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On the Number of Limit Cycles Which Appear by Perturbation of Two-Saddle Cycles of Planar Vector Fields

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Abstract. We prove that the number of limit cycles which bifurcate from a two-saddle loop of an analytic planar vector field X_0 under an arbitrary finite-parameter analytic deformation X_λ, $\lambda \in (\mathbb{R}^N, 0)$, is uniformly bounded with respect to λ.

Key words: limit cycles, finite cyclicity, heteroclinic loop, two-saddle loop.

1. Introduction

Consider a finite-parameter analytic family of analytic planar vector fields

$$X_\lambda = P(x,y,\lambda)\frac{\partial}{\partial x} + Q(x,y,\lambda)\frac{\partial}{\partial y}, \qquad \lambda \in \mathbb{R}^N, \tag{1}$$

such that X_0 has a limit periodic set Γ. The cyclicity of Γ is, roughly speaking, the maximal number of limit cycles of X_λ which tend to Γ as $\lambda \to 0$. Roussarie's finite cyclicity conjecture claims that *every limit periodic set occurring in an analytic finite-parameter family of planar analytic vector fields has finite cyclicity* [23]. If true, this conjecture would imply the finiteness of the maximal number $H(n)$ of limit cycles which a planar polynomial vector field of degree n can have. Therefore, it plays a fundamental role in all questions related to the second part of Hilbert's 16th problem and its ramifications.

Recall that a polycycle of the vector field X_0 is a topological polygon composed of separatrices and singular points. A k-saddle cycle of X_0 (or a hyperbolic k-graph), denoted by Γ_k, is a polycycle composed of k distinct saddle-type singular points $p_1, \dots, p_k$, $p_{k+1} = p_1$ and separatrices (heteroclinic orbits) connecting p_i to p_{i+1} as in Fig. 1. The simplest limit periodic sets are k-saddle cycles, periodic orbits, and weak foci or centers. The finite cyclicity of periodic orbits and weak foci is well known and follows from Gabrielov's theorem [23, p. 68]. The finite cyclicity of one-saddle loops was proved by Roussarie in [21] and [22].

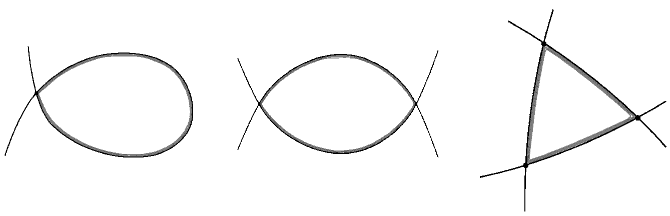

Fig. 1. One-, two-, and three-saddle cycles.

The purpose of the present paper is to prove the finite cyclicity of a two-saddle cycle under a finite-parameter analytic deformation (see Theorem 4).

Several special cases of this result were earlier proved, under various genericity assumptions on X_0 or on the family X_λ, by Cherkas, Mourtada, El Morsalani, Dumortier, Roussarie, Rousseau, Jebrane, Żołądek, Li, Caubergh, Luca, and other authors in [6]–[8], [13], and [16] (see also [23, Sec. 5.4.1] for a survey of the results and references up to 1996). The finite cyclicity of a k-saddle cycle (for any k) under a finite-parameter analytic deformation was recently announced by Mourtada [18].

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For *generic families* of vector fields, analyticity can be relaxed. As shown by Ilyashenko and Yakovenko [11] and Kaloshin [12], *any nontrivial elementary polycycle occurring in a generic k-parameter family of C^∞ vector fields has finite cyclicity.*

In contrast to the aforementioned authors, we shall not use asymptotic expansions of the corresponding Dulac maps. Instead of this, we evaluate the number of limit cycles near Γ_2 in a complex domain by making use of a suitable version of the argument principle. This approach was initiated by the author in [10], where we studied the cyclicity of Hamiltonian two-loops. As is well known, the limit cycles of planar systems close to Hamiltonian are closely related to the zeros of the associated Abelian integrals depending on a parameter (see the so-called weakened 16th Hilbert problem in Arnold's book [1, p. 313]). Zeros of complete elliptic integrals in a complex domain have been successfully studied by topological methods (the argument principle) after Petrov's pioneering works [19] and [20]; see also Żoładek's book [25, Sec. 6] for a description of the method. This method was used in a more general context by several authors (see, e.g. [3] and [2]); in [10] the idea was used to replace Abelian integrals by the true Poincaré return map.

In the present paper we shall find a relation between the fixed points of the Poincaré first return map and the fixed points of holonomies of the separatrices of the saddle points, which correspond to complex limit cycles. Counting such fixed points reduces to counting the zeros of families of *analytic functions*, which is easy. The main technical tool is Lemma 2, in which we show that the connected components of the zero locus of the imaginary part of the Dulac map are smooth semianalytic curves. This allows us to estimate the variation of the argument of the displacement map along the boundary of an appropriate complex domain and apply the argument principle to evaluate the zeros of this map in the domain.

Note that previously the relation between the monodromy and the Dulac map was used by Roussarie to compute the Bautin ideal associated to the Dulac map [24]. This result, combined with [21] and [22], also implies the finite cyclicity of one-saddle cycles.

The paper is organized as follows. In Section 2 we provide the necessary technical background and prove the main technical Lemma 2. In Section 3 we give a new self-contained proof of Roussarie's theorem about the finite cyclicity of one-saddle cycles. The origin of our method is explained in Section 4, where we give a brief account of a local version of the so-called "Petrov trick." The same method is easily adapted in Section 5 to show that the cyclicity of Γ_2 is finite.

2. The Dulac Map

Consider an analytic family of plane real analytic foliations $\mathscr{F}_\lambda$, $\lambda \in \mathbb{R}^N$, having a nondegenerate isolated saddle point. An appropriate translation analytically depending on λ will place the saddle point at the origin. The foliation $\mathscr{F}_\lambda$ has two analytic separatrices, intersecting transversally at the saddle point and depending analytically on λ (see [5] and [17]). Therefore, a further real bi-analytic change of the variables x and y analytically depending on λ will identify these separatrices with the axes $\{x = 0\}$ and $\{y = 0\}$ as in Fig. 2; thus,

$$\mathscr{F}_\lambda : x(1 + \cdots)\, dy + \alpha(\lambda) y(1 + \cdots)\, dx, \qquad \alpha(0) > 0, \tag{2}$$

where the dots replace higher-order terms in x and y with coefficients depending on λ. The number $\alpha(\lambda)$ is the hyperbolic ratio of the saddle point. From now on we shall suppose that the foliation (2) is analytic and depends analytically on λ in a neighborhood of the origin in $\mathbb{R}^2 \times \mathbb{R}^N$.

