Доклады Болгарской академии наук
Comptes rendus de l'Académie bulgare des Sciences
Tome 41, N° 3, 1988

MATHEMATIQUES
Equations différentielles

ON THE NON-INTEGRABILITY OF A CLASS OF DIFFERENTIAL EQUATIONS WHICH ARE NOT OF PAINLEVÉ TYPE

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(Submitted by Corresponding Member V. Popov on August 20, 1987)

1. Introduction. The central object of the present announcement is the connection between the asymptotic behaviour of the solutions of a certain type of two-dimensional systems of ordinary differential equations (ODE) and the existence of rational first integrals. One of the most successful approaches for the detection of integrability in the last few years has been the so called singularity analysis (Painlevé method) [6,7,9]. A system of ODE is said to be of Painlevé type (P-type) or to possess the Painlevé property if the only movable singularities of its solutions are poles. It turns out that most of the systems of ODE of P-type are integrable with rational first integrals. According to recent results of Yoshida [11] if at least one Kowalevski's exponent of a similarity invariant system of ODE is not a rational number (and thus the system is not of P-type) then this system is not algebraically integrable (see [11] for definitions).

However, if all possible Kowalevski's exponents are rational numbers it is still possible that a system of ODE is not of P-type. We prove that if a two-dimensional systems of ODE is not of P-type because the asymptotic expansions around the singularity contain $\log(t)$ then the system does not possess a rational first integral (Theorem 1 and Theorem 2 of section 3).

In section 2 we formulate two lemmas which connect the integrability of a (n-dimensional) system of ODE and the integrability of its reductions — suitable similarity invariant systems of ODE.

2. Meromorphic and Rational First Integrals. Consider the system of autonomous ODE

$$dx_i/dt=F_i(x_1,\ x_2,\dots,\ x_n),\quad i=1,2,\dots,n, \tag{2.1}$$

where $F_1, F_2,\dots, F_n$ are rational functions of $x_1, x_2,\dots, x_n$. For an arbitrary monomial $y=\prod_{i=1}^{n} x_i^{k_i}$ let the weighed degree $\deg(y)$ of y be $\sum_{i=1}^{n} k_i\cdot g_i$, where $g_1, g_2\dots, g_n$ are rational numbers.

Any analytic function can be represented as a sum of weight-homogeneous polynomials $\Phi=\sum_{i=s}^{\infty}\Phi_i$ where $\deg(\Phi_i)=i$. We denote $\Phi^*=\Phi_s$. If the above sum is a finite one, i. e. $\Phi=\sum_{i=s}^{r}\Phi_i$, we denote $\Phi^0=\Phi_r$. If $\Phi=\widetilde{\Phi}_1/\widetilde{\Phi}_2$ is a rational (meromorphic) function we denote $\Phi^0=\widetilde{\Phi}_1^0/\widetilde{\Phi}_2^0$ ($\Phi^*=\widetilde{\Phi}_1^*/\widetilde{\Phi}_2^*$). Consider the following two systems of ODE

$$dx_i/dt=F_i^*(x_1,\ x_2,\dots,\ x_n),\ i=1,\ 2,\dots,n; \tag{2.1*}$$

(2.1⁰) $dx_i/dt = F_i^0(x_1, x_2, \dots, x_n)$, $i=1, 2, \dots, n$.

Definition. If the system of ODE (2.1*) or (2.1⁰) is invariant under the similarity transformation

(2.2) $$t \to \alpha^{-1} . t, \quad x_i \to \alpha^{g_i} . x_i, \quad i=1, 2, \dots, n$$

then this system is called reduction of the system (2.1) with respect to the transformation (2.2).

To check whether the system (2.1) possesses reduction for a given similarity transformation (2.2) it is convenient to substitute (2.2) into (2.1). Thus we obtain the system

(2.3) $$dx_i/dt = \alpha^{-g_i-1} . F_1(\alpha^{g_1} . x_1, \dots, \alpha^{g_n} . x_n), \quad i=1, 2, \dots, n.$$

If the system (2.3) reduces to an autonomous system for the limit $\alpha \to 0$ $(\alpha \to \infty)$ then the reduced system is (2.1*) ((2.1⁰)). The last system becomes automatically invariant under the transformation (2.2) and thus it is reduction of the system (2.1).

