The holonomy group at infinity of the Painlevé VI equation

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We prove that the holonomy group at infinity of the Painlevé VI equation is virtually commutative. © *2012 American Institute of Physics.* [doi:10.1063/1.3681897]

I. INTRODUCTION

The sixth Painlevé equation (PVI)

$$\frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}$$

$$+ \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[\alpha - \beta\frac{t}{\lambda^2} + \gamma\frac{t-1}{(\lambda-1)^2} + (\frac{1}{2}-\delta)\frac{t(t-1)}{(\lambda-t)^2}\right] \tag{1}$$

is a family of differential equations parameterized by $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$.[8] *The purpose of the present paper is to show that the holonomy group of* (PVI) *at infinity is virtually commutative.* The precise meaning is as follows. It is straightforward to check that (1) is equivalent to a non-autonomous Hamiltonian system (the so called sixth Painlevé system)

$$\begin{cases} \dfrac{d\lambda}{dt} = \dfrac{\partial H}{\partial \mu}, \\ \dfrac{d\mu}{dt} = -\dfrac{\partial H}{\partial \lambda}, \end{cases} \tag{2}$$

where

$$H := \frac{1}{t(t-1)}\left[\lambda(\lambda-1)(\lambda-t)\mu^2 + \{\kappa_0(\lambda-1)(\lambda-t)\right.$$

$$\left. +\kappa_1\lambda(\lambda-t) + (\kappa_t+1)\lambda(\lambda-1)\}\mu + \kappa(\lambda-t)\right] \tag{3}$$

and

$$\alpha = \frac{1}{2}\kappa_\infty^2, \ \beta = \frac{1}{2}\kappa_0^2, \ \gamma = \frac{1}{2}\kappa_1^2, \ \delta = \frac{1}{2}\kappa_t^2, \kappa = \frac{1}{2}\left[(\kappa_0+\kappa_1+\kappa_t+1)^2 - \kappa_\infty^2\right]. \tag{4}$$

The phase space of the above system is

$$\{(\lambda, \mu, t) \in \mathbb{C}^3 : t \neq 0, 1\},$$

which we partially compactify to $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \{\mathbb{C} \setminus \{0, 1\}\}$. It is immediately seen that the projective lines

$$\Gamma_c = \{\mu = \infty, t = c\} \subset M, c \neq 0, 1$$

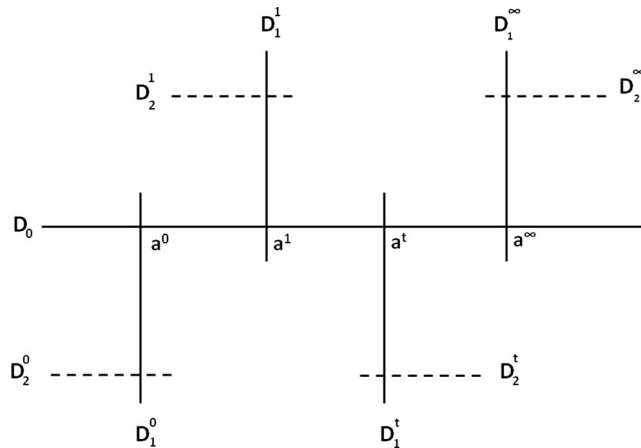a)Author to whom correspondence should be addressed. Electronic mail: lubomir.gavrilov@math.univ-toulouse.fr.

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FIG. 1. The divisor $\bar{E}_t \setminus E_t$.

are leaves of the one-dimensional foliation induced by (2) on M. On each leaf Γ_c the foliation has four singular points defined by $\lambda = 0, 1, c, \infty$. Let $P \neq 0, 1, c, \infty$ be a point on Γ_c and consider a germ of a cross-section $(\mathbb{C}^2, 0)$ to Γ_c at P. The holonomy group G at infinity is then the image of the holonomy representation

$$\pi_1(\Gamma_c \setminus \{0, 1, t, \infty\}, P) \to Diff(\mathbb{C}^2, 0). \tag{5}$$

It is defined up to a conjugation by a diffeomorphism, depending on the germ of cross-section and the initial point P. Our main result is

Theorem 1: *The holonomy group at infinity of the sixth Painlevé equation is virtually commutative.*

Recall that a group G is said to be virtually commutative, provided that there is a normal commutative subgroup $G^0 \subset G$, such that G/G^0 is finite. The isomorphism class of the holonomy group G along the leaf Γ_c has in fact a canonical meaning. As we shall see in Sec. II, the leaf Γ_c coincides with the divisor $D_0(c)$ in the Okamoto compactification[15] of the phase space of **PVI**, see Fig. 1. In particular, the holonomy group along Γ_c is isomorphic to the holonomy group along the Okamoto divisor $D_0(c)$. The remaining divisors shown on Fig. 1 are topological cylinders, the associated holonomy has therefore one generator and is commutative.