For $c_1, c_2 \in \mathbb{R}$ sufficiently small, let $\sigma \subset \{y = c_1\}$ and $\tau \subset \{x = c_2\}$ be open complex discs centered at $(0, c_1)$ and $(c_2, 0)$ and parameterized by x and y, respectively. The (real) Dulac map

$$\mathscr{D}_\lambda \colon \sigma \cap \mathbb{R}^+ \to \tau \cap \mathbb{R}^+, \qquad \mathscr{D}_\lambda(0) = 0,$$

is the germ of an analytic map at $x = 0$ defined as follows: if $x \in \sigma \cap \mathbb{R}^+_*$, then $\mathscr{D}_\lambda(x) \in \tau \cap \mathbb{R}^+_*$ is the intersection with $\tau \cap \mathbb{R}^+_*$ of the orbit $\gamma_\lambda(x)$ of (2) passing through x; see Fig. 2 (ii). This geometric definition of $\mathscr{D}_\lambda$ allows us to control, to a certain extent, its analytic continuation to a complex domain.

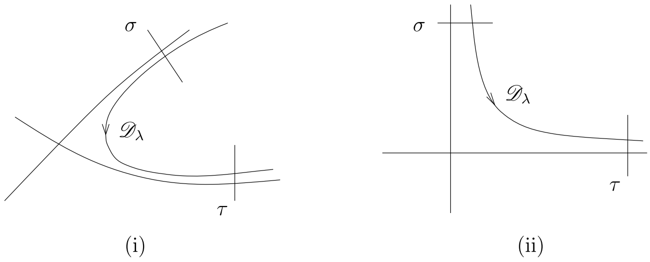

Fig. 2. The Dulac map.

2.1. Analytic continuation. The Dulac map admits an analytic continuation to some open subset, depending on λ, of the universal covering space $\sigma_\bullet$ of $\sigma\setminus\{0\}$. Let us parameterize $\sigma_\bullet$ by polar coordinates $\rho>0$, $\varphi\in\mathbb{R}$, and $z=\rho\exp i\varphi$. The following result is well known (see, e.g., [10, Appendix A]).

Theorem 1. *There exists an* $\varepsilon_0>0$ *and a continuous function*

$$\rho\colon\mathbb{R}\to\mathbb{R}_*^+,\qquad\varphi\mapsto\rho(\varphi),$$

such that the Dulac map admits an analytic continuation to the domain

$$\{(\lambda,\rho,\varphi)\in\mathbb{C}^N\times\sigma_\bullet : |\lambda|<\varepsilon_0,\ 0<\rho<\rho(\varphi)\}. \tag{3}$$

The geometric content of Theorem 1 is as follows. Let $\{\gamma_\lambda(z)\}_{z,\lambda}$ be a continuous family of paths contained in the leaves of $\mathscr{F}_\lambda$ and connecting $z\in\sigma$ to τ.

Given $z\in\sigma\cap\mathbb{R}_*^+$, we suppose that $\gamma_\lambda(z)$ is the real orbit of $\mathscr{F}_\lambda$ contained in the first quadrant $x\geqslant 0$, $y\geqslant 0$ and connecting z to τ; see Fig. 2 (ii). The above theorem claims that this family of orbits admits an extension to a continuous family of paths $\{\gamma_\lambda(z)\}_{z,\lambda}$ contained in the leaves of $\mathscr{F}_\lambda$ and connecting $z\in\sigma_\bullet$ to $\tau_\bullet$. The family is defined for all (λ,ρ,φ) which belong to the domain (3). Each path starts at z and terminates at a unique point on σ, denoted by $\mathscr{D}_\lambda(z)$. Although the paths $\{\gamma_\lambda(z)\}_{z,\lambda}$ are not unique, their relative homotopy classes are determined uniquely.

2.2. The monodromy of the Dulac map and the holonomy of separatrices. With the axes $\{x=0\}$ and $\{y=0\}$ parameterized by y and x we associate the holonomy maps

$$h_\sigma^\lambda\colon\sigma\to\sigma\qquad\text{and}\qquad h_\tau^\lambda\colon\tau\to\tau$$

defined by two closed paths contained in the axes $\{x=0\}$ and $\{y=0\}$ and based at $(0,c_1)$ and $(c_2,0)$, respectively. We shall use the convention that each closed path makes one turn around the origin of the axis in which it is contained in the positive direction (recall that the axes are parameterized by y and x). It is easily seen that, in the case of a linear foliation of the form

$$x\,dy+\alpha y\,dx=0,\qquad\alpha\in\mathbb{R}^+, \tag{4}$$

we have

$$\mathscr{D}_\alpha\colon x\mapsto y=c_1c_2^{-\alpha}x^\alpha,\quad h_\sigma\colon x\mapsto xe^{-2\pi i/\alpha},\quad\text{and}\quad h_\tau\colon y\mapsto ye^{-2\pi i\alpha}. \tag{5}$$

In the general case of a nonlinear foliation of the form (2), the Dulac map $\mathscr{D}_\lambda$ is only asymptotic to $c_1c_2^{-\alpha}x^\alpha$, while the holonomy maps are analytic in x, y, and λ and, moreover,

$$h_\sigma^\lambda\colon x\mapsto xe^{-2\pi i/\alpha}+\dots\quad\text{and}\quad h_\tau^\lambda\colon y\mapsto ye^{-2\pi i\alpha}+\dots,\qquad\alpha=\alpha(\lambda). \tag{6}$$

The Dulac map $\mathscr{D}_\lambda$ is a transcendental multivalued map. For $x>0$, let $\mathscr{D}_\lambda(e^{2\pi i}x)$ be the result of the analytic continuation of $\mathscr{D}_\lambda$ along an arc of radius x subtending an angle $2\pi i$. Similarly, for $y>0$, let $\mathscr{D}_\lambda(e^{2\pi i}y)$ be the result of the analytic continuation of $\mathscr{D}_\lambda$ along an arc of radius y subtending an angle $2\pi i$.

