}{21a^2} I_1 + \tfrac{4}{7a} h I_0, \\ I_5 &= (\tfrac{572}{63a^3} - \tfrac{209}{42a^2}) I_2 - (\tfrac{286}{63a^3} - \tfrac{5}{4a^2} - \tfrac{1}{a} h) I_1 + \tfrac{26}{21a^2} h I_0. \end{aligned} \tag{13}$$

Replacing, we finally derive formula (9) with

$$\begin{aligned} \alpha_2 &= -q_0 - \tfrac{4}{7a} q_4 h, \\ \beta_2 &= -q_1 + \tfrac{1}{a} q_3 + \tfrac{44}{21a^2} q_4, \\ \gamma_2 &= -q_2 - \tfrac{2}{a} q_3 + (\tfrac{8}{7a} - \tfrac{88}{21a^2}) q_4. \end{aligned}$$

Then $M_2(h) \equiv 0$ is equivalent to $\alpha_2 = \beta_2 = \gamma_2 = 0$ (see Corollary 2 below). Taking the coefficients at h zero, we obtain that either $\mu = q_{03} = q_{13} = 0$ or $\mu \neq 0$ and $q_{31} = q_{03} = q_{13} = 0$. In the first case, $\lambda \neq 0$ and taking the coefficients at 1 zero, we easily obtain $q_{01} = 0$, $q_{21} = -2q_{11}$, $q_{31} = aq_{11}$ which is case (a). In the second case above, if $\lambda = 0$, then one easily obtains $q_{01} = -\frac{1}{2} q_{11}$, $q_{21} = -\frac{a}{2} q_{11}$ which is case (b). If $\lambda \neq 0$, then taking the coefficients at 1 zero yields $q_{01} = 0$ and equations $-\lambda q_{11} + \frac{2}{a}\mu q_{21} = 0$, $-2\mu q_{11} - (\lambda + \frac{4}{a}\mu) q_{21} = 0$. Provided that $a\lambda^2 + 4\lambda\mu + 4\mu^2 = 0$ (it is possible for $a \leq 1$ only), one has $q_{21} = \frac{a\lambda}{2\mu} q_{11}$ which is case (c). Otherwise, one obtains $q_{11} = q_{21} = 0$ which is case (d). Proposition 2 is proved. □

2.3. *The coefficient $M_3(h)$*

It turns out that if $Q_1 = 0$ then the perturbation is integrable. This is because the perturbed system (2) becomes time-reversible in this case. Below we are going to consider the three cases (a), (b), (c) when $q_{11} \neq 0$. For them, the next coefficient $M_3(h)$ in (3) should be calculated. For this purpose, we need to express the one-form $\omega_2 = (\lambda x + \mu x^2)\omega$ as $dS_2 + s_2 dH$ and then integrate the one-form $\omega_3 = s_2 \omega$.

Proposition 3. *Assume that $q_{11} \neq 0$.*

(i) *If $M_1(h) = M_2(h) \equiv 0$, then the function $M_3(h)$ has the form*

$$M_3(h) = \alpha_3 I_0(h) + \beta_3 I_1(h) + \gamma_3 I_2(h), \tag{14}$$

where α_3, β_3 are first-degree polynomials in h with coefficients depending on the perturbation and γ_3 is a constant.

(ii) *If $M_1(h) = M_2(h) = M_3(h) \equiv 0$, then the even part of $Q(x, y)$ becomes*

$$Q_2 = q_{20} x^2 - (\tfrac{4}{3} q_{20} + \tfrac{1}{3}\lambda) x^3 + (\tfrac{a}{2} q_{20} + \tfrac{1}{2}\lambda - \tfrac{1}{4}\mu) x^4 + (q_{02} - \tfrac{1}{3}\lambda x - \tfrac{1}{3}\mu x^2) y^2 + q_{04} y^4$$

where $\mu = 0$ in case (a), *$\lambda = 0$ in case* (b) *and $a\lambda^2 + 4\lambda\mu + 4\mu^2 = 0$ in case* (c).

Proof. To find s_2, is suffices to perform the calculations modulo exact forms. Let us handle first case (a). By (5) one obtains (neglecting the exact forms)

$$\omega_2 = \lambda x d[(q_{02} + q_{12}x + q_{22}x^2)y^2 + q_{04}y^4 + q_{11}(x - 2x^2 + ax^3)y] + \lambda^2 x^2 dH.$$

Then $xdq_{11}(x - 2x^2 + ax^3)y = -q_{11}(x - 2x^2 + ax^3)ydx = -q_{11}yd(H - \frac{1}{2}y^2) = -q_{11}ydH$. Similarly,

$$\begin{aligned} &xd(q_{02} + q_{12}x + q_{22}x^2)y^2 = 2xd(q_{02} + q_{12}x + q_{22}x^2)H \\ &= -2(q_{02} + q_{12}x + q_{22}x^2)Hdx = 2(q_{02}x + \tfrac{1}{2}q_{12}x^2 + \tfrac{1}{3}q_{22}x^3)dH. \end{aligned}$$

Finally,

$$\begin{aligned} &xdq_{04}y^4 = q_{04}xd[4H^2 - 4H(x^2 - \tfrac{4}{3}x^3 + \tfrac{a}{2}x^4)] = 8q_{04}xHdH \\ &+ 4q_{04}H(x^2 - \tfrac{4}{3}x^3 + \tfrac{a}{2}x^4)dx = 4q_{04}(2xH - \tfrac{1}{3}x^3 + \tfrac{1}{3}x^4 - \tfrac{a}{10}x^5)dH. \end{aligned}$$

Summing up all terms together, we obtain for case (a)

$$s_2 = \lambda^2 x^2 - \lambda q_{11}y + 2\lambda(q_{02}x + \tfrac{1}{2}q_{12}x^2 + \tfrac{1}{3}q_{22}x^3) + 4\lambda q_{04}(2xH - \tfrac{1}{3}x^3 + \tfrac{1}{3}x^4 - \tfrac{a}{10}x^5).$$

In a similar way, we consider (b). In this case,

$$\omega_2 = \mu x^2 d[(q_{02} + q_{12}x + q_{22}x^2)y^2 + q_{04}y^4 - \tfrac{1}{2}q_{11}(1 - 2x + ax^2)y] + \mu^2 x^4 dH.$$

Then $-\frac{1}{2}x^2 d(1 - 2x + ax^2)y = (x - 2x^2 + ax^3)ydx = ydH$,

$$\begin{aligned} &x^2 d(q_{02} + q_{12}x + q_{22}x^2)y^2 = 2x^2 d(q_{02} + q_{12}x + q_{22}x^2)H \\ &= -4(q_{02}x + q_{12}x^2 + q_{22}x^3)Hdx = 4(\tfrac{1}{2}q_{02}x^2 + \tfrac{1}{3}q_{12}x^3 + \tfrac{1}{4}q_{22}x^4)dH, \end{aligned}$$

$$\begin{aligned} &x^2 dy^4 = x^2 d[4H^2 - 4H(x^2 - \tfrac{4}{3}x^3 + \tfrac{a}{2}x^4)] = 8x^2 HdH \\ &+ 8H(x^3 - \tfrac{4}{3}x^4 + \tfrac{a}{2}x^5)dx = 8(x^2 H - \tfrac{1}{4}x^4 + \tfrac{4}{15}x^5 - \tfrac{a}{12}x^6)dH. \end{aligned}$$

Summing up all needed terms, we obtain in case (b) the formula

$$s_2 = \mu^2 x^4 + \mu q_{11}y + 4\mu(\tfrac{1}{2}q_{02}x^2 + \tfrac{1}{3}q_{12}x^3 + \tfrac{1}{4}q_{22}x^4) + 8\mu q_{04}(x^2 H - \tfrac{1}{4}x^4 + \tfrac{4}{15}x^5 - \tfrac{a}{12}x^6).$$

Finally, in case (c) we have $a\lambda^2 + 4\lambda\mu + 4\mu^2 = 0$ and

$$\omega_2 = (\lambda x + \mu x^2)d[(q_{02} + q_{12}x + q_{22}x^2)y^2 + q_{04}y^4 + q_{11}(x + \tfrac{a\lambda}{2\mu}x^2)y] + (\lambda x + \mu x^2)^2 dH.$$

As above,

$$\begin{aligned} &(\lambda x + \mu x^2)dq_{11}(x + \tfrac{a\lambda}{2\mu}x^2)y = -q_{11}(x + \tfrac{a\lambda}{2\mu}x^2)(\lambda + 2\mu x)ydx \\ &= -q_{11}yd(\tfrac{\lambda}{2}x^2 + \tfrac{a\lambda^2 + 4\mu^2}{6\mu}x^3 + \tfrac{a\lambda}{4}x^4) = -\lambda q_{11}yd(H - \tfrac{1}{2}y^2) = -\lambda q_{11}ydH, \end{aligned}$$

$$
\begin{aligned}
&(\lambda x+\mu x^2)d(q_{02}+q_{12}x+q_{22}x^2)y^2=-2H(q_{02}+q_{12}x+q_{22}x^2)(\lambda+2\mu x)dx\\
&=[2\lambda q_{02}x+(\lambda q_{12}+2\mu q_{02})x^2+\tfrac{2}{3}(\lambda q_{22}+2\mu q_{12})x^3+\mu q_{22}x^4]dH,
\end{aligned}
$$

$$
\begin{aligned}
&(\lambda x+\mu x^2)dq_{04}y^4=q_{04}(\lambda x+\mu x^2)d[4H^2-4H(x^2-\tfrac{4}{3}x^3+\tfrac{a}{2}x^4)]\\
&=8q_{04}(\lambda x+\mu x^2)HdH+4q_{04}H(x^2-\tfrac{4}{3}x^3+\tfrac{a}{2}x^4)(\lambda+2\mu x)dx\\
&=4q_{04}[2(\lambda x+\mu x^2)H-\tfrac{1}{3}\lambda x^3+(\tfrac{1}{3}\lambda-\tfrac{1}{2}\mu)x^4-(\tfrac{a}{10}\lambda-\tfrac{8}{15}\mu)x^5-\tfrac{a}{6}\mu x^6]dH.
\end{aligned}
$$