The proof of Theorem 1 is based on Lemma 2 which claims that the local holonomies near the singular points of the leaf Γ_c are involutions, as well on the algebraic Lemma 1. Lemma 2 and Lemma 1 suggest that Theorem 1 is related to the fact that the vertical divisor shown on Fig. 1 belongs to the Kodaira list of degenerate elliptic curves.

Let E_k be the kth order variational equation along Γ_c and G_k the associated differential Galois group. E_k defines a connection on the Riemann sphere $\Gamma_c = \mathbb{P} \setminus \{0, 1, c, \infty\}$ with four regular singular points at the punctures $\{0, 1, c, \infty\}$. The monodromy group of E_k represents the kth order jet of the holonomy group along $D_0(c)$. We describe these monodromy groups in the simplest cases $k = 1, 2$ in Sec. III. It follows, for instance, that the monodromy group of E_1 is isomorphic to a semi direct product $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$, while $G_1 = \mathbb{C}^2 \rtimes \mathbb{Z}_2$. In particular, G_1 as well G_2 are virtually commutative. This is a particular case of a general fact. According to Theorem 1, the monodromy group of E_k is virtually commutative for all k. As its Zarisky closure is G_k, then we also have the following

Theorem 2: *For every k the differential Galois group G_k is virtually commutative.*

The present paper was motivated by the study of the Liouville non-integrability of the PVI system through the Ziglin-Morales-Ramis-Simo theory of non-integrability.[10, 11, 13, 14] This theory asserts that integrability in a Liouville sense along a particular solution Γ_c implies that the variational

equation E_1, as well all higher order variational equations E_k along this solution, has virtually commutative differential Galois groups. Indeed, in such a way the "semi-local" non-integrability in a neighborhood of some particular solutions and parameter values of the PVI system has been recently proved by Horozov and Stoyanova,[7,16] see also Morales-Ruiz.[12] To prove the non-integrability for all parameters we need, however, an explicitly known particular solution which exists for all parameter values. The only such appropriate solution is the vertical divisor $\Gamma_c = D_0(c)$, defined in Theorem 1. The result of Theorem 2 shows that, contrary to what we expected, one can not prove the absence of a first integral of the PVI equation, by making use of the Ziglin-Morales-Ramis-Simo theory. It is an open question, whether the PVI equation has a first integral, meromorphic along the divisor Γ_c "at infinity." This question, but in a more general setting, has been raised in Ref. 11, Sec. 7.

Non-integrability or transcendency of solutions is one of the central subjects in the study of the PVI equation. The fact that its general solution can not be reduced to a solution of a first order differential equation has been claimed already by Painlevé, and proved more recently by Watanabe[18] and others. A different approach to the transcendency, going back to Drach and Vessiot, is to interpret it as an irreducibility of the Galois groupoid defined by Malgrange, see Refs. 2–4, and 9. The irreducibility of the PVI equation in the sense of Drach-Vessiot-Malgrange has been shown by Cantat and Loray (Ref. 1, Theorem 7.1). It follows from these results that the PVI equation does not allow an additional rational first integral. The relation between the irreducibility of the Galois groupoid of a Hamiltonian system and the differential Galois group along a given algebraic solution is studied recently by Casale.[5] In this context, our Theorem 1 comes at a first sight as a surprise. The solution Γ_c which we use is however rather special: it is an irreducible component of the anti-canonical divisor of the space of initial conditions, and hence it is invariant under the action of the Galois groupoid. This leads to special properties of the Galois groupoid along Γ_c too.

The paper is organized as follows. In Sec. II, we resume briefly the Okamoto compactification of the phase space of PVI equation.[15] In Sec. III, we describe the monodromy group of the first and the second variational equation along Γ_c, in terms of complete elliptic integrals of first and second kind. These groups provide an approximation of the holonomy group along Γ_c. Our main result, Theorem 1, is proved in Sec. IV.