Lemma 1. *For every sufficiently small* $x>0$, $y>0$, *and* $|\lambda|$,

$$h_\tau^\lambda\circ\mathscr{D}_\lambda(e^{2\pi i}x)=\mathscr{D}_\lambda(x)\qquad\text{and}\qquad h_\sigma^\lambda\circ\mathscr{D}_\lambda^{-1}(e^{2\pi i}y)=\mathscr{D}_\lambda^{-1}(y).$$

Proof. Consider the underlying path γ_λ instead of $\mathscr{D}_\lambda$. The loop $\gamma_\lambda(e^{2\pi i}x)$ has the same origin as $\gamma_\lambda(x)$, so these loops can be composed, and the resulting loop $\widetilde{\gamma}_\lambda(y)$ starts at $y = \mathscr{D}_\lambda(e^{2\pi i}x) \in \tau$ and terminates at $\mathscr{D}_\lambda(x) \in \tau$. In the special linear case (4) with $\alpha = 1$, the foliation is a fibration, the paths $\gamma_\lambda(\,\cdot\,)$ represent relative cycles in the fibers of xy, and the path $\widetilde{\gamma}_\lambda(y)$ is closed and represents a vanishing cycle. The claim of Lemma 1 is then the classical Picard–Lefschetz formula. In the general case, the result follows "by deformation." Indeed, in the linear case (4) with $\alpha = 1$, the family of closed paths $\{\widetilde{\gamma}_\lambda(y)\}_y$ is defined for all sufficiently small y, and $\widetilde{\gamma}_\lambda(0) \subset \{y = 0\}$ is a closed path which makes one turn around the origin of the axis $\{y = 0\}$ in the positive direction. Note that the paths $\widetilde{\gamma}_\lambda(0)$ are bounded away from the origin in $\mathbb{C}^2$. It follows that $\widetilde{\gamma}_\lambda$ defines the holonomy h_τ^λ of the separatrix $\{y = 0\}$, and this property holds true also for every sufficiently small deformation of (4). The homothety $(x, y) \to (\varepsilon x, \varepsilon y)$ transforms (2) into a small deformation of (4), which completes the proof of the first identity (see also [15]). The second identity in Lemma 1 is proved in a similar way. □

2.3. The zero locus of the imaginary part of the Dulac map. Consider the universal covering

$$\mathbb{C}_\bullet \xrightarrow{\pi} \mathbb{C} \setminus \{0\} \tag{7}$$

and the zero locus $\mathscr{H}_\lambda \subset \mathbb{C}_\bullet$ of the imaginary part of the Dulac map $\mathscr{D}_\lambda$ corresponding to the domain (3). We have

$$\mathscr{H}_\lambda = \{z = (\rho, \varphi) \in \mathbb{C}_\bullet : \operatorname{Im} \mathscr{D}_\lambda(z) = 0,\ 0 < \rho < \rho(\varphi),\ \varphi \in \mathbb{R}\}. \tag{8}$$

In the case of a linear foliation of the form (4), the zero locus is a union of half-lines:

$$\mathscr{H}_\alpha = \{z \in \mathbb{C}_\bullet : \operatorname{Im} z^\alpha = 0\} = \bigcup_{k \in \mathbb{Z}} \mathscr{H}_{\alpha,k}, \qquad \mathscr{H}_{\alpha,k} = \{(\rho, \varphi) \in \mathbb{C}_\bullet : \varphi = k\pi/\alpha\}.$$

To describe $\mathscr{H}_\lambda$ in the case of a general foliation of the form (2) with hyperbolic ratio $\alpha(\lambda) > 0$, consider the following germs of real analytic sets at the origin in $\mathbb{R}^2 = \mathbb{C}$:

$$C_{\lambda,k} = \{z \in \mathbb{C} = \mathbb{R}^2 : (h_\sigma^\lambda)^k(z) = \bar{z}\}, \tag{9}$$

where h_σ^λ is the holonomy map associated to the separatrix $\{x = 0\}$.

Lemma 2. *The zero locus $\mathscr{H}_\lambda \subset \mathbb{C}_\bullet$ of the imaginary part of the Dulac map in the domain* (3) *is a union of connected components $\mathscr{H}_{\lambda,k}$ indexed by $k \in \mathbb{Z}$.*

• *Each set $C_{\lambda,k}$ in* (9) *is the germ of a real analytic curve in $\mathbb{R}^2$ which is smooth at the origin and tangent to the line*

$$\{z = s e^{ik\pi/\alpha(\lambda)} : s \in (\mathbb{R}, 0)\} \tag{10}$$

at the origin.

• *The projection of each connected component $\mathscr{H}_{\lambda,k}$ on the plane $\mathbb{C} = \mathbb{R}^2$ under the map π* (7) *is the connected component of $C_{\lambda,k} \setminus \{0\}$ tangent to the half-line* (10) *with $s > 0$ at the origin.*

Remark. For a general bi-holomorphic map h_σ^λ vanishing at the origin, the set (9) coincides with the origin itself. The above lemma shows, however, that, for the monodromy map h_σ^λ of a saddle point of a real analytic planar vector field, the set $C_{\lambda,k}$ defined by (9) is the germ of a real analytic curve in $\mathbb{R}^2$ smooth at the origin. The position of the connected components of $C_{\lambda,k} \setminus \{0\}$ tangent to the half-lines (10) with $s > 0$ is shown in Fig. 3.

The above lemma is the main technical result of the present paper. The analyticity of the zero locus $\mathscr{H}_\lambda$ is responsible for the algebraic-like behavior of the Dulac map.

Proof of Lemma 2. Let $x \in \sigma \cap \mathbb{R}^+$ and suppose that, for some $\varphi > 0$, $\mathscr{D}_\lambda(e^{i\varphi}x) \in \mathbb{R}$. As the Dulac map is real along $\sigma \cap \mathbb{R}^+$, it follows that $\mathscr{D}_\lambda(e^{-i\varphi}x)$ is complex conjugate to $\mathscr{D}_\lambda(e^{i\varphi}x)$, and hence

$$\mathscr{D}_\lambda(e^{-i\varphi}x) = \mathscr{D}_\lambda(e^{i\varphi}x).$$

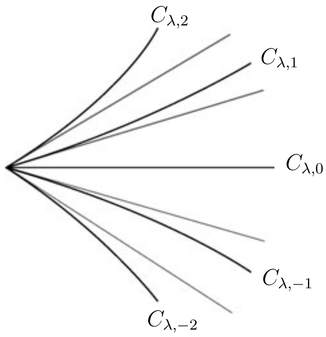

Fig. 3. The zero locus $\mathscr{H}_\lambda$ of the imaginary part of the Dulac map projected on the complex plane $\mathbb{C}$.