Summing up all terms, we obtain in case (c) the respective formula

$$
\begin{aligned}
s_2&=(\lambda x+\mu x^2)^2-\lambda q_{11}y+2\lambda q_{02}x+(\lambda q_{12}+2\mu q_{02})x^2+\tfrac{2}{3}(\lambda q_{22}+2\mu q_{12})x^3\\
&+\mu q_{22}x^4+4q_{04}[2(\lambda x+\mu x^2)H-\tfrac{1}{3}\lambda x^3+(\tfrac{1}{3}\lambda-\tfrac{1}{2}\mu)x^4-(\tfrac{a}{10}\lambda-\tfrac{8}{15}\mu)x^5-\tfrac{a}{6}\mu x^6].
\end{aligned}
$$

In order to calculate M_3 at once for all three cases (a), (b), (c), we shall use the formula of s_2 for case (c) from which the other two cases are obtained by taking μ or λ zero. Indeed, let us denote by s_2^0 the even part of s_2 with respect to y. Then $s_2=\kappa y+s_2^0$ where $\kappa=-\lambda q_{11}$ in cases (a), (c) and $\kappa=\mu q_{11}$ in case (b). Then

$$
M_3(h)=\oint_{\delta(h)} s_2\omega=\oint_{\delta(h)}\kappa y d[Q_2+(\tfrac{2}{5}\mu-\tfrac{a}{5}\lambda)x^5\ -\tfrac{a}{6}\mu x^6]+\oint_{\delta(h)} s_2^0 dQ_1=I+J.
$$

We further have

$$
\begin{aligned}
I&=\kappa\oint_{\delta(h)}[(q_{10}+2q_{20}x+3q_{30}x^2+4q_{40}x^3+(2\mu-a\lambda)x^4-a\mu x^5)y+(\tfrac{1}{3}q_{12}+\tfrac{2}{3}q_{22}x)y^3]dx\\
&=\kappa(q_0I_0+q_1I_1+q_2I_2+q_3I_3+q_4I_4+q_5I_5)
\end{aligned}
$$

with

$$
\begin{aligned}
q_0&=q_{10}+(\tfrac{4}{7}q_{12}+\tfrac{2}{21a}q_{22})h,\\
q_1&=2q_{20}-\tfrac{2}{21a}q_{12}-\tfrac{1}{63a^2}q_{22}+q_{22}h,\\
q_2&=3q_{30}+(\tfrac{4}{21a}-\tfrac{1}{7})q_{12}+(\tfrac{2}{63a^2}-\tfrac{4}{21a})q_{22},\\
q_3&=4q_{40}+(\tfrac{1}{3a}-\tfrac{1}{4})q_{22},\\
q_4&=2\mu-a\lambda,\\
q_5&=-a\mu
\end{aligned}
$$

(we used (8) and (10) as well). On the other side, integrating by parts one can rewrite J as $J=-\oint_{\delta(h)}(s_2^0)'Q_1dx=J_1+J_2$ where J_2 is the part corresponding to the expression in s_2^0 which contains q_{04}. Let us first verify that $J_2=0$. Indeed, one can establish by easy calculations that

$$\begin{aligned}&4q_{04}[(2\lambda+4\mu x)H-\lambda x^2+(\tfrac{4}{3}\lambda-2\mu)x^3-(\tfrac{a}{2}\lambda-\tfrac{8}{3}\mu)x^4-a\mu x^5]\\&=8q_{04}(\lambda+2\mu x)(H-\tfrac{1}{2}x^2+\tfrac{2}{3}x^3-\tfrac{a}{4}x^4)=4q_{04}(\lambda+2\mu x)y^2,\\&-Q_1(\lambda+2\mu x)=\kappa y(x-2x^2+ax^3).\end{aligned}$$

Hence,

$$J_2=4\kappa q_{04}\oint_{\delta(h)}(x-2x^2+ax^3)y^3dx=4\kappa q_{04}\oint_{\delta(h)}y^3d(H-\tfrac{1}{2}y^2)=0.$$

What J_1 concerns, another easy calculation shows that

$$\begin{aligned}&-2[(\lambda x+\mu x^2)(\lambda+2\mu x)+\lambda q_{02}+(\lambda q_{12}+2\mu q_{02})x+(\lambda q_{22}+2\mu q_{12})x^2+2\mu q_{22}x^3]Q_1\\&=2\kappa(x-2x^2+ax^3)[q_{02}+(q_{12}+\lambda)x+(q_{22}+\mu)x^2]y\end{aligned}$$

for all three cases. Therefore, by integrating, one obtains

$$J=J_1=\kappa(r_1I_1+r_2I_2+r_3I_3+r_4I_4+r_5I_5)$$

where

$$\begin{aligned}r_1&=2q_{02},\\r_2&=2\lambda-4q_{02}+2q_{12},\\r_3&=2\mu-4\lambda+2aq_{02}-4q_{12}+2q_{22},\\r_4&=2a\lambda-4\mu+2aq_{12}-4q_{22},\\r_5&=2a\mu+2aq_{22}.\end{aligned}$$

Combining with the formula of I and using (13), one obtains expression (14) with coefficients

$$\begin{aligned}\alpha_3&=\kappa[q_0+\tfrac{4}{7a}h(q_4+r_4)+\tfrac{26}{21a^2}h(q_5+r_5)],\\\beta_3&=\kappa[q_1+r_1-\tfrac{1}{a}(q_3+r_3)-\tfrac{44}{21a^2}(q_4+r_4)-(\tfrac{286}{63a^3}-\tfrac{5}{4a^2}-\tfrac{1}{a}h)(q_5+r_5)],\\\gamma_3&=\kappa[q_2+r_2+\tfrac{2}{a}(q_3+r_3)+(\tfrac{88}{21a^2}-\tfrac{8}{7a})(q_4+r_4)+(\tfrac{572}{63a^3}-\tfrac{209}{42a^2})(q_5+r_5)].\end{aligned}$$

It is seen that α_3 and β_3 are first-degree polynomials while γ_3 is a constant polynomial. This proves part (i) of the statement. To prove part (ii), assume that $M_3(h)$ vanishes, which is equivalent to $\alpha_3=\beta_3=\gamma_3=0$ (see Corollary 2 below). Then by straightforward calculations one obtains that this is equivalent to

$$q_{10}=0,\quad q_{30}=-\tfrac{4}{3}q_{20}-\tfrac{1}{3}\lambda,\quad q_{40}=\tfrac{a}{2}q_{20}+\tfrac{1}{2}\lambda-\tfrac{1}{4}\mu,\quad q_{12}=-\tfrac{1}{3}\lambda,\quad q_{22}=-\tfrac{1}{3}\mu$$

which yields the needed formula of Q_2. Proposition 3 is proved. □

2.4. The coefficient $M_4(h)$

Replacing the values of the coefficients we just calculated, we obtain

$$\begin{aligned}\omega =\ & (2q_{20} - \lambda x - \mu x^2)(x - 2x^2 + ax^3)dx \\ & + d[Q_1 + (q_{02} - \tfrac{1}{3}\lambda x - \tfrac{1}{3}\mu x^2)y^2 + q_{04}y^4] + (\lambda x + \mu x^2)dH, \\ s_2 =\ & \tfrac{2}{3}(\lambda x + \mu x^2)^2 + \kappa y + 2q_{02}(\lambda x + \mu x^2) \\ & + 4q_{04}[2(\lambda x + \mu x^2)H - \lambda(\tfrac{1}{3}x^3 - \tfrac{1}{3}x^4 + \tfrac{a}{10}x^5) - \mu(\tfrac{1}{2}x^4 - \tfrac{8}{15}x^5 + \tfrac{a}{6}x^6)].\end{aligned}$$

Proposition 4. *Assume that $q_{11} \neq 0$ and $M_1(h) = M_2(h) = M_3(h) \equiv 0$. Then the function $M_4(h)$ has the form*

$$\begin{aligned} & M_4(h) = \lambda q_{11}^3[2hI_0(h) - (3ah + \tfrac{3}{4} - \tfrac{2}{3a})I_1(h) + (\tfrac{3}{2} - \tfrac{4}{3a})I_2(h)], \quad \mu = 0, \\ & M_4(h) = -\tfrac{1}{2}\mu q_{11}^3[I_0(h) - 2I_1(h) + aI_2(h)], \quad \lambda = 0, \\ & M_4(h) = -(\tfrac{\lambda^2}{\mu^2} + \tfrac{3\lambda}{2\mu})q_{11}^3[2\mu I_1(h) + a\lambda I_2(h)], \quad \lambda\mu \neq 0, \quad a\lambda^2 + 4\lambda\mu + 4\mu^2 = 0.\end{aligned}$$

Moreover, $M_4(h) \not\equiv 0$.