II. THE OKAMOTO COMPACTIFICATION

Let (E, π, B) be a complex-analytic fibration with base B, total space E, and projection $\pi : E \to B$. Consider a foliation $\mathcal{F}$ on E of dimension equal to the dimension of B. Following Ref. 15, we say that $\mathcal{F}$ is P-uniform if for every leaf $\Gamma \subset E$ the induced map

$$\pi : \Gamma \to B$$

is an analytic covering. Thus, for every initial point $e \in E$, and every continuous path $\gamma \subset B$ starting at $b = \pi(e)$, there is a unique continuous path $\tilde{\gamma} \subset E$ starting at e, which is a lift of γ with respect to π (the "Painlevé property" of the foliation). The analyticity of π implies moreover that at each point $e \in E$ the leaf of the foliation is transversal to the corresponding fiber of the fibration.

From now on, we put

$$E = \{(\lambda, \mu, t) \in \mathbb{C}^3 : t \neq 0, 1\}, B = \mathbb{C} \setminus \{0, 1\}$$

$$\pi : E \to B : (\lambda, \mu, t) \mapsto t$$

being the natural projection. The system (2) defines a one-dimensional foliation $\mathcal{F}$ on the total space E which is not P-uniform, but can be completed to a P-uniform foliation after an appropriate partial compactification $\bar{E}$ of E.

The main result of Ref. 15 may be formulated as follows.

Theorem 3: *There exists a canonical compact complex-analytic fibration $(\bar{E}, \bar{\pi}, B)$, such that*

- $E \subset \bar{E}, \bar{\pi}|_E = \pi$.
- *Each fiber $\bar{E}_t = \bar{\pi}^{-1}(t)$ is compact.*

- $\bar{E}_t \setminus E_t$ *is a union of nine transversal projective lines, as it is shown on Fig. 1. The intersection points of the lines depend analytically on t.*
- *Let D_t be the union of five solid lines shown on Fig. 1. The foliation induced by (2) on $\tilde{E} = \bar{E} \setminus \cup_{t \in B} D_t$ is P-uniform with respect to the induced projection.*

A similar result holds true for the remaining Painlevé equations.[15]

Remark: $\bar{E}_t \setminus D_t$ *is the so called "space of initial conditions" of the Painlevé VI equation which we describe next.*

Sketch of the proof of Theorem 3: Following Ref. 15, define first the Hirzebruch surface $\Sigma_{(\epsilon)}^{(2)}$, $\varepsilon \in \mathbb{C}$, using four charts $W_i = \mathbb{C}^2$, with local coordinates (λ_i, μ_i), $i = 1, \ldots, 4$, where

$$\begin{cases} \lambda_2 = \lambda_1, \ \mu_2 = \dfrac{1}{\mu_1} & \text{in} \quad W_1 \cap W_2, \\[2mm] \lambda_3 = \dfrac{1}{\lambda_1}, \ \mu_3 = \epsilon\lambda_1 - \lambda_1^2\mu_1 & \text{in} \quad W_1 \cap W_3, \\[2mm] \lambda_4 = \lambda_3, \ \mu_4 = \dfrac{1}{\mu_3} & \text{in} \quad W_3 \cap W_4. \end{cases} \tag{6}$$

If $\epsilon \neq 0$, then the Hirzebruch surface $\Sigma_{(\epsilon)}^{(2)}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; otherwise, it is isomorphic to the tangent projective bundle of $\mathbb{P}^1$ with projection

$$\Sigma_{(\epsilon)}^{(2)} \to \mathbb{P}^1,$$

$$(\lambda_i, \mu_i) \mapsto \lambda_i \ .$$

The vector field (2) extends on the total space of the trivial bundle

$$\Sigma_{(\epsilon)}^{(2)} \times B \overset{\pi}{\to} B, \ B = \mathbb{P}^1 \setminus \{0, 1, \infty\}, \tag{7}$$

where $\epsilon = -(\kappa_0 + \kappa_1 + \kappa_t + \kappa_\infty + 1)$. For instance, in the chart W_2 it takes the form

$$\begin{cases} \mu_2\lambda_2' &= \dfrac{1}{t(t-1)}\left[2E(t, \lambda_2) + F(t, \lambda_2)\mu_2\right], \\[2mm] \mu_2' &= \dfrac{1}{t(t-1)}\left[E_\lambda(t, \lambda_2) + F_\lambda(t, \lambda_2)\mu_2 + G\mu_2^2\right], \end{cases} \tag{8}$$

where

$$\begin{cases} E(t, \lambda) = \lambda(\lambda - 1)(\lambda - t), \\[2mm] F(t, \lambda) = \kappa_0(\lambda - 1)(\lambda - t) + \kappa_1\lambda(\lambda - t) + (\kappa_t + 1)\lambda(\lambda - 1), \\[2mm] E_\lambda = \dfrac{\partial E}{\partial \lambda}, \ F_\lambda = \dfrac{\partial F}{\partial \lambda}, \\[2mm] G = -\dfrac{1}{2}\epsilon\left(\kappa_0 + \kappa_1 + \kappa_t - \kappa_\infty + 1\right) = \kappa. \end{cases} \tag{9}$$