If the point $e^{-i\varphi}x$ is seen as the inverse image of $\mathscr{D}_\lambda(e^{-i\varphi}x)$ under the Dulac map $\mathscr{D}_\lambda^{-1}$, then the point $e^{i\varphi}x$ is the result of the analytic continuation of the map $\mathscr{D}_\lambda^{-1}$ along a suitable closed path of τ, starting and terminating at $\mathscr{D}_\lambda(e^{-i\varphi}x)$. If we put

$$y = \mathscr{D}_\lambda(e^{-i\varphi}x), \qquad \text{i.e., } e^{-i\varphi}x = \mathscr{D}_\lambda^{-1}(y),$$

then $e^{\pm i\varphi}x$ are two values of the multivalued map $\mathscr{D}_\lambda^{-1}(y)$; hence, by Lemma 1, they differ by a power of the monodromy h_σ^λ, that is,

$$(h_\sigma^\lambda)^k(e^{i\varphi}x) = e^{-i\varphi}x,$$

or, equivalently,

$$(h_\sigma^\lambda)^k(z) = \bar{z}, \quad z = e^{i\varphi}x, \quad \text{for some } k \in \mathbb{Z}.$$

Clearly, every such relation corresponds to a connected component $\mathscr{H}_{\lambda,k}$ of $\mathscr{H}_\lambda$. As $\mathscr{H}_{\lambda,k}$ is an analytic set of real dimension 1, it follows that $C_{\lambda,k}$ is an analytic set of dimension 1 too. It can be defined, therefore, by each of the equivalent relations

$$C_{\lambda,k} \subset \{z \in \mathbb{C} = \mathbb{R}^2 : \mathrm{Re}[(h_\sigma^\lambda)^k(z)] = \mathrm{Re}(\bar{z})\}$$

and

$$C_{\lambda,k} \subset \{z \in \mathbb{C} = \mathbb{R}^2 : \mathrm{Im}[(h_\sigma^\lambda)^k(z)] = \mathrm{Im}(\bar{z})\}.$$

As $\frac{\partial}{\partial \bar{z}}[(h_\sigma^\lambda)^k(z) - \bar{z}] = -1$, the linear part of the complex analytic function

$$\mathbb{R}^2 \to \mathbb{C}, \qquad (z, \bar{z}) \mapsto (h_\sigma^\lambda)^k(z) - \bar{z},$$

can not be identically zero, and therefore $C_{\lambda,k} \subset \mathbb{R}^2$ is a real analytic curve smooth at the origin. It follows from (6) that the projection of $\mathscr{H}_{\lambda,k}$ under π on the plane $\mathbb{C} = \mathbb{R}^2$ is tangent to the half-line (10) with $s > 0$ at the origin. $\square$

2.4. The argument principle. Let $\mathbf{D} \subset \mathbb{C}$ be a relatively compact domain with piecewise smooth boundary, and let $\psi\colon \mathbf{D} \to \mathbb{C}$ be an analytic function which admits a continuation to the closure $\overline{\mathbf{D}}$. We denote the number of zeros of ψ in $\mathbf{D}$ counted with multiplicities by $Z_{\mathbf{D}}(\psi)$. If we assume that ψ does not vanish on the boundary $\partial\mathbf{D}$, then the increment $\mathrm{Var}_{\partial\mathbf{D}}(\arg(\psi))$ of the argument of ψ along $\partial\mathbf{D}$ oriented counterclockwise is well defined and equals the winding number of the curve $\psi(\partial\mathbf{D}) \subset \mathbb{C}$ around the origin; the classical argument principle states that

$$2\pi Z_{\mathbf{D}}(\psi) = \mathrm{Var}_{\partial\mathbf{D}}(\arg(\psi)). \tag{11}$$

In the general case, where ψ has zeros on $\partial\mathbf{D}$, isolated or not, the variation of the argument $\mathrm{Var}_{\partial\mathbf{D}}(\arg(\psi))$ is not necessarily well defined.

Definition. We say that $z \in \partial\mathbf{D}$ is a regular zero of ψ if $\psi(z) = 0$ and ψ admits an analytic continuation to a neighborhood of z in $\mathbb{C}$.

If we assume that ψ has only regular zeros in $\overline{\mathbf{D}}$, then $\mathrm{Var}_{\partial\mathbf{D}}(\arg(\psi))$ is well defined as the sum of the increments of the argument of $\psi|_{\partial\mathbf{D}}$ between consecutive zeros of ψ. Indeed, the increments are finite, because the boundary ∂D is piecewise smooth. The argument principle can be reformulated as follows.

Proposition 1. *Let $\mathbf{D} \subset \mathbb{C}$ be a relatively compact domain with piecewise smooth boundary. If $\psi\colon \overline{\mathbf{D}} \to \mathbb{C}$ is a continuous function analytic in $\mathbf{D}$ and having only regular zeros in $\overline{\mathbf{D}}$, then*

$$2\pi Z_{\mathbf{D}}(\psi) \leqslant \mathrm{Var}_{\partial\mathbf{D}}(\arg(\psi)) \leqslant 2\pi Z_{\mathbf{D}}(\psi) + 2\pi Z_{\partial\mathbf{D}}(\psi). \tag{12}$$

Proof. There always exists a polynomial P such that ψ/P has no zeros in $\overline{\mathbf{D}}$, so we need to verify (12) only for polynomials. The set $\mathbf{D}$ is open, connected, and oriented, and it has piecewise smooth boundary, which is, therefore, self-avoiding and has the induced orientation. The inequality

$$0 \leqslant \mathrm{Var}_{\partial\mathbf{D}}(\arg(z)) \leqslant 2\pi$$

allows us to "remove" the zeros from $\partial\mathbf{D}$, and hence formula (11) implies (12). $\square$

In the present paper the first inequality in (12) will be used to bound the number of the zeros $Z_{\mathbf{D}}(\,\cdot\,)$. For this purpose, we shall need estimates on the variation $\mathrm{Var}_{\mathbf{l}}(\arg(\,\cdot\,))$ of the argument along any compact segment l of a curve. More precisely, let $l \subset \mathbb{R}^2 = \mathbb{C}$ be a compact segment of a smooth real analytic curve. Let $U \subset \mathbb{C}$ be an open set containing l, and let $\psi_\lambda(z)$, $\lambda \in (\mathbb{C}^N, 0)$, be the germ of a family of complex analytic functions in U at $\lambda = 0$. For every fixed λ such that the function ψ_λ is not identically zero, the variation

$$|\,\mathrm{Var}_l(\arg(\psi_\lambda)|$$

of its argument is well defined.