Proof. In what follows, it is useful to introduce notations

$$\begin{aligned} & A = \lambda(\tfrac{1}{3}x^3 - \tfrac{1}{3}x^4 + \tfrac{a}{10}x^5) + \mu(\tfrac{1}{2}x^4 - \tfrac{8}{15}x^5 + \tfrac{a}{6}x^6), \\ & B = \tfrac{1}{2}x^2 - \tfrac{2}{3}x^3 + \tfrac{a}{4}x^4, \qquad L = \lambda x + \mu x^2.\end{aligned}$$

Then $dA = 2BdL$, $(2q_{20} - L)dB = d[(2q_{20} - L)B + \frac{1}{2}A]$ and one can rewrite the expressions of ω and s_2 as follows:

$$\begin{aligned}\omega &= (2q_{20} - L)B'dx + d[Q_1 + (q_{02} - \tfrac{1}{3}L)y^2 + q_{04}y^4] + LdH, \\ &= d[Q_1 + (2q_{20} - L)B + \tfrac{1}{2}A + (q_{02} - \tfrac{1}{3}L)y^2 + q_{04}y^4] + LdH, \\ s_2 &= \tfrac{2}{3}L^2 + \kappa y + 2q_{02}L + 4q_{04}(2LH - A).\end{aligned}$$

Below, we are going to express the one-form $\omega_3 = s_2\omega$ in the form $\omega_3 = dS_3 + s_3dH$ in order to calculate $M_4(h) = \oint_{\delta(h)} \omega_4$ where $\omega_4 = s_3\omega$. As above, we can perform our calculations modulo exact forms. Thus,

$$\omega_3 = s_2\omega = s_2LdH + \text{(odd part)} + \text{(even part)},$$

$$\begin{aligned}
\text{(odd part)} = {} & \kappa y[(2q_{20}-L)dB + d((q_{02}-\tfrac{1}{3}L)y^2+q_{04}y^4)] \\
& + [\tfrac{2}{3}L^2+2q_{02}L+4q_{04}(2LH-A)]dQ_1 \\
& = \kappa y[(2q_{20}-L)dB - \tfrac{1}{3}d(Ly^2)] \\
& - Q_1(\tfrac{4}{3}LL'+2q_{02}L'+8q_{04}HL'-8q_{04}BL')dx - 8q_{04}LQ_1dH \\
& = \kappa y(2q_{20}+\tfrac{1}{3}L+2q_{02}+4q_{04}y^2)dB - \tfrac{1}{3}\kappa y d(Ly^2) - 8q_{04}LQ_1dH \\
& = [\kappa y(2q_{20}+\tfrac{1}{3}L+2q_{02}+4q_{04}y^2) - 8q_{04}LQ_1]dH \\
& - \tfrac{1}{3}\kappa y d(Ly^2) - \tfrac{1}{3}\kappa y^2 L dy \\
& = [\kappa y(2q_{20}+\tfrac{1}{3}L+2q_{02}+4q_{04}y^2) - 8q_{04}LQ_1]dH.
\end{aligned}$$

We used that $-Q_1L' = \kappa yB'$ and $\frac{1}{2}y^2 = H - B$. Similarly, by using the identity $(2q_{20}-L)dB = d[(2q_{20}-L)B + \frac{1}{2}A]$ one obtains

$$\begin{aligned}
\text{(even part)} = {} & \kappa y dQ_1 + [\tfrac{2}{3}L^2+2q_{02}L+4q_{04}(2LH-A)]\times \\
& \times[(2q_{20}-L)dB + d((q_{02}-\tfrac{1}{3}L)y^2+q_{04}y^4)] \\
& = -\kappa Q_1 dy - [(2q_{20}-L)B+\tfrac{1}{2}A+(q_{02}-\tfrac{1}{3}L)y^2+q_{04}y^4]\times \\
& \times d[\tfrac{2}{3}L^2+2q_{02}L+4q_{04}(2LH-A)] \\
& = -[(2q_{20}-L)B+\tfrac{1}{2}A+(q_{02}-\tfrac{1}{3}L)y^2+q_{04}y^4][\tfrac{4}{3}L+2q_{02}+4q_{04}y^2]dL \\
& - 8q_{04}L[(2q_{20}-L)B+\tfrac{1}{2}A+(q_{02}-\tfrac{1}{3}L)y^2+q_{04}y^4]dH - \kappa y^{-1}Q_1dH \\
& = -[2q_{02}^2+\tfrac{2}{3}q_{02}L-\tfrac{4}{9}L^2+4q_{04}((2q_{20}-L)B+\tfrac{1}{2}A)]y^2dL \\
& - (6q_{02}q_{04}y^4+4q_{04}^2y^6)dL \\
& - 8q_{04}L[(2q_{20}-L)B+\tfrac{1}{2}A+(q_{02}-\tfrac{1}{3}L)y^2+q_{04}y^4]dH - \kappa y^{-1}Q_1dH \\
& = [4q_{02}^2L+\tfrac{2}{3}q_{02}L^2-\tfrac{8}{27}L^3+8q_{04}X]dH \\
& + [24q_{02}q_{04}(2LH-A)+96q_{04}^2(LH^2-AH+Y)]dH \\
& - 8q_{04}L[(2q_{20}-L)B+\tfrac{1}{2}A+(q_{02}-\tfrac{1}{3}L)y^2+q_{04}y^4]dH - \kappa y^{-1}Q_1dH
\end{aligned}$$

where $dX = [(2q_{20}-L)B + \frac{1}{2}A]dL$ and $dY = B^2dL$. Finally, summing up all terms with dH, we obtain the expression

$$\begin{aligned}
s_3 = {} & \kappa y(2q_{20}+\tfrac{4}{3}L+2q_{02}+4q_{04}y^2) - 8q_{04}LQ_1 \\
& + 4q_{02}^2L+\tfrac{8}{3}q_{02}L^2+\tfrac{10}{27}L^3+4q_{04}(L+6q_{02})(2LH-A) \\
& + 8q_{04}X+96q_{04}^2(LH^2-AH+Y) \\
& - 8q_{04}L[(2q_{20}-L)B+\tfrac{1}{2}A+(q_{02}-\tfrac{1}{3}L)y^2+q_{04}y^4] - \kappa y^{-1}Q_1.
\end{aligned}$$

It should be mentioned that some terms in s_3 were included in s_2 and $s_1 = L$, too. Since $M_2(h) = \oint_{\delta(h)} s_1\omega \equiv 0$, the terms H^kL have no impact in the values of $M_3(h) = \oint_{\delta(h)} s_2\omega$ and $M_4(h) =$

$\oint_{\delta(h)} s_3\omega$. In the proof of Proposition 3, we have established that $J_2 = \oint_{\delta(h)}(2LH - A)\omega \equiv 0$. By $M_3(h) \equiv 0$, one obtains that the terms $H^k A$ and $\frac{2}{3}L^2 + \kappa y$ will have no impact on the value of $M_4(h)$, too. Using these facts, one can rewrite $M_4(h)$ in the form

$$M_4(h) = \oint\limits_{\delta(h)} (\sigma_1\omega + \sigma_2\omega + \sigma_3\omega) = K_1 + K_2 + K_3$$

where

$$\begin{aligned}
\sigma_1 &= \kappa y(2q_{20} - 2q_{02} + \tfrac{4}{3}L + 4q_{04}y^2) - 8q_{04}LQ_1 \\
&\quad - 8q_{04}L[(2q_{20} - L)B + \tfrac{1}{2}A + (q_{02} - \tfrac{1}{3}L)y^2 + q_{04}y^4], \\
\sigma_2 &= \tfrac{10}{27}L^3 + 4q_{04}L(2LH - A) + 8q_{04}X + 96q_{04}^2 Y, \\
\sigma_3 &= -\kappa y^{-1}Q_1.
\end{aligned}$$

Below, we are going to verify that $K_1 + K_2 = 0$. Therefore

$$M_4(h) = \oint\limits_{\delta(h)} \sigma_3\omega = \oint\limits_{\delta(h)} \sigma_3 dQ_1 = \kappa \oint\limits_{\delta(h)} y^{-1}Q_{1x}Q_1 dx.$$

Then the three formulas in Proposition 4 follow by simple calculations making use of (13). Since it is assumed that $(|\lambda| + |\mu|)q_{11} \neq 0$, $M_4(h)$ is not identically zero. Note that the coefficient at the third formula in Proposition 4 vanishes for $2\lambda + 3\mu = 0$, however this is equivalent to $a = \frac{8}{9}$, a value corresponding to the symmetric eight loop, which was excluded from consideration here.