As Yoshida [11], § 6, II has noted, there is a connection between the first integrals of the system (2.1) and the first integrals of its reductions — the systems (2.1⁰) and (2.1*). The following two lemmas constitute the main result of this section.

Lemma 1. If the system (2.1) possesses m rational functionally independent first integrals then any reduction of the system (2.1) also possesses m rational functionally independent first integrals.

If we suppose that $g_1, g_2, \dots, g_n$ are positive rational numbers and the system (2.1*) is reduction of (2.1) then the following (local) version of Lemma 1 holds

Lemma 2. If the system (2.1) possesses m meromorphic functionally independent first integrals in a neighbourhood of the origine in C^n then the system (2.1*) possesses m rational functionally independent first integrals.

To prove Lemma 1 we note that if (2.1*) ((2.1⁰)) is reduction of (2.1), and Φ is a rational first integral of (2.1), then Φ^* (Φ^0) is a rational first integral of (2.1*) ((2.1⁰)). Indeed, if $\frac{d}{dt}$ and $\frac{d^*}{dt}$ are the time derivatives along the phase curves of (2.1) and (2.1*) respectively, and $\frac{d^*}{dt}\Phi^* \not\equiv 0$ then $\frac{d^*}{dt}\Phi^* = (\frac{d}{dt}\Phi)^* \equiv 0$. In a similar way we prove that if (2.1⁰) is reduction of (2.1) then Φ^0 is a first integral of (2.1⁰). To prove that the rational first integrals of (2.1*) or (2.1⁰) are functionally independent we use the Ziglin's algebraic lemma [5], § 1.2. Lemma 2 is proved along the same lines.

Example. Consider the equations describing the motion of a rigid body about a fixed point under the action of gravity and gyroscopic forces [2].

(2.4) $$I\dot{\omega} = (I\omega + \lambda) \times \omega + \varepsilon . e \times r$$
$$\dot{e} = e \times \omega,$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity, $I\omega = (A\omega_1, B\omega_2, C\omega_3)$ is the kinetic momentum, $e = (e_1, e_2, e_3)$ is the unit vector along the vertical axes of the inertial frame, $r = (x_0, y_0, z_0)$ is the center of mass, ε is the mass of the body, A, B, C are the principal moments of inertia and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is the gyrostatic moment of intrinsic cyclic motions in the body (due to symmetric rotors or holes completely filled with an ideal incompressible fluid). If $\lambda = \vec{0}$ we obtain the Euler-Poisson equations [1]. Denote these equations by (2.4⁰). If $\deg(\omega_i) = 1$, $\deg(e_i) = 2$, $i = 1, 2, 3$, then the system (2.4⁰) is invariant under similarity transformation (2.2) and thus it coincides with its reduction. According to Lemma 2 if the system (2.4⁰) possesses m first integrals, meromorphic in a neighbourhood of the origin in C^n, then it possesses also m rational first integrals. As the system (2.4⁰) is rationally integrable (i. e. all first integrals are rational functions) only in the three well known cases of Euler, Lagrange, and Kowalevski [1] then it is concluded that the Euler-Poisson equations are meromorphically integrable (i. e. the first integrals are meromorphic functions) only in the three cases mentioned above. A more general result has been shown recently by Ziglin [5]. He has proved the non-existence of sufficient number meromorphic first integrals for the system (2.4⁰) on the invariant hypersurface $\{e.e=1\}$ (and thus far from the origin in C^6) except in the classical cases of Euler, Lagrange and Kowalevski.

Noting that the system (2.4⁰) is a reduction of (2.4) we conclude, with the help of Lemma 1, that the system (2.4) may be rationally integrable only when the corresponding reduction (2.4⁰) is rationally integrable (see [3] for complete discussion).