Theorem 2. *Let l be a compact segment of a real analytic curve, and let $\{\psi_\lambda\}_\lambda$ be a family of functions analytic in a neighborhood of l and depending analytically on λ. Then there exists an $\varepsilon_0 > 0$ such that*

$$\sup_{|\lambda|<\varepsilon_0,\, \psi_\lambda \neq 0} |\,\mathrm{Var}_l(\arg(\psi_\lambda)| < \infty.$$

The above result is implied by the following theorem due to Gabrielov (see [14] and [9]).

Theorem 3. *Let M and N be real analytic varieties and consider the canonical projection $\pi\colon M \times N \to N$. For every relatively compact semianalytic set $E \subset M \times N$, the number of the connected components of the preimages $\pi^{-1}(n)$ is bounded from above uniformly over $n \in N$.*

Proof of Theorem 2. The number of the isolated zeros of ψ_λ along l counted with multiplicities is uniformly bounded in λ at $\lambda = 0$ (see the Françoise–Yomdin theorem in [14]). On any interval between two zeros of $\psi_\lambda(\,\cdot\,)$ the variation of the argument divided by 2π is bounded by the number of zeros of the imaginary part of ψ_λ divided by 2 plus the sum of the multiplicities of the zeros of ψ_λ at the endpoints of the interval. The imaginary part of ψ_λ is a real analytic function in $U \subset \mathbb{R}^2$, and the Gabrielov theorem implies that the number of connected components of $\{\mathrm{Im}(\psi_\lambda) = 0\} \cap l$ is uniformly bounded in λ at $\lambda = 0$. $\square$

3. Cyclicity of One-Saddle Cycles

Let X_λ, $\lambda \in (\mathbb{R}^N, 0)$, be the germ of an analytic family of analytic planar vector fields such that X_0 has a one-saddle cycle (homoclinic saddle loop) Γ_1. The first return map associated with Γ_1 is the composition of the Dulac map $\mathscr{D}_\lambda(z)\colon \sigma \to \tau$ and the transport map $\mathscr{T}_\lambda(z)$ (see Fig. 4). We assume that the Dulac map is in the normal form, as in Section 2.1. The limit cycles of X_λ near Γ_1 correspond to the zeros of the displacement map

$$\psi_\lambda(z) = \mathscr{D}_\lambda(z) - \mathscr{T}_\lambda(z)$$

near $z = 0$. An appropriate choice of the local coordinates on the cross-sections σ and τ brings the transport map to the form $\mathscr{T}_\lambda(z) \equiv z$. Alternatively, we could choose simply $\sigma = \tau$ (without supposing that the Dulac map is in the normal form of Section 2.1). We shall bound the number of zeros of ψ_λ in the domain $\mathbf{D}_R \subset \mathbb{C}_\bullet$ enclosed by the circle $\{\rho = R\}$ and the connected components $\mathscr{H}_{\lambda,1}$ and $\mathscr{H}_{\lambda,-1}$ of the zero locus of the imaginary part of the Dulac map, as shown in Fig. 5. We shall suppose that $R > 0$ is so small that $\psi_\lambda(\,\cdot\,)$ is analytic in $\mathbf{D}_R$ for all $\lambda \in \mathbb{R}^N$ such that $|\lambda| \leqslant \varepsilon_0$ (see Theorem 1) and that it is analytic even on the closure of $\mathbf{D}_R$ except, of course, at $z = 0$, where $\psi_\lambda(\,\cdot\,)$ is only continuous. Indeed,

$$\lim_{z \to 0,\, z \in \mathbf{D}_R} \mathscr{D}_\lambda(z) = 0,$$

while $\mathscr{T}_\lambda(z)$ is holomorphic at $z = 0$, so that

$$\lim_{z \to 0,\, z \in \mathbf{D}_R} \psi_\lambda(z) = c(\lambda),$$

where $c(\lambda)$ is analytic and $c(0) = 0$. If the family of functions ψ_λ is sufficiently generic, then $c(\lambda) \not\equiv 0$, and in the case when $c(\lambda) \equiv 0$, we can replace ψ_λ by the new family $\psi_\lambda + \lambda_{N+1}$, $\lambda_{N+1} \in \mathbb{R}$, for which the limit at $z = 0$ is the parameter λ_{N+1}. After this preparation, we can prove the finite cyclicity of the homoclinic loop Γ_1. For this purpose, we apply Proposition 1 (the argument principle) to the family of functions ψ_λ in the domain $\mathbf{D}_R$. In the course of computation, it will be assumed that $R > 0$ is sufficiently small, ε_0 is sufficiently small with respect to R, and λ is such that $|\lambda| < \varepsilon_0$. For this choice of parameters we use the "physical" notation

$$0 < |\lambda| < \varepsilon_0 \ll R \ll 1. \tag{13}$$

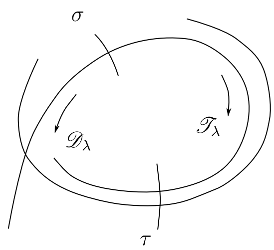

Fig. 4. The Dulac map $\mathscr{D}_\lambda(z)$ and the transport map $\mathscr{T}_\lambda(z)$.

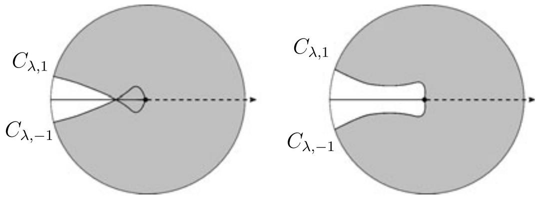

Fig. 5. Examples of domains $\mathbf{D}_R \subset \mathbb{C}_\bullet$ projected on the complex plane $\mathbb{C}$ under π (see (7)).

The hyperbolic ratio of the saddle point does not exceed 1 only in a suitable semianalytic set in the parameter space, and it is larger than 1 in another (complementary) semianalytic set. After the eventual interchange of σ and τ, it will also be assumed that the hyperbolic ratio of the saddle point is not larger than 1 for all parameter values.

The variation of the argument of ψ_λ along the circle $\{z : |z| = R\}$ through an angle close to or strictly less than 2π is uniformly bounded in λ (see Theorem 2).