To finish the proof, it remains to calculate K_2 and K_1. We obtain (modulo one-forms $dR + rdH$ which yield zero integrals)

$$\begin{aligned}
\sigma_2\omega &= \sigma_2 dQ_1 = -Q_1 d[\tfrac{10}{27}L^3 + 4q_{04}(2L^2H - LA) + 8q_{04}X + 96q_{04}^2 Y] \\
&= -Q_1[\tfrac{10}{9}L^2L' + 8q_{04}(2LH - \tfrac{1}{2}A - BL)L' \\
&\quad + 8q_{04}((2q_{20} - L)B + \tfrac{1}{2}A)L' + 96q_{04}^2B^2L']dx \\
&= -Q_1L'[\tfrac{10}{9}L^2 + 8q_{04}(2q_{20}B + Ly^2) + 96q_{04}^2B^2]dx \\
&= \kappa y[\tfrac{10}{9}L^2 + 8q_{04}(2q_{20}B + Ly^2) + 96q_{04}^2B^2]dB \\
&= \kappa y(\tfrac{10}{9}L^2 + 8q_{04}Ly^2)d(H - \tfrac{1}{2}y^2) = -\kappa(\tfrac{10}{9}L^2y^2 + 8q_{04}Ly^4)dy \\
&= \kappa(\tfrac{20}{27}Ly^3 + \tfrac{8}{5}q_{04}y^5)L'dx.
\end{aligned}$$

Finally (again modulo one-forms $dR + rdH$),

$$
\begin{aligned}
\sigma_1\omega &= [\kappa y(2q_{20} - 2q_{02} + \tfrac{4}{3}L + 4q_{04}y^2) - 8q_{04}LQ_1]\omega \\
&- 8q_{04}L[(2q_{20} - L)B + \tfrac{1}{2}A + (q_{02} - \tfrac{1}{3}L)y^2 + q_{04}y^4]dQ_1 \\
&= [\kappa y(2q_{20} - 2q_{02} + \tfrac{4}{3}L + 4q_{04}y^2) - 8q_{04}LQ_1]\omega \\
&+ 8q_{04}Q_1L'[(2q_{20} - L)B + \tfrac{1}{2}A + (q_{02} - \tfrac{1}{3}L)y^2 + q_{04}y^4]dx + 8q_{04}LQ_1\omega \\
&= \kappa y(2q_{20} - 2q_{02} + \tfrac{4}{3}L + 4q_{04}y^2)\omega \\
&- 8q_{04}\kappa y[(2q_{20} - L)B + \tfrac{1}{2}A + (q_{02} - \tfrac{1}{3}L)y^2 + q_{04}y^4]dB \\
&= \kappa y(2q_{20} - 2q_{02} + \tfrac{4}{3}L + 4q_{04}y^2)\omega \\
&+ 8q_{04}\kappa y^2[(2q_{20} - L)B + \tfrac{1}{2}A + (q_{02} - \tfrac{1}{3}L)y^2 + q_{04}y^4]dy \\
&= \kappa y(2q_{20} - 2q_{02} + \tfrac{4}{3}L + \tfrac{4}{3}q_{04}y^2)\omega.
\end{aligned}
$$

Since

$$
\begin{aligned}
\omega &= d[(2q_{20} - L)B + \tfrac{1}{2}A + (q_{02} - \tfrac{1}{3}L)y^2 + q_{04}y^4] \\
&= [(2q_{02} - 2q_{20} + \tfrac{1}{3}L)y + 4q_{04}y^3]dy - \tfrac{1}{3}y^2dL,
\end{aligned}
$$

we last obtain by easy calculations that $\sigma_1\omega = -\kappa(\frac{20}{27}Ly^3 + \frac{8}{5}q_{04}y^5)dL$. Proposition 4 is proved. $\square$

3. The Petrov module

In this section we recall the algebraic structure of the set of Abelian integrals and differentials, related to the bivariate polynomial H, defined by (1). The polynomial H is semiweighted homogeneous, with weighted degrees $\deg x = 1$, $\deg y = 2$, and highest weighted homogeneous part $\frac{1}{2}y^2 + \frac{a}{4}x^4$ ($a \neq 0$). For such polynomials the theory developed in [3] applies. Namely, define as in the preceding sections the differential one-forms and Abelian integrals

$$
\omega_k = x^k y\, dx, \quad I_k = I_k(h) = \int_{\delta(h)} \omega_k, \; k = 0, 1, 2
$$

and consider the function space

$$
\mathcal{A}_H = \left\{ \int_{\delta(h)} \omega : \omega = P dx + Q dy,\, P, Q \in \mathbb{R}[x, y] \right\}
$$

where $\mathbb{R}[x, y]$ is the ring of polynomials in x, y. It is an infinite dimensional real vector space, having an additional structure of a $\mathbb{R}[h]$-module, defined by the multiplication

$$
h \cdot \int_{\delta(h)} \omega = \int_{\delta(h)} H(x, y)\omega.
$$

Clearly, if $\omega = dA + BdH$ for suitable functions A, B, then the corresponding Abelian integral $\int_{\delta(h)} \omega$ vanishes identically. This motivates the introduction of the quotient vector space of differential one-forms

$$\mathcal{P}_H = \frac{\Omega^1}{d\Omega^0 + \Omega^0 dH}$$

where Ω^1 is the vector space of polynomial one-forms on $\mathbb{R}^2$ and $\Omega^0 = \mathbb{R}[x, y]$. Similarly to $\mathcal{A}_H$, the infinite dimensional vector space $\mathcal{P}_H$ has an additional structure of $\mathbb{R}[h]$-module with multiplication defined by

$$h \cdot \omega = H(x, y)\omega.$$

Recall that a free module is a module having a basis (a generating set, linearly independent over the ring of coefficients $\mathbb{R}[h]$). The next Proposition has been known to Petrov [11], but its proof in a more general setting goes back to Ilyashenko [9]. For more details we refer to [3].

Proposition 5.

- *The $\mathbb{R}[h]$-module $\mathcal{P}_H$ is freely generated by $\omega_0, \omega_1, \omega_2$.*
- *The $\mathbb{R}[h]$-module $\mathcal{A}_H$ is freely generated by I_0, I_1, I_2.*
- *The natural map*

$$\mathcal{P}_H \to \mathcal{A}_H \tag{15}$$

$$\omega \mapsto \int_{\delta(h)} \omega \tag{16}$$

is an isomorphism of $\mathbb{R}[h]$ modules.

The precise meaning that $\mathcal{A}_H$ is freely generated by I_0, I_1, I_2 is as follows.

Corollary 2. *Let $\alpha(h), \beta(h), \gamma(h)$ be (real or complex) polynomials in h. The Abelian integral*

$$I(h) = \alpha(h)I_0(h) + \beta(h)I_1(h) + \gamma(h)I_2(h) \tag{17}$$

is identically zero, if and only if $\alpha(h), \beta(h), \gamma(h)$ are identically zero.

From now on we denote by $\mathcal{A}_n$ the vector space of Abelian integrals of the form (17), with

$$\deg \alpha \le n, \deg \beta \le n, \deg \gamma \le n.$$

Clearly the dimension of the vector space $\mathcal{A}_n$ is at most $3(n + 1)$. The vector space of Abelian integrals $\mathcal{A}_n$ coincides, however, with the space of Abelian integrals

$$\int_{\delta(h)} P(x, y)dx + Q(x, y)dy \tag{18}$$

where P, Q are real polynomials of weighted degree $4n+5$, where the weight of x is 1 and the weight of y is 2. Therefore, according to [3, p. 582] we obtain

Corollary 3.

$$\dim \mathcal{A}_H = 3 + [n + \frac{2}{4}] + [n + \frac{1}{4}] + [n] = 3(n+1).$$

Let h be a non-critical value of H. The complex algebraic curve

$$\Gamma_h = \{x, y) \in \mathbb{C}^2 : H(x, y) = h\} \tag{19}$$

has the topological type of a torus with two punctures. It follows that its first homology (co-homology) group $H_1(\Gamma_h, \mathbb{Z})$ of the Riemann surface Γ_h is of dimension three. The algebraic form of De Rham theorem says that the first cohomology group $H^1(\Gamma_h, \mathbb{C})$ of Γ_h is generated by polynomial one-forms restricted to Γ_h. Proposition 5 then implies

Corollary 4. *The cohomology classes of the restrictions of the one-forms $\omega_0, \omega_1, \omega_2$ on Γ_h generate the vector space $H^1(\Gamma_h, \mathbb{C})$.*

Proof of Proposition 5. As

$$d\omega_0 = -dx \wedge dy, d\omega_1 = -xdx \wedge dy, d\omega_0 = -x^2 dx \wedge dy,$$

and the monomials $1, x, x^2$ generate the quotient ring $\mathbb{C}[x, y]/ < H_x, H_y >$, then $\mathcal{P}_H$ is a free $\mathbb{R}[h]$-module generated by $\omega_0, \omega_1, \omega_2$, see [3, Theorem 1.1]. It is straightforward to check that the natural map (15) $\mathcal{P}_H \to \mathcal{A}_H$ is a surjective morphism of $\mathbb{R}[h]$-modules. To complete the proof, it remains to show that the kernel of this morphism is trivial.