3. Logarithmic Singularities and Non-Existence of Rational First Integrals. In what follows we suppose that the system (2.1⁰) is reduction of the system (2.1) with respect to the similarity transformation (2.2) ($g_1, g_2, \ldots, g_n$ are rational numbers) and it possesses a particular solution

$$x_1=c_1 . t^{-g_1}, \quad x_2=c_2 . t^{-g_2}, \ldots, \ x_n=c_n . t^{-g_n}. \tag{3.1}$$

After a change of the variables

$$x_1=(c_1+z_1) . t^{-g_1}, \quad x_2=(c_2+z_2) . t^{-g_2}, \ldots, \ x_n=(c_n+z_n) . t^{-g_n} \tag{3.2}$$

in (2.1) we obtain the following system of ODE for $z_1, z_2, \ldots, z_n$;

$$t . \frac{d}{dt} z_i=G_i(z_1, z_2, \ldots, z_n, t), \quad i=1, 2, \ldots, n, \tag{3.3}$$

where $G_1, \ldots, G_n$ are analytic functions with respect to $z_1, z_2, \ldots, z_n, t$ in a neighbourhood of the origin in C^{n+1} and $G_i(0, \ldots, 0)=0$ for $i=1, 2, \ldots, n$. According to [11] the matrix

$$K=\left(\frac{\partial G_i}{\partial z_j}(0, \ldots, 0)\right)_{i,j=1}^{n,n}=\left(\frac{\partial F_i^0}{\partial x_j}(c_1, c_2, \ldots, c_n)+g_i . \delta_{ij}\right)_{i,j=1}^{n,n}$$

(δ_{ij} is the Kronecker delta)
will be called the Kowalevski's matrix of the system (2.1). In view of Lemma 1 the main result of [11] can be formulated in the following way:

Theorem (H. Yoshida [11], II). If the system of ODE (2.1) is rationally integrable then every Kowalevski's exponent (i. e. an eigenvalue of the Kowalevski's matrix) is a rational number.

Suppose the Kowalevski's exponents are rational numbers. Let $\rho_1, \rho_2, \ldots, \rho_k$ be the positive Kowalevski's exponents, $N=\sum_{i=1}^{k} \operatorname{mult}(\rho_i)$ and q is the least common multiple of the denominators of $\rho_1, \rho_2, \ldots, \rho_k$.

Lemma 3. Either the system (2.1) possesses a family of solutions of the type

$$x_i=\left(c_i+\sum_{j=1}^{\infty} a_{ij} . t^{j/q}\right) . t^{-g_i}, \quad i=1, 2, \ldots, n, \tag{3.4}$$

where the coefficients $a_{i,j}$ of the expansion (3.4) depend upon N arbitrary constants of integration, or the system (2.1) possesses a 1-parameter family of solutions of the type

$$x_i=\left(c_i+\sum_{j=1}^{\infty} b_{i,j}(\alpha+\log t) . t^{j/q}\right) . t^{-g_i}, \quad i=1, 2, \ldots, n \tag{3.5}$$

such that the coefficients $b_{i,j}$ are polynomials of degree j or less in variables $(\alpha+\log t)$, and α is the parameter.

We note that the (formal) asymptotic expansions (3.4) and (3.5) are convergent for sufficiently small $|t|$ [8]. Till the end of this section the system (2.1) will be a 2-dimensional one. Suppose that the Kowalevski's matrix possesses a positive Kowalevski's exponent. Then the following theorem holds

Theorem 1. If the two-dimensional system of ODE (2.1) possesses a rational first integral then this system possesses a 1-parameter family of solutions of the type (3.4).

Using Lemma 3 it is not difficult to prove Theorem 1. Indeed, if Φ is a rational first integral of (2.1) but the system does not possess a 1-parameter family of solutions (3.4), then there is a 1-parameter family of solutions of the type (3.5). It is clear that the substitution of this 1-parameter family of solutions into Φ must be a function $f(\alpha)$ of α and $f(\alpha)\not\equiv$const. However, if f depends upon α then it has to depend upon $(\alpha+\log t)$ and it is a contradiction (Φ is a first integral).

For the practical use of the above theorem it is convenient to apply the long known test whether a given system of ODE is of P-type or not (see [6] for example). However, for our purposes this test has to be slightly modified to include the systems of the so called 'weak' P-type [9] (i. e. movable algebraic points are also allowed). The last means that besides integer Kowalevski's exponents rational ones are also possible. In view of the Yoshida's theorem and Theorem 1 of the present paper we can state the following theorem

Theorem 2. If the two-dimensional system of ODE (2.1) does not pass the weak Painlevé property test [6,9] then this system does not possess a rational first integral.

Acknowledgements are due to Emil Horozov for the useful discussions.

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