On the curve $C_{\lambda,1}$ the imaginary part of ψ_λ equals the imaginary part of the transport map $-\mathscr{T}_\lambda(z) = -z$. Therefore, the zeros of $\mathrm{Im}(\psi_\lambda)$ on $C_{\lambda,1}$ are exactly the intersection points of $C_{\lambda,1}$ and the interval $(-R, 0)$. According to Lemma 2, we have

$$C_{\lambda,1} \cap \mathbb{R} = \{x \in \mathbb{R} : h_\sigma^\lambda(x) = x\} = C_{\lambda,-1} \cap \mathbb{R}. \tag{14}$$

As $h_\sigma^\lambda(x)$ is an analytic family of analytic functions, it follows by Gabrielov's theorem that the number of such fixed points is uniformly bounded in λ on $[-R, 0]$. To conclude, we have only to check that the family $\{\psi_\lambda\}_\lambda$ has regular zeros on the boundary of the domain $\mathbf{D}_R$. This is indeed the case when $c(\lambda) \neq 0$, because $\psi_\lambda(0) = c(\lambda)$. We conclude that the number of isolated zeros of the family of functions

$$\{\psi_\lambda : c(\lambda) \neq 0,\ |\lambda| \leqslant \varepsilon_0\}$$

in the domain $\mathbf{D}_R$ is uniformly bounded by some integer, say C. Finally, note that the condition $c(\lambda) \neq 0$ can be removed. Indeed, if, for some λ_0 such that $|\lambda_0| \leqslant \varepsilon_0$ and $c(\lambda_0) = 0$, the function ψ_{λ_0} has at least $C+1$ zeros in $\mathbf{D}_R$, then it has at least $C+1$ zeros in $\mathbf{D}_R$ in a sufficiently small neighborhood of λ_0, in contradiction with the preceding estimate.

Summarizing, we have proved the following classical result.

Theorem (Roussarie [21], [22], and [24]). *Every homoclinic saddle loop* (*a one-saddle cycle*) *occurring in an analytic finite-parameter family of planar analytic vector fields can generate no more than a finite number of limit cycles within the family.*

Let us note that our method, exactly as Roussarie's theorem, allows us to compute the cyclicity of Γ_1 more accurately. We shall not go into details here. We only mention by way of illustration that if the hyperbolic ratio $\alpha(0)$ is strictly larger than 1, then the total increment of the argument of the displacement map along the boundary of $\mathbf{D}_R$ is strictly less than 2π (we omit the computation), and the cyclicity of Γ_1 is zero.

4. The Petrov Trick

The content of this section is not necessary for the proof of our main result (Theorem 4), but it aims at shedding some light on the origin of the method used to bound the limit cycles near the saddle loop in the preceding section.

With the same notation as in Section 3, consider the analytic family of analytic vector fields

$$X_\lambda, \qquad \lambda = (\lambda_1, \dots, \lambda_N) \in (\mathbb{R}^N, 0),$$

defining a holomorphic foliation $\mathscr{F}_\lambda$ of the form

$$\mathscr{F}_\lambda = \{dH + \lambda_1 \omega_\lambda = 0\}, \qquad \omega_0 \neq 0,$$

where H is a function and ω_λ is an analytic family of differential 1-forms, both analytic in a neighborhood of the saddle loop Γ_1. For definiteness, we put the saddle point at the origin in $\mathbb{R}^2$, so that $dH(0) = 0$. We shall further suppose that the saddle loop Γ_1 is contained in the level set $\{H(x,y) = 0\}$ and the interior of Γ_1 is filled with a continuous family of periodic orbits $\gamma_0(h) \subset \{H(x,y) = h\}$ parameterized by $h > 0$, where $h = H(x,y)|_\sigma$ is the restriction of H to the cross-section σ. The displacement map is approximated by the usual Poincaré–Pontryagin formula as

$$\psi_\lambda(h) = \lambda_1 \int_{\gamma_0(h)} \omega_\lambda + o(\lambda_1), \tag{15}$$

where, as λ tends to zero, $o(\lambda_1)/\lambda_1$ tends to zero uniformly in h in every compact interval in which the displacement map is defined. The zeros of $\psi_\lambda(\,\cdot\,)$ correspond to limit cycles, and, at least far from $\Gamma_1 \subset \{H(x,y) = 0\}$, they are approximated by the zeros of the complete Abelian integral

$$h \mapsto I_\lambda(h) = \int_{\gamma_0(h)} \omega_\lambda, \qquad h \geqslant 0.$$

We make the assumption (which is in fact justified by Roussarie's theorem [21]) that this is also so in a neighborhood of $h = 0$ (corresponding to limit cycles close to the saddle loop Γ_1). Thus, it makes sense to prove the finiteness of the maximal number of those zeros of the Abelian integral $I_\lambda(h)$ which tend to $h = 0$ as λ tends to the origin in the parameter space. This readily follows from a well known general result of Varchenko and Khovansky. We shall use, however, a different idea due to Petrov [20], who showed that a similar global problem for complete elliptic integrals of the second kind is of algebraic nature. This observation has been used in several papers by Petrov to evaluate the precise number of zeros of complete elliptic integrals and, thereby, of limit cycles of perturbed Hamiltonian vector fields; see, e.g., Żoładek's book [25, Sec. 6]. We are ready to describe the local version of the Petrov method.

Consider the sector

$$S_R = \{z = \rho e^{i\varphi} \in \mathbb{C} : 0 < \rho < R,\ 0 < \varphi < 2\pi\}.$$

For a fixed sufficiently small $R > 0$ and all sufficiently small $\|\lambda\|$, the Abelian integral $I_\lambda(z)$ admits an analytic continuation to S_R. To bound the number of its zeros on S_R (and hence on $(0, R)$), we apply the argument principle to the domain S_R. Along the circle $\{\rho = R\}$ the increment of the argument of I_λ is bounded uniformly in λ (due to Gabrielov's theorem). Along the interval $[-R, 0]$ the Abelian integral has two analytic continuations $I_\lambda^\pm(h)$. As $I_\lambda(\,\cdot\,)$ is real analytic on $(0, R)$, it follows that

$$I_\lambda^+(h) = \overline{I_\lambda^-(h)}, \qquad h \in (-R, 0),$$

and by the Picard–Lefschetz formula we have

$$2\sqrt{-1}\,\mathrm{Im}\, I_\lambda^+(h) = I_\lambda^+(h) - I_\lambda^-(h) = \int_{\delta(h)} \omega_\lambda, \qquad h \in (-R, 0), \tag{16}$$

where $\delta(h) \subset \{H(x, y) = h\}$ is a continuous family of cycles vanishing at the origin as h tends to zero.