The above meromorphic vector field induces a singular foliation on $\Sigma_{(\epsilon)}^{(2)}$ having four one-parameter families of singular points S^θ, $\theta = 0, 1, \infty, t$ defined by

$$S^\theta \cap \pi^{-1}(t) = a^\theta(t),$$

$$a^0(t) = \{(\lambda_2, \mu_2) = (0, 0)\},$$

$$a^1(t) = \{(\lambda_2, \mu_2) = (1, 0) \text{ or } (\lambda_4, \mu_4) = (1, 0)\},$$

$$a^t(t) = \left\{(\lambda_2, \mu_2) = (t, 0) \text{ or } (\lambda_4, \mu_4) = (\frac{1}{t}, 0)\right\},$$

$$a^\infty(t) = \{(\lambda_4, \mu_4) = (0, 0)\}.$$

Replace the Hirzebruch surface $\Sigma^{(2)}_{(\epsilon)}$ by $\Sigma^{(2)}_{(\epsilon)}$ blown up at $a^\theta(t)$ for every $t \in B$. This replaces each $a^\theta(t)$ by a projective line denoted $D_1^\theta(t)$. The induced foliation has still four one-parameter families of singular points which belong to $D_1^\theta(t)$. We blow up once again the surfaces at these singular points to obtain the fibers $\bar{E}_t$ of the fibration described in Theorem 3, see Fig. 1. The remaining claims of the Theorem follow by computation.   □

III. HIGHER ORDER VARIATIONAL EQUATIONS AND THEIR MONODROMY GROUPS

In this section, we consider the foliation $\mathcal{F}$ defined by the vector field (2) on the total space of the fibration $(\bar{E}, \bar{\pi}, B)$, see Theorem 3. This foliation has in each fiber $\pi^{-1}(t)$ a vertical leaf $D_0(t)$, which in the chart W_2 takes the form

$$D_0(t) : \mu_2 = 0.$$

According to (8) the foliation $\mathcal{F}$ in the local chart W_2 is defined by

$$
\begin{cases}
d\mu &= \dfrac{\left[E_\lambda(t, \lambda) + F_\lambda(t, \lambda)\mu + G\mu^2 \right]\mu}{2E(t, \lambda) + F(t, \lambda)\mu} d\lambda, \\[4mm]
dt &= \dfrac{t(t-1)\mu}{2E(t, \lambda) + F(t, \lambda)\mu} d\lambda,
\end{cases}
\tag{10}
$$

where E, F, and G are given by (9). Here, as well until the end of the paper, we replace for simplicity μ_2, λ_2 by μ, λ.

In this section, we compute the first and the second variational equations of (10) along $D_0(c)$ and study the corresponding monodromy groups. For this purpose, we put, following Refs. 7 and 11,

$$t = c + \varepsilon\eta_1 + \frac{\varepsilon^2}{2}\eta_2 + \cdots, \quad \mu = \varepsilon\xi_1 + \frac{\varepsilon^2}{2}\xi_2 + \cdots, \quad \varepsilon \sim 0,$$

where $\eta_k = \eta_k(\lambda)$, $\xi_k = \xi_k(\lambda)$ are unknown functions and substitute these expressions in (10). Equating the coefficients of ϵ^k, we get a recursive system of linear non-homogeneus equations on (η_k, ξ_k) – the higher order variational equations. We note that these equations, except in the case $k = 1$, are non-linear. In order to obtain a linear system, we add suitable monomials in η_i, ξ_j, e.g., Refs. 11 and 14. The fundamental matrices of solutions of these equations are then explicitly computed by the Picard method in terms of iterated integrals. This implies also a description of the corresponding monodromy matrices. In Secs. III A and III B, we carry out this procedure in the particular case of the first and the second variational equation.