Recall that if $A_c = \{h_1, h_2, h_3\}$ is the set of critical values of H, and if h_{reg} is some regular critical value, then any closed loop $l \subset \mathbb{C} \setminus A_c$ starting and terminating at h_{reg} induces an automorphism (monodromy)

$$l_* : H_1(\Gamma_{h_{reg}}, \mathbb{Z}) \to H_1(\Gamma_{h_{reg}}, \mathbb{Z}).$$

This automorphism depends only on the homotopy class of l so it defines a group morphism (monodromy representation)

$$\pi_1(A_c, h_{reg}) \to Aut(H_1(\Gamma_{h_{reg}}, \mathbb{Z})).$$

The key observation is that the orbit of the cycle $\delta(h)$, under the action of the fundamental group $\pi_1(A_c, h)$, spans the first homology group $H_1(\Gamma_h, \mathbb{Z})$. Indeed, if this were true, it would imply that if an Abelian integral $I(h) = \int_{\{\delta(h)} \omega$ vanishes identically, then $\int_{\gamma(h)} \omega = 0$ for any $\gamma(h) \in H_1(\Gamma_h, \mathbb{Z})$. Thus the cohomology class of the differential ω on Γ_h is trivial, which on its turn implies that $\omega = 0$ in $\mathcal{P}_H$, see [3, Theorem 1.2]. Therefore the kernel of the morphism (15) is trivial.

To this end, we check that the orbit of $\delta(h)$ generates $H_1(\Gamma_h, \mathbb{Z})$. Suppose first that the oval corresponding to $\delta(h)$ surrounds a single elliptic critical point (a center) of H. If H has three

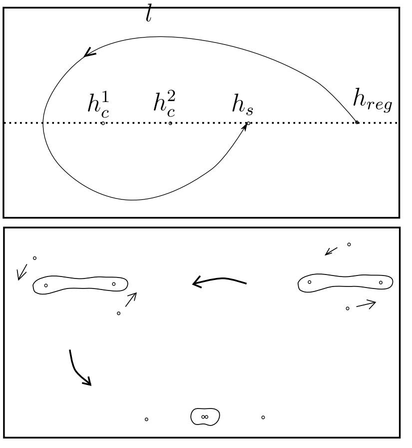

Fig. 1. The path l along which the cycle $\delta(h)$ vanishes, and deformation of the roots of the polynomial $\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{a}{4}x^4 - h$.

distinct critical values, the claim that the orbit of $\delta(h)$ spans the first homology group $H_1(\Gamma_h, \mathbb{Z})$ follows from the proof of [3, Proposition 3.2]. In the degenerate cuspidal case ($a = 1$), the local analysis of the branching of $\delta(h)$ near $h = 0$ leads to the same conclusion.

Consider finally the case when $\delta(h)$ is represented by an oval, which belongs to the exterior period annulus of $\{dH = 0\}$ (the so-called exterior eight-loop case, in which $0 < a < 1$). Clearly, $\delta(h)$ does not surround a single elliptic critical point, but this condition can be relaxed. The arguments from the proof of [3, Proposition 3.2] hold true also for cycles $\delta(h)$, which vanish at any Morse critical point, along suitable path in $\mathbb{C}$. The cycle $\delta(h)$ turns out to be vanishing at the saddle point $(0, 0)$ of $\{dH = 0\}$ along a suitable path in the complex h-plane. Indeed, with the self-explaining notations of Fig. 5, let $h_{reg} \in (h_s, \infty)$ be a real regular value of H and consider the path l on the complex h-plane, shown in Fig. 1. Let $\delta(h)$, be represented by a closed loop on the algebraic curve Γ_h. The projection of this closed loop on the complex x-plane, and its deformation as h follows the path l is shown in Fig. 1. The conclusion is that the cycle $\delta(h)$ vanishes at the saddle point when h tends to the corresponding critical value h_s along the path l. This completes the proof of the proposition. $\square$

4. Zeroes of Abelian integrals

In this section we find upper bounds for the number of the zeroes of the Abelian integrals $\mathcal{A}_n$ defined in (17) on the interval of existence of the ovals $\delta(h)$. The dimension of $\mathcal{A}_n$ equals $3(n+1)$, see Corollary 3. Similar results were earlier obtained for the space of non-weighted Abelian integrals (18) (deg P, deg $Q \le n$) by Petrov [11] and Liu [10], see the survey of Christopher and Li [1]. Our upper bounds however do not follow from the aforementioned papers. They will be proved along the lines, given in [4, section 7].

All families of cycles will depend continuously on a parameter h and will be defined without ambiguity in the complex half-plane $\{h : Im(h) > 0\}$. This will allow a continuation on $\mathbb{C}$ along any curve avoiding the real critical values of H. In particular, it will be supposed that all three critical values of H are real.

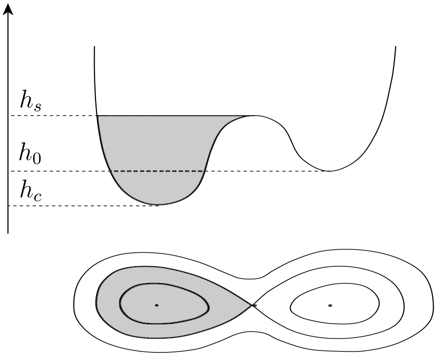

Fig. 2. The graph of the polynomial $\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{a}{4}x^4$, $\frac{8}{9} < a < 1$, and the level sets $\{H = h\}$.

4.1. The interior eight-loop case

Using the normal form (1) we can suppose that $\frac{8}{9} < a < 1$. The Hamiltonian system $dH = 0$ has an eight-loop containing in its interior two continuous families of ovals, vanishing at a singular point, which is a center, as it is shown on Fig. 2. We denote by $\delta(h) \subset \{H = h\}$ either of these two families.

Theorem 6. *The space of Abelian integrals $\mathcal{A}_n$ corresponding to Fig. 2 is Chebyshev on the interval of existence of $\delta(h)$.*

Proof. H has three critical values $h_c = 0 < h_0 < h_s$, where h_s corresponds to a saddle, and h_c, h_0 to centers of the Hamiltonian system $dH = 0$. Let $\delta(h) \subset \{H = h\}$ be the continuous family of ovals defined on a maximal open interval $\Sigma = (h_c, h_s)$, where for $h = h_c = 0$ the oval degenerates to a point $\delta(h_c)$ which is a center and for $h = h_s > 0$ the oval becomes a homoclinic loop of the Hamiltonian system $dH = 0$. The family $\{\delta(h)\}$ represents a continuous family of cycles vanishing at the center $\delta(h_c)$.

We note first that $I_0(h)$, $I_1(h)$, $I_2(h)$ can be expressed as linear combinations of $I_0'(h)$, $I_1'(h)$, $I_2'(h)$, whose coefficients are polynomials in h of degree one. Therefore the vector space

$$\mathcal{A}_n' = \{I'(h) : I(h) \in \mathcal{A}_n\}$$

coincides with the vector space of Abelian integrals

$$\{\alpha(h)I_0'(h) + \beta(h)I_1'(h) + \gamma(h)I_2'(h) : \deg\alpha \le n, \deg\beta \le n, \deg\gamma \le n\}.$$

We shall prove the Chebyshev property of $\mathcal{A}_n'$ in the complex domain

$$\mathcal{D} = \mathbb{C} \setminus [h_s, \infty),$$

in which $I'(h)$ has an analytic extension, see Fig. 3. For this purpose we apply the argument principle to the function

$$F(h) = \frac{I'(h)}{I_0'(h)}.$$

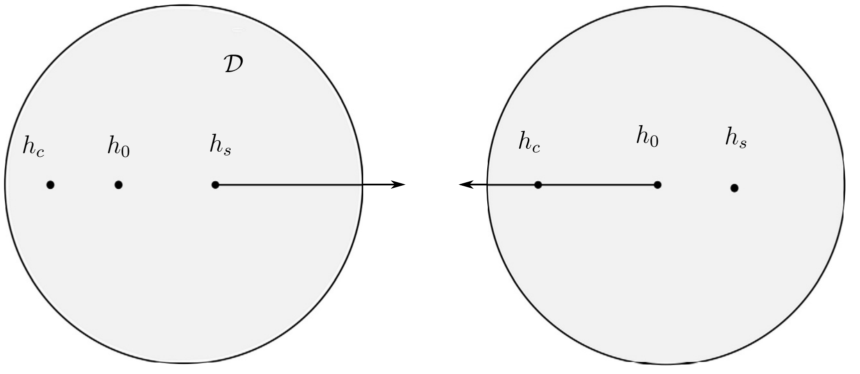

Fig. 3. The branch cuts and the domains of the Abelian integral $I(h)$ and $W_{\delta,\delta_s}(\omega', \omega_0')$ respectively.

We note that $I_0'(h)$ is a complete elliptic integral of first kind and hence cannot vanish in $\mathcal{D}$. For sufficiently big $|h|$ the function $F(h)$ behaves as $h^{n+\frac{1}{2}}$ and hence the increment of the argument of F along a circle with a sufficiently big radius is close to $(2n+1)\pi$. Along the interval $[h_s, \infty)$ the imaginary part of $F(h)$ can be computed by making use of the Picard–Lefschetz formula. Namely, let $\{\delta_s(h)\}_h$ be a continuous family of cycles, vanishing at the saddle point as h tends to h_s. Then along $[h_s, \infty)$ the family $\delta(h)$ has two analytic complex-conjugate continuations $\delta^\pm(h)$, $\delta^+ = \bar{\delta^-}$ and moreover, by the Picard–Lefschetz formula the cycle

$$\delta^+(h) - \delta^-(h) = \delta_s(h)$$

where the identity should be understood up to homology equivalence. This implies the following identity along $[h_s, \infty)$

$$2Im(F(h)) = \frac{\int_{\delta^+(h)} \omega'}{\int_{\delta_0^+(h)} \omega_0'} - \frac{\int_{\delta^-(h)} \omega'}{\int_{\delta_0^-(h)} \omega_0'} = \frac{W_{\delta,\delta_s}(\omega', \omega_0')}{|I_0'(h)|^2}$$

where

$$W_{\delta,\delta_s}(\omega', \omega_0') = \det \begin{pmatrix} \int_{\delta(h)} \omega' & \int_{\delta_s(h)} \omega' \\ \int_{\delta(h)} \omega_0' & \int_{\delta_s(h)} \omega_0' \end{pmatrix}.$$

Following [4, section 7] we may use the reciprocity law on the elliptic curve $\{H = h\}$ to compute

$$W_{\delta,\delta_s}(\omega', \omega_0') = p(h) + q(h) \int_{-\infty}^{+\infty} \frac{dx}{y}$$

where $p(h), q(h)$ are suitable degree n polynomials, $\pm\infty$ are the two "infinite" points on the compactified Riemann surface Γ_h, and the integration is along some path connecting $\pm\infty$ on Γ_h.