The imaginary part of $I_\lambda(h)$ on $(-R, 0)$ is therefore an analytic function, and, again by Gabrielov's theorem, its zeros are uniformly bounded in λ on the closed interval $[-R, 0]$. This implies that the increment of the argument of $I_\lambda(h)$ on $(-R, 0)$ is also uniformly bounded in λ, which, combined with the argument principle, shows the finiteness of the maximal number of zeros.

The proof of the finite cyclicity of the one-saddle loop from the preceding section may be seen as a generalization of the Petrov method. Indeed, the Picard–Lefschetz formula corresponds to the claim of Lemma 1, and by Lemma 2 the zeros of the analytic Abelian integral (16) correspond to the fixed points (complex limit cycles) of the holonomy map h_σ^λ of the separatrix. As is well known, the holonomy map of a separatrix is analytic, which implies the finite cyclicity of the saddle loop Γ_1.

5. Cyclicity of Two-Saddle Cycles

The main result of the paper is the following theorem.

Theorem 4. *Every heteroclinic saddle loop* (*a two-saddle cycle*) *occurring in an analytic finite-parameter family of planar analytic vector fields can generate no more than a finite number of limit cycles within the family.*

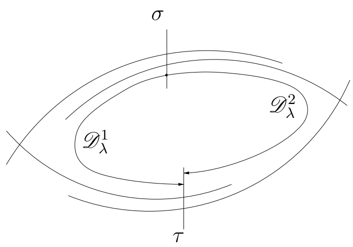

Fig. 6. The Dulac maps $\mathscr{D}_\lambda^1$ and $\mathscr{D}_\lambda^2$.

Using the notation of the preceding sections, suppose that the vector field X_0 has a two-saddle loop Γ_2. Consider the Dulac maps

$$\mathscr{D}_\lambda^i \colon \sigma \to \tau, \qquad i = 1, 2,$$

associated with the corresponding foliation, as in Fig. 5. Each map $\mathscr{D}_\lambda^i$ is a composition of a "local" Dulac map (as in Section 2) and two real analytic transport maps. It follows that Lemma 2 applies to $\mathscr{D}_\lambda^i$, $i = 1, 2$, too. From now on we choose a real analytic local variable z on the cross-section

σ, thus identifying σ with an open disc centered at $0 \in \mathbb{C}$. We shall also suppose that $0 = \sigma \cap \Gamma_2$. The functions $\mathscr{D}^i_\lambda(z)$, $i = 1, 2$, are multivalued on the cross-section σ and have critical points at $s_i(\lambda) \in \mathbb{R}$ ($s_i(0) = 0$). The functions s_i are real analytic. The limit cycles of X_λ near Γ_2 correspond to the zeros of the displacement map

$$\psi_\lambda(z) = \mathscr{D}^1_\lambda(z) - \mathscr{D}^2_\lambda(z)$$

near $z = 0$. Let $\alpha_i(\lambda) > 0$, $i = 1, 2$, be the hyperbolic ratios of the saddles. Interchanging σ and τ if necessary, we can assume that $\alpha_1(0)\alpha_2(0) \geqslant 1$. Let us denote the zero loci of the imaginary parts of the Dulac maps $\mathscr{D}^1_\lambda(z)$ and $\mathscr{D}^2_\lambda(z)$ by $\mathscr{H}^1_\lambda$ and $\mathscr{H}^2_\lambda$, respectively. We shall bound the number of zeros of ψ_λ in the complex domain $\mathbf{D}_R$ of the universal covering of $\mathbb{C} \setminus \{s_1(\lambda), s_2(\lambda)\}$ defined as follows (without loss of generality, we assume that $s_1(\lambda) \leqslant s_2(\lambda)$).

• If $\alpha_2(0) > 1$, then the domain $\mathbf{D}_R$ is bounded by the circle

$$S_R = \{z : |z| = R\} \tag{17}$$

and by

$$\mathscr{H}^1_{\lambda,1}, \quad \mathscr{H}^1_{\lambda,-1}, \quad \mathscr{H}^2_{\lambda,1}, \quad \text{and} \quad \mathscr{H}^2_{\lambda,-1},$$

as shown in Fig. 7.

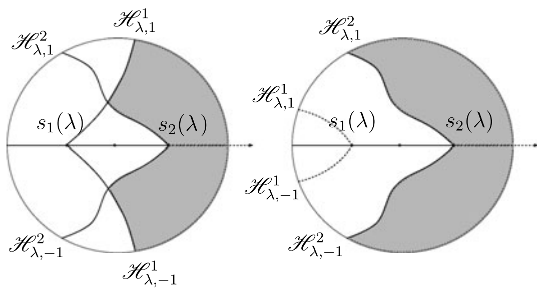

Fig. 7. The domain $\mathbf{D}_R \subset \mathbb{C}_\bullet$ projected on the complex plane $\mathbb{C}$ in the case $\alpha_2(0) > 1$.

• If $\alpha_2(0) \leqslant 1$, then necessarily $\alpha_1(0) \geqslant 1$. The domain $\mathbf{D}_R$ is bounded by the circle S_R, by the interval $[s_1(\lambda), s_2(\lambda)]$, and by $\mathscr{H}^1_{\lambda,1}$, $\mathscr{H}^1_{\lambda,-1}$, as shown in Fig. 8.

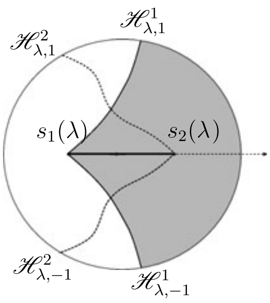

Fig. 8. The domain $\mathbf{D}_R \subset \mathbb{C}_\bullet$ projected on the complex plane $\mathbb{C}$ in the case $\alpha_2(0) \leqslant 1$, $\alpha_1(0) \geqslant 1$.