A. The first variational equation

The first variational equation E_1 along D_0 is the linear system

$$
\begin{pmatrix} \dot{\eta}_1 \\ \dot{\xi}_1 \end{pmatrix} = \begin{pmatrix} 0 & b(\lambda) \\ 0 & a(\lambda) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \xi_1 \end{pmatrix},
\tag{11}
$$

where

$$a(\lambda) = \frac{E_\lambda(c, \lambda)}{2E(c, \lambda)},$$

$$b(\lambda) = \frac{c(c-1)}{2E(c, \lambda)}.$$

The general solution of the system (11) is given by

$$\eta_1(\lambda) = c_1 \int_p^\lambda \frac{c(c-1)d\lambda}{2\sqrt{\lambda(\lambda-1)(\lambda-c)}} + c_2,$$

$$\xi_1(\lambda) = c_1 \sqrt{\lambda(\lambda-1)(\lambda-c)}.$$

where $(c_1, c_2) \in \mathbb{C}^2$ and $p \in \mathbb{C}$ is a fixed initial point. The fundamental matrix of solutions

$$X(\lambda) = \begin{pmatrix} \int_p^\lambda \dfrac{c(c-1)d\lambda}{2\sqrt{\lambda(\lambda-1)(\lambda-c)}} & 1 \\ \\ \sqrt{\lambda(\lambda-1)(\lambda-c)} & 0 \end{pmatrix}$$

is multivalued, and the result of the analytic continuation of $X(.)$ along small loops making one turn around $\lambda = 0, 1, c$ respectively is

$$X \to XT_0, \, X \to XT_1, \, X \to XT_c.$$

The matrices T_0, T_1, T_c generate the monodromy group of (11) and can be computed as follows. Let S_c be the compact elliptic Riemann surface of the algebraic function $\sqrt{\lambda(\lambda-1)(\lambda-c)}$. It has an affine equation

$$\{(\lambda, y) : y^2 = \lambda(\lambda-1)(\lambda-c)\}. \tag{12}$$

The one-form

$$\frac{d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-c)}}$$

is holomorphic on S_c and hence $X(.)$ can be seen as a globally multivalued, but locally meromorphic matrix function on S_c. This implies that

$$T_0^2 = T_1^2 = T_c^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and hence

$$T_0 = \begin{pmatrix} -1 & 0 \\ \alpha_0 & 1 \end{pmatrix}, \, T_1 = \begin{pmatrix} -1 & 0 \\ \alpha_1 & 1 \end{pmatrix}, \, T_c = \begin{pmatrix} -1 & 0 \\ \alpha_c & 1 \end{pmatrix}. \tag{13}$$

The constants $\alpha_0, \alpha_1, \alpha_c$ depend on the initial point p and can be determined as follows. The matrix

$$T_0 T_1 = \begin{pmatrix} 1 & 0 \\ \alpha_1 - \alpha_0 & 1 \end{pmatrix}$$

represents the monodromy of $X(.)$ along a closed loop on the λ-plane, which lifts, on on the Riemann surface of $\sqrt{\lambda(\lambda-1)(\lambda-c)}$ to a closed loop too, which we denote γ. The monodromy of the fundamental matrix X along this loop is $T_0 T_1$, and we have

$$X \to XT_0 T_1 = X + \begin{pmatrix} \Pi & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\Pi = \int_\gamma \frac{c(c-1)d\lambda}{2\sqrt{\lambda(\lambda-1)(\lambda-c)}} = \int_0^1 \frac{c(c-1)d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-c)}}$$

is a period of the holomorphic one-form on S_c. Therefore,

$$\alpha_1 - \alpha_0 = \int_0^1 \frac{c(c-1)d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-c)}}$$

and in a similar way

$$\alpha_c - \alpha_0 = \int_0^c \frac{c(c-1)d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-c)}}.$$

Finally, taking the limit $p \to 0$ we obtain $\alpha_0 = 0$. The monodromy group of (11) is therefore

$$< T_0, T_1, T_c > = \{ \begin{pmatrix} \pm 1 & 0 \\ p\Pi_1 + q\Pi_2 & 1 \end{pmatrix} : p, q \in \mathbb{Z} \}, \tag{14}$$

where

$$\Pi_1 = \int_0^1 \frac{c(c-1)d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-c)}}, \quad \Pi_2 = \int_0^c \frac{c(c-1)d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-c)}}$$

are the fundamental periods of the elliptic surface S_c. As Π_1, Π_2 are linearly independent over $\mathbb{R}$, then we obtain

Proposition 1: For every $c \neq 0$, 1, t, the monodromy group (14) of the first variational equation (11) is isomorphic to the semidirect product $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

It is well known that for a Fuchs type equation the Zariski closure of the monodromy group is the differential Galois group (e.g., Ref. 17). Therefore, the Galois group G_1 of (11) is

$$G_1 = \mathbb{C} \rtimes \mathbb{Z}_2 = \{ \begin{pmatrix} \pm 1 & 0 \\ z & 1 \end{pmatrix} : z \in \mathbb{C} \}.$$

In particular, G_1 is virtually commutative.