It is easy to check now that the function $p(h) + q(h) \int_{-\infty}^{+\infty} \frac{dx}{y}$ can have at most $2n+1$ zeroes on $[h_s, \infty)$. For this, consider an analytic continuation of this function to the complex domain

$\mathbb{C} \setminus (-\infty, h_0]$ where h_0 is a critical value of H, $h_0 < h_s$, see Fig. 3. By the Picard–Lefschetz formula, the imaginary part of $p(h) + q(h) \int_{-\infty}^{+\infty} \frac{dx}{y}$ along the branch cut $(-\infty, h_0)$ equals

$$\tilde{q}(h) \int\limits_{\delta(h)} \frac{dx}{y}$$

where $\tilde{q}$ differs from q by an addition of a constant. We conclude that the imaginary part of this function vanishes at most n times. This combined to the asymptotic behavior

$$p(h) + q(h) \int\limits_{-\infty}^{+\infty} \frac{dx}{y} \sim h^n \times const.$$

gives that the increase of the argument along a big circle is close to $2\pi n$ and finally, that our function can have at most $2n + 1$ zeroes on $\mathbb{C} \setminus (-\infty, h_0]$. Now we come back to the function $F(h)$ and conclude that it can have at most $3n + 2$ zeroes in the complex domain $\mathcal{D}$, counted with the multiplicity. As $I(0) = 0$ the same conclusion holds true for $I(h)$ on the real interval $(-\infty, h_s)$.

Finally, note that the second continuous family of ovals $\{\delta(h)\}_{h\in\Sigma}$, $\Sigma = (h_0, h_s)$, $h_0 > 0$, is easier to study. This is so, because the corresponding Wronskian $W_{\delta,\delta_s}(\omega', \omega_0')$ has only one singular point (instead of two). Therefore we do not consider this case in details. □

Remark 1. Through the proof we did not inspect the behavior of $F(h)$ near the branch point h_s. In the original papers of Petrov a small circle centered at h_s is removed and the behavior of F along it is taken into account. It is important to note that, we do not remove a small circle here, because we use a slightly improved version of the argument principle, as explained in section 2.4 of [5]. It allows one to apply the argument principle, even if F is not analytic at $F(h_s)$, provided that F has a continuous limit at h_s, which is not zero. The case when $F(h_s) = 0$ is studied then by a small perturbation (by adding a real constant for instance) – this does not decrease the number of zeroes of F in the complement of the branch cut. Of course, the same considerations hold true for the function $\int_{-\infty}^{+\infty} \frac{dx}{y}$ in its respective domain of analyticity.

4.2. The saddle-loop case

In the normal form (1) we suppose that $a < 0$. As before, we let $\delta(h) \subset \{H = h\}$ be a continuous family of ovals defined on a maximal open interval $\Sigma = (h_c, h_s)$, where for $h = h_c = 0$ the oval degenerates to a point $\delta(h_c)$ which is a center and for $h = h_s > 0$ the oval becomes a homoclinic loop of the Hamiltonian system $dH = 0$. The family $\{\delta(h)\}$ represents a continuous family of cycles vanishing at the center $\delta(h_c)$.

Theorem 7. *The space of Abelian integrals $\mathcal{A}_n$ corresponding to the shadowed area in Fig. 4 is of dimension $3n + 3$, and each Abelian integral from $\mathcal{A}_n$ can have at most $4n + 3$ zeroes.*

Proof. We shall prove the Chebyshev property of $\mathcal{A}_n'$ in the complex domain

$$\mathcal{D} = \mathbb{C} \setminus [h_s, \infty),$$

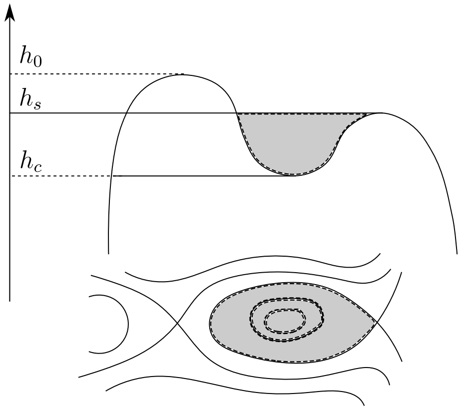

Fig. 4. The graph of the polynomial $\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{a}{4}x^4$, $a < 0$, and the level sets $\{H = h\}$.

in which $I'(h)$ has an analytic extension. For this purpose we apply the argument principle to the function

$$F(h) = \frac{I'(h)}{I_0'(h)}.$$

Indeed, a local analysis shows that at h_s, h_0 the function $F|_{\mathcal{D}}$ has continuous limits, which we assume to be non-zero. $I_0'(h)$ is a complete elliptic integral of first kind and hence cannot vanish in $\mathcal{D}$. For sufficiently big $|h|$ the function $F(h)$ behaves as $h^{n+\frac{1}{2}}$ and hence the increment of the argument of F along a circle with a sufficiently big radius is close to $(2n+1)\pi$. Along the intervals (h_s, h_0) and (h_0, ∞) the imaginary part of $F(h)$ can be computed by making use of the Picard–Lefschetz formula. Namely, let $\{\delta_s(h)\}_h$, $\{\delta_0(h)\}_h$ be the continuous family of cycles, vanishing at the saddle points h_s and h_0 respectively, as h tends to h_s and h_0. As in the preceding section we deduce that along $[h_s, h_0)$,

$$2Im(F(h)) = \frac{W_{\delta,\delta_s}(\omega', \omega_0')}{|I_0'(h)|^2},\ \ h \in (h_s, h_0)$$

while along (h_0, ∞)

$$2Im(F(h)) = \frac{W_{\delta,\delta_s}(\omega', \omega_0')}{|I_0'(h)|^2} + \frac{W_{\delta,\delta_0}(\omega', \omega_0')}{|I_0'(h)|^2},\ \ h \in (h_0, \infty).$$

The function

$$W_{\delta,\delta_s}(\omega', \omega_0'),\ h \in (h_s, h_0)$$

allows an analytic continuation in $\mathbb{C} \setminus [h_0, \infty)$ and exactly as in the preceding section we compute that it can have at most $2n+1$ zeroes there. More precisely, $W_{\delta,\delta_s}(\omega', \omega_0')$ has an analytic continuation in $\mathbb{C} \setminus [h_0, \infty)$. The number of its zeroes in this domain is bounded by n (coming from the behavior at infinity) plus one plus the number of the zeroes of

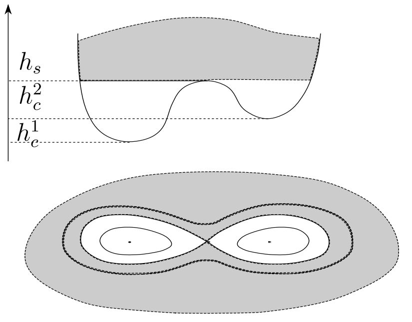

Fig. 5. The graph of the polynomial $\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{a}{4}x^4$, $\frac{8}{9} < a < 1$, and the level sets $\{H = h\}$.

$$2Im(W_{\delta,\delta_s}(\omega', \omega_0')) = W_{\delta_0,\delta_s}(\omega', \omega_0') = q(h) \int_{-\infty}^{+\infty} \frac{dx}{y}, \quad h \in (h_0, \infty)$$

where q is a degree n polynomial. Similarly, the function

$$W_{\delta,\delta_s}(\omega', \omega_0') + W_{\delta,\delta_0}(\omega', \omega_0'), \ (h_0, \infty)$$

allows an analytic continuation in $\mathbb{C} \setminus [h_s, h_0]$ and its zeroes there are bounded by n plus plus one plus the number of the zeroes of

$$2Im(W_{\delta,\delta_s}(\omega', \omega_0') + W_{\delta,\delta_0}(\omega', \omega_0')) = W_{\delta_s,\delta_0}(\omega', \omega_0') = q(h) \int_{-\infty}^{+\infty} \frac{dx}{y}, \quad h \in (h_s, h_0).$$

Summing up the above information, we get that the function $F(h)$ can have at most $4n + 3$ zeroes in the complex domain $\mathcal{D}$, counted with the multiplicity. As $I(0) = 0$ the same conclusion holds true for $I(h)$ on the real interval $(-\infty, h_s)$. $\square$

4.3. The exterior eight-loop case

In this section we consider the exterior eight-loop case, with period annulus as shown in Fig. 5 and $\frac{8}{9} < a < 1$. Let $\delta(h) \subset \{H = h\}$ be the continuous family of ovals defined on the maximal open interval $\Sigma = (h_s, \infty)$