In the course of the proof the parameters R and λ will be chosen as in the one-saddle case: the constant R will be sufficiently small, $\varepsilon_0 > 0$ will be sufficiently small with respect to R, and $\lambda \in \mathbb{R}^N$ will be such that $|\lambda| < \varepsilon_0$ (see (13)). Like in Section 3, we shall suppose, without loss of generality, that the analytic functions $c_1(\lambda)$ and $c_2(\lambda)$, where

$$\lim_{z \to s_1(\lambda), z \in \mathbf{D}_R} \psi_\lambda(z) = c_1(\lambda) \qquad \text{and} \qquad \lim_{z \to s_2(\lambda), z \in \mathbf{D}_R} \psi_\lambda(z) = c_2(\lambda),$$

are not identically zero. This will guarantee that, for generic values of λ, the displacement map will have only regular zeros in the closure of $\mathbf{D}_R$, so that the argument principle (Proposition 1) can be applied.

Proof of Theorem 4. It follows from the definition of the domain $\mathbf{D}_R \subset \mathbb{C}_\bullet$ that the displacement map $\psi_\lambda(z)$ is analytic there. To count the zeros (corresponding to real and complex limit cycles) of the displacement map in $\mathbf{D}_R$, we apply Proposition 1 (the argument principle) to the family of functions ψ_λ. To evaluate the variation of the argument of the displacement map along the boundary of $\mathbf{D}_R$, we repeat the argument of Section 3.

Consider first the case $\alpha_2(0) > 1$ (see Fig. 7). The connected component of the zero locus of the imaginary part of $\mathscr{D}^2_\lambda$ which is tangent to the line $\varphi = \pi/\alpha_2(\lambda)$ and passes through $s_2(\lambda)$ intersects the circle S_R transversally, and along this circle the variation of the argument of ψ_λ is uniformly bounded in λ (by Theorem 2). The imaginary part of $\psi_\lambda(z)$ restricted to $\mathscr{H}^1_\lambda$ equals the imaginary part of $-\mathscr{D}^2_\lambda$, and hence $\operatorname{Im}\psi_\lambda$ vanishes along $\mathscr{H}^1_{\lambda,1}$, $\mathscr{H}^1_{\lambda,-1}$ exactly at the intersection points

$$\mathscr{H}^1_{\lambda,1} \cap \mathscr{H}^2_{\lambda,1} \quad \text{and} \quad \mathscr{H}^1_{\lambda,-1} \cap \mathscr{H}^2_{\lambda,-1}.$$

According to Lemma 2, these intersection points are the solutions of the equation

$$h^\lambda_2(z) = h^\lambda_1(z), \tag{18}$$

where h^λ_1 and h^λ_2 are the holonomies of the separatrices intersecting σ and related to the saddle points $s_1(\lambda)$ and $s_2(\lambda)$. By Gabrielov's theorem, the number of such fixed points is uniformly bounded in the disc $\{z : |z| < R\}$.

Consider now the second case $\alpha_2(0) \leqslant 1$, $\alpha_1(0) \geqslant 1$ (see Fig. 8). Along this circle S_R the variation of the argument of ψ_λ is uniformly bounded in λ (by Theorem 2). Along the interval $[s_1(\lambda), s_2(\lambda)]$ the imaginary part of $\mathscr{D}^1_\lambda$ vanishes identically, and the imaginary part of $\psi_\lambda(z)$ restricted to this interval equals the imaginary part of $-\mathscr{D}^2_\lambda$. Therefore, the zeros of $\operatorname{Im}(\psi_\lambda)$ along $[s_1(\lambda), s_2(\lambda)]$ are exactly the intersection points of $\mathscr{H}^2_{\lambda,1}$ and $[s_1(\lambda), s_2(\lambda)]$. By Lemma 2, like in (14), these intersection points are the solutions of the equation

$$h^\lambda_2(z) = z,$$

where h^λ_2 is the holonomy of the separatrix intersecting σ and related to the saddle point $s_2(\lambda)$. By Gabrielov's theorem, the number of such fixed points is uniformly bounded. Finally, the zeros of $\operatorname{Im}(\psi_\lambda)$ along $\mathscr{H}^1_{\lambda,1}$ and $\mathscr{H}^1_{\lambda,-1}$ are evaluated as in the case $\alpha_2(0) > 1$. This completes the proof of Theorem 4. $\square$

6. Concluding Remarks

Identity (18), which determines complex limit cycles "responsible" for the cyclicity of the double loop Γ_2, is the main new ingredient of the proof in the one-saddle case. Indeed, the solutions of (18) are fixed points of the holonomy $h^\lambda_2 \circ (h^\lambda_1)^{-1}$, which, for $\lambda = 0$, is generated by a closed loop γ contained in the complexified separatrix of Γ_2 intersecting the cross-section σ. The topological type of this separatrix near Γ_2 is a disc with two punctures corresponding to two saddle points $S_1(\lambda)$ and $S_2(\lambda)$. Clearly, γ makes one turn around each of them, but depending on the orientation, we have the two possibilities shown in Fig. 9, (i) and (ii). A simple computation of a model example shows that the loop γ associated with the holonomy $h^\lambda_2 \circ (h^\lambda_1)^{-1}$ is the figure-eight loop in Fig. 9 (i). In the loop γ the reader will recognize a key ingredient in the proof of the local boundedness of the number of zeros of pseudo-Abelian integrals given in [3] and [4].

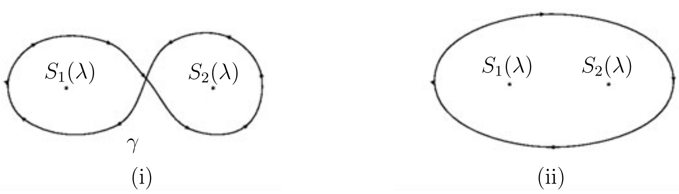

Fig. 9. The figure-eight loop γ.

Although Theorem 4 is existential, our proof of this theorem leads to effective upper bounds for the number of bifurcating limit cycles. This possibility is explored in [10], where we show that the cyclicity of a Hamiltonian two-loop is bounded by the number of zeros of *a pair* of associated Abelian integrals; this phenomenon also explains the appearance of alien limit cycles in [8].

It is worth noting that our finiteness result holds true, with the same proof, for other hyperbolic polycycles (in the plane or on an analytic surface), such as those shown in Fig. 10.

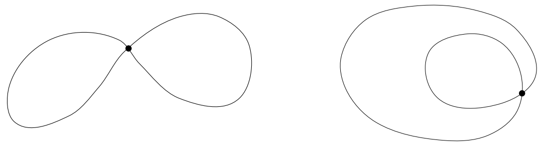

Fig. 10. Hyperbolic planar polycycles with finite cyclicity.

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