Theorem 8. *The space of Abelian integrals $\mathcal{A}_n$ corresponding to the shadowed area in Fig. 5 is of dimension $3n + 3$, and each Abelian integral from $\mathcal{A}_n$ can have at most $4n + 4$ zeroes.*

Proof. We shall evaluate the number of the zeroes of a function from $\mathcal{A}_n'$ in the complex domain

$$\mathcal{D} = \mathbb{C} \setminus (-\infty, h_s],$$

in which $I'(h)$ has an analytic extension. For this purpose we apply the argument principle to the function

$$F(h) = \frac{I'(h)}{I_0'(h)}.$$

As before, a local analysis shows that at h_s, h_c^1, h_c^2 the function $F|_{\mathcal{D}}$ has continuous limits, which we assume to be non-zero. $I_0'(h)$ is a complete elliptic integral of first kind and hence cannot vanish in $\mathcal{D}$. For sufficiently big $|h|$ the function $F(h)$ behaves as $h^{n+\frac{1}{2}}$ and hence the increment of the argument of F along a circle with a sufficiently big radius is close to $(2n+1)\pi$. It remains to study the number of the zeroes of the imaginary part of $F(h)$ along the intervals

$$(-\infty, h_c^1),\ (h_c^1, h_c^2),\ (h_c^2, h_s).$$

Namely, let $\{\delta_s(h)\}_h$, $\{\delta_c^1(h)\}_h$, $\{\delta_c^2(h)\}_h$, where $Im h \geq 0$, be the continuous family of cycles, vanishing at the saddle points as h tends to h_s, and h_c^1 or h_c^2, respectively. These cycles are defined up to an orientation, and we consider their continuation to $\mathcal{D} = \mathbb{C} \setminus (-\infty, h_s]$, as well the limits along the branch cut $(-\infty, h_s]$. The family of exterior loops $\{\delta(h)\}$ is expressed in terms of these vanishing cycles as follows

$$\delta(h) = \delta_c^1(h) + \delta_c^2(h) + \delta_s(h),\ h \in \mathcal{D}$$

(the orientations of the vanishing cycles are fixed from this identity). Let $\delta^+(h) = \delta(h)$ be the continuation of $\delta(h)$ on $(-\infty, h_s]$, along paths contained in the upper complex half-plane, and $\delta^-(h)$ be the continuation on $(-\infty, h_s]$ along paths contained in the lower complex half-plane. The Picard–Lefschetz formula easily implies

$$\delta^-(h) = \delta_c^1(h) + \delta_c^2(h) - \delta_s(h), h \in (h_c^2, h_s)$$
$$\delta^-(h) = \delta_c^1(h) - \delta_s(h), h \in (h_c^1, h_c^2)$$
$$\delta^-(h) = -\delta_s(h), h \in (-\infty, h_c^1)$$

As in the preceding section we deduce that along the branch cut $(-\infty, h_s)$ we have

$$2Im(F(h)) = \frac{W_{\delta,2\delta_s}(\omega', \omega_0')}{|I_0'(h)|^2},\ \ h \in (h_c^2, h_s) \tag{20}$$

and

$$2Im(F(h)) = \frac{W_{\delta,2\delta_s}(\omega', \omega_0')}{|I_0'(h)|^2} + \frac{W_{\delta,\delta_c^2}(\omega', \omega_0')}{|I_0'(h)|^2},\ \ h \in (h_c^1, h_c^2) \tag{21}$$

and

$$2Im(F(h)) = \frac{W_{\delta,2\delta_s}(\omega', \omega_0')}{|I_0'(h)|^2} + \frac{W_{\delta,\delta_c^1+\delta_c^2}(\omega', \omega_0')}{|I_0'(h)|^2} = \frac{W_{\delta,\delta_s}(\omega', \omega_0')}{|I_0'(h)|^2},\ \ h \in (-\infty, h_c^1). \tag{22}$$

Clearly, the function $W_{\delta,\delta_s}(\omega', \omega_0')$ has an analytic continuation in $\mathbb{C} \setminus [h_c^1, h_c^2]$. Its number of zeroes in this domain depends on the zeroes of

$$2Im(W_{\delta,\delta_s}(\omega',\omega_0'))=W_{\delta_c^1,\delta_c^2}(\omega',\omega_0')=q(h)\int\limits_{-\infty}^{+\infty}\frac{dx}{y},\ \ h\in(h_c^1,h_c^2).$$

Thus, the total number of the zeroes of the functions (20), (22) is bounded by $n+1$ plus the number of the zeroes of $q(h)$ on the interval (h_c^1,h_c^2). Finally, similar considerations show that the function (21) has an analytic continuation in

$$\mathbb{C}\setminus\{(-\infty,h_c^1)\cup(h_c^2,\infty)\}$$

and its zeroes in this domain are bounded by $n+1$ plus the number of the zeroes of the polynomial $q(h)$ on the interval $(-\infty,h_c^1)\cup(h_c^2,\infty)$.

Summing up the above information, we get that the function $F(h)$ can have at most $4n+3$ zeroes in the complex domain $\mathcal{D}$, counted with the multiplicity. Therefore the Abelian integral $I(h)$ has at most $4n+4$ zeroes on the real interval $(-\infty,h_s)$. □

5. Lower bounds for the number of zeroes of $M_k(h)$

In this section we provide examples which show that Chebyshev's property would not hold in the saddle-loop case. For this purpose, we study the number of small-amplitude limit cycles bifurcating around the center at the origin.

We begin with the system satisfied by the basic integrals $I_k(h)$. It is derived in a standard way by using (1), (13) and the formula $I_k'(h)=\oint_{\delta(h)}(x^k/y)dx$.

Lemma 1. *The integrals $I_0(h)$, $I_1(h)$ and $I_2(h)$ satisfy the system*

$$\begin{aligned}&\tfrac{4}{3}hI_0'-\tfrac{2}{9a}I_1'-(\tfrac{1}{3}-\tfrac{4}{9a})I_2'=I_0,\\&\tfrac{2}{9a}hI_0'+(h+\tfrac{1}{4a}-\tfrac{10}{27a^2})I_1'-(\tfrac{13}{18a}-\tfrac{20}{27a^2})I_2'=I_1,\\&-(\tfrac{4}{15a}-\tfrac{56}{135a^2})hI_0'+(\tfrac{4}{15a}h+\tfrac{29}{45a^2}-\tfrac{56}{81a^3})I_1'+(\tfrac{4}{5}h+\tfrac{4}{15a}-\tfrac{46}{27a^2}+\tfrac{112}{81a^3})I_2'=I_2.\end{aligned}$$

We use this system to find the expansions of integrals I_k, $k=0,1,2$ near $h=0$. Denoting $c=I_0'(0)\neq0$, one obtains

Lemma 2. *The following expansions hold near $h=0$:*

$$\begin{aligned}I_0(h)&=c[h+(\tfrac{5}{3}-\tfrac{3}{8}a)h^2+(\tfrac{385}{27}-\tfrac{35}{4}a+\tfrac{35}{64}a^2)h^3\\&\quad+(\tfrac{85085}{486}-\tfrac{25025}{144}a+\tfrac{5005}{128}a^2-\tfrac{1155}{1024}a^3)h^4\\&\quad+1001(\tfrac{7429}{2916}-\tfrac{2261}{648}a+\tfrac{1615}{1152}a^2-\tfrac{85}{512}a^3+\tfrac{45}{16384}a^4)h^5+\ldots],\\I_1(h)&=c[h^2+(\tfrac{70}{9}-\tfrac{35}{12}a)h^3+(\tfrac{5005}{54}-\tfrac{5005}{72}a+\tfrac{1155}{128}a^2)h^4\\&\quad+1001(\tfrac{323}{243}-\tfrac{323}{216}a+\tfrac{85}{192}a^2-\tfrac{15}{512}a^3)h^5\\&\quad+1001(\tfrac{185725}{8748}-\tfrac{185725}{5832}a+\tfrac{52003}{3456}a^2-\tfrac{11305}{4608}a^3+\tfrac{1615}{16384}a^4)h^6+\ldots],\\I_2(h)&=c[\tfrac{1}{2}h^2+(\tfrac{35}{9}-\tfrac{5}{8}a)h^3+(\tfrac{5005}{108}-\tfrac{385}{16}a+\tfrac{315}{256}a^2)h^4\\&\quad+1001(\tfrac{323}{486}-\tfrac{85}{144}a+\tfrac{15}{128}a^2-\tfrac{3}{1024}a^3)h^5\\&\quad+1001(\tfrac{185725}{17496}-\tfrac{52003}{3888}a+\tfrac{11305}{2304}a^2-\tfrac{1615}{3072}a^3+\tfrac{255}{32768}a^4)h^6+\ldots].\end{aligned}$$

Proof. We rewrite the system from Lemma 1 in the form $(\mathbf{A}h+\mathbf{B})\mathbf{I}'=\mathbf{EI}$ where $\mathbf{I}=(I_0,I_1,I_2)^\top$. As $\mathbf{I}(h)$ is a solution which is analytical near zero and $\mathbf{I}(0)=0$, one can replace

$$\mathbf{I}(h)=\sum_{k=1}^{\infty}\mathbf{V}_k h^k,\quad \mathbf{V}_k=(V_{0,k},V_{1,k},V_{2,k})^\top$$

in the system. Then the coefficient at h^k should be zero, which yields the equation

$$(k+1)\mathbf{B}\mathbf{V}_{k+1}=(\mathbf{E}-k\mathbf{A})\mathbf{V}_k.$$

Since $\mathbf{I}'(0)=(c,0,0)^\top=\mathbf{V}_1$, one can solve the system above with respect to $(V_{0,k},V_{1,k+1},V_{2,k+1})$ and thus to obtain via recursive procedure formulas for all $\mathbf{V}_k$, $k=2,3,\dots$. Explicitly,

$$\begin{aligned}(8-9a)V_{1,k+1}&=[8-9a+(88-87a)\tfrac{k-1}{k+1}]V_{0,k}-(48a-36a^2)\tfrac{k-1}{k+1}V_{1,k},\\(8-9a)V_{2,k+1}&=[4-\tfrac{9}{2}a+(44-\tfrac{63}{2}a)\tfrac{k-1}{k+1}]V_{0,k}-24a\tfrac{k-1}{k+1}V_{1,k},\\V_{0,k+1}&=\tfrac{6k-1}{3k}V_{1,k+1}-a\tfrac{4k-1}{4k}V_{2,k+1},\quad k=1,2,3,\dots\end{aligned}$$

Applying these formulas, we obtain the expansions in Lemma 2. □

Proof of Theorem 5. Consider the following linear combinations

$$\begin{aligned}&J_0=I_0, && J_3=\alpha_1 hI_0+\beta_1 I_1+\gamma_1 I_2,\\&J_1=I_1, && J_4=\alpha_2 hI_0+(\beta_2+\delta_2 h)I_1+\gamma_2 I_2,\\&J_2=I_1-2I_2, && J_5=\alpha_3 hI_0+(\beta_3+\delta_3 h)I_1+(\gamma_3+\eta_3 h)I_2,\end{aligned}$$

where

$$\begin{aligned}&\alpha_1=a, && \delta_2=\tfrac{8}{3}a^2+a^3,\\&\beta_1=-\tfrac{11}{3}+\tfrac{21}{40}a, && \alpha_3=\tfrac{17}{81}a-\tfrac{775}{5148}a^2+\tfrac{63}{9152}a^3,\\&\gamma_1=\tfrac{22}{3}-\tfrac{61}{20}a, && \beta_3=-\tfrac{187}{243}+\tfrac{55}{72}a-\tfrac{1085}{9152}a^2-\tfrac{189}{73216}a^3,\\&\alpha_2=\tfrac{208}{63}a-\tfrac{2}{3}a^2, && \gamma_3=\tfrac{374}{243}-\tfrac{631}{324}a+\tfrac{155}{288}a^2-\tfrac{315}{36608}a^3,\\&\beta_2=-\tfrac{2288}{189}+\tfrac{52}{9}a+\tfrac{1}{4}a^2, && \delta_3=\tfrac{119}{702}a^2-\tfrac{147}{1144}a^3-\tfrac{189}{18304}a^4,\\&\gamma_2=\tfrac{4576}{189}-\tfrac{1144}{63}a+\tfrac{5}{6}a^2, && \eta_3=\tfrac{49}{234}a^3.\end{aligned}$$

The coefficients above are chosen so that $J_k(h)=O(h^{k+1})$ near zero for $0\le k\le 5$. Their explicit values are determined from the respective linear systems. By calculation, then one obtains

$$\begin{aligned}&J_0=c[h+\dots], && J_3=c[\tfrac{49}{32}a^2(a+\tfrac{8}{3})h^4+(\tfrac{68992}{405}+O(a+\tfrac{8}{3}))h^5+\dots],\\&J_1=c[h^2+\dots], && J_4=c[\tfrac{154}{9}a^4h^5+\dots],\\&J_2=c[-\tfrac{5}{3}ah^3+\dots], && J_5=c[\tfrac{49}{128}a^5(a+\tfrac{8}{9})h^6+(-119(\tfrac{2}{3})^{14}+O(a+\tfrac{8}{9}))h^7+\dots].\end{aligned}$$

Let us fix the Hamiltonian parameter a be a little bit smaller that $-\frac{8}{3}$, so that we would have $J_3 = c[\delta_4 h^4 + \delta_5 h^5 + O(h^6)]$ with $|\delta_4| \ll |\delta_5|$ and $\delta_4 < 0 < \delta_5$. Then, one can choose a linear combination $J(h)$ of J_k, $0 \le k \le 3$, such that $J(h) = c\sum_{k=1}^{5} \delta_k h^k[1 + O(h)]$ will satisfy $\delta_k \delta_{k+1} < 0$ and $|\delta_k| \ll |\delta_{k+1}|$. Therefore, $J(h)$ would have 4 small positive zeroes. As the four coefficients in (7) are independently free, one can take a small perturbation such that $M_1(h) = J(h)$ will produce 4 small-amplitude limit cycles around the center at the origin. The proof of the claim concerning $M_2(h)$ is the same, as long as we fix the parameter a a little bit smaller than $-\frac{8}{9}$ and construct in the same way a linear combination $J(h) = c\sum_{k=1}^{7} \delta_k h^k[1 + O(h)]$ with coefficients having the same properties, thus $M_2(h)$ producing 6 small positive zeroes in the saddle-loop case.

For all other $a \in \mathbb{R}$ different from 0, $-\frac{8}{3}$ and $\pm\frac{8}{9}$, any linear combination of J_k, $0 \le k \le m$ where $m = 3, 4, 5$, will have at most m small positive zeroes. Moreover, $M_k(h)$, $k = 1, 2, 3$ can be expressed as linear combination of the respective J_k, thus having as much zeroes at its dimension minus one. It is easy to see that $M_4(h)$ has no small positive zeroes at all. □

Acknowledgments

We thank the referee for useful comments. Part of the paper was written while the second author was visiting the Institut of Mathematics, University of Toulouse III (Paul Sabatier). He is obliged for the hospitality.

Appendix A

For readers' convenience, below we present the explicit expressions of the parameters λ, μ and the coefficients q_{ij} of Q in (5) in terms of the original coefficients of perturbation in (2). First of all, note that condition $M_1(h) \equiv 0$ is equivalent to

$$a_{10} + b_{01} = 0, \quad 2a_{20} + b_{11} = 0, \quad 3a_{30} + b_{21} = 0, \quad a_{12} + 3b_{03} = 0.$$

Then, a simple calculation yields

$$\begin{array}{ll}
\lambda = -a_{11} - 2b_{02} & q_{21} = -a_{20} \\
\mu = -a_{21} - b_{12} & q_{12} = b_{02} \\
q_{10} = b_{00} & q_{03} = -\frac{1}{3}a_{02} \\
q_{01} = -a_{00} & q_{40} = \frac{1}{4}(-2a_{11} + a_{21} - 4b_{02} + b_{30} + b_{12}) \\
q_{20} = \frac{1}{2}b_{10} & q_{31} = -a_{30} \\
q_{11} = -a_{10} & q_{22} = \frac{1}{2}b_{12} \\
q_{02} = -\frac{1}{2}a_{01} & q_{13} = -\frac{1}{3}a_{12} \\
q_{30} = \frac{1}{3}(a_{11} + b_{20} + 2b_{02}) & q_{04} = -\frac{1}{4}a_{03}
\end{array}$$

References

[1] Colin Christopher, Chengzhi Li, Limit Cycles of Differential Equations, Birkhäuser, Basel, 2007.

[2] J.P. Francoise, Successive derivatives of a first return map, application to the study of quadratic vector fields, Ergodic Theory Dynam. Systems 16 (1) (1996) 87–96.
[3] Lubomir Gavrilov, Petrov modules and zeros of Abelian integrals, Bull. Sci. Math. 122 (8) (1998) 571–584.
[4] Lubomir Gavrilov, Abelian integrals related to Morse polynomials and perturbations of plane Hamiltonian vector fields, Ann. Inst. Fourier (Grenoble) 49 (2) (1999) 611–652.
[5] Lubomir Gavrilov, On the number of limit cycles which appear by perturbation of two-saddle cycles of planar vector fields, Funct. Anal. Appl. 47 (3) (2013) 174–186.
[6] Lubomir Gavrilov, Iliya D. Iliev, The displacement map associated to polynomial unfoldings of planar Hamiltonian vector fields, Amer. J. Math. 127 (6) (2005) 1153–1190.
[7] I.D. Iliev, Higher-order Melnikov functions for degenerate cubic Hamiltonians, Adv. Differential Equations 1 (4) (1996) 689–708.
[8] I.D. Iliev, On second order bifurcations of limit cycles, J. Lond. Math. Soc. (2) 58 (2) (1998) 353–366.
[9] Yu.S. Ilyashenko, The appearance of limit cycles under a perturbation of the equation $dw/dz = -R_z/R_w$, where $R(z, w)$ is a polynomial, Mat. Sb. (N.S.) 78 (120) (1969) 360–373.
[10] Changjian Liu, Estimate of the number of zeros of Abelian integrals for an elliptic Hamiltonian with figure-of-eight loop, Nonlinearity 16 (3) (2003) 1151–1163.
[11] G.S. Petrov, Nonoscillation of elliptic integrals, Funktsional. Anal. i Prilozhen. 24 (3) (1990) 45–50, 96.