B. The second variational equation

The second variational equation E_2 along the divisor $D_0(c)$ reads

$$\dot{\xi}_2 = d(\lambda)(\xi_1)^2 + e(\lambda)\xi_1\eta_1 + a(\lambda)\xi_2,$$
$$\dot{\eta}_2 = f(\lambda)(\xi_1)^2 + g(\lambda)\xi_1\eta_1 + b(\lambda)\xi_2, \tag{15}$$

where

$$a(\lambda) = \frac{E_\lambda(c, \lambda)}{2E(c, \lambda)},$$

$$b(\lambda) = \frac{c(c-1)}{2E(c, \lambda)},$$

$$d(\lambda) = \frac{2E(c, \lambda)F_\lambda(c, \lambda) - E_\lambda(c, \lambda)F(c, \lambda)}{4(E(c, \lambda))^2},$$

$$e(\lambda) = \frac{-(2\lambda - 1)E(c, \lambda) + \lambda(\lambda - 1)E_\lambda(c, \lambda)}{2(E(c, \lambda))^2},$$

$$f(\lambda) = -\frac{c(c-1)F(c, \lambda)}{4(E(c, \lambda))^2},$$

$$g(\lambda) = \frac{(2c - 1)E(c, \lambda) + c(c-1)\lambda(\lambda - 1)}{2(E(c, \lambda))^2}.$$

Having computed μ_1, ξ_1, this is a linear non-homogeneous equation in μ_2, ξ_2, but it is also equivalent to the linear system

$$\begin{pmatrix} \dot{\eta}_2 \\ \dot{\xi}_2 \\ \dot{u}_1 \\ \dot{v}_1 \end{pmatrix} = \begin{pmatrix} 0 & b(\lambda) & g(\lambda) & f(\lambda) \\ 0 & a(\lambda) & e(\lambda) & d(\lambda) \\ 0 & 0 & a(\lambda) & b(\lambda) \\ 0 & 0 & 0 & 2a(\lambda) \end{pmatrix} \begin{pmatrix} \eta_2 \\ \xi_2 \\ u_1 \\ v_1 \end{pmatrix}, \tag{16}$$

where $u_1 = \xi_1\eta_1$ et $v_1 = (\xi_1)^2$.

The substitution

$$\sigma_2 = \frac{\xi_2}{\sqrt{\lambda(\lambda-1)(\lambda-c)}}, \ \ u_2 = \frac{u_1}{\sqrt{\lambda(\lambda-1)(\lambda-c)}}, \ \ v_2 = \frac{v_1}{\lambda(\lambda-1)(\lambda-c)},$$

transforms (16) to a strictly upper triangular form

$$\begin{pmatrix} \dot{\eta}_2 \\ \dot{\sigma}_2 \\ \dot{u}_2 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} 0 & a_{12}(\lambda) & a_{13}(\lambda) & a_{14}(\lambda) \\ 0 & 0 & a_{23}(\lambda) & a_{24}(\lambda) \\ 0 & 0 & 0 & a_{34}(\lambda) \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_2 \\ \sigma_2 \\ u_2 \\ v_2 \end{pmatrix}, \tag{17}$$

where

$$a_{12}(\lambda) = \sqrt{\lambda(\lambda-1)(\lambda-c)}\, b(\lambda),$$
$$a_{13}(\lambda) = \sqrt{\lambda(\lambda-1)(\lambda-c)}\, g(\lambda),$$
$$a_{14}(\lambda) = \lambda(\lambda-1)(\lambda-c)\, f(\lambda),$$
$$a_{23}(\lambda) = e(\lambda),$$
$$a_{24}(\lambda) = \sqrt{\lambda(\lambda-1)(\lambda-c)}\, d(\lambda)$$
$$a_{34}(\lambda) = a_{12}(\lambda).$$

The linear system (17) is solved recursively in terms of iterated integrals. Namely, for differential forms $\omega_i(x) = f_i(x)dx$ on the interval $[0, 1]$ define the linear iterated integrals

$$\int_0^1 \omega_1\omega_2 = \int_0^1 f_1(y)\Big(\int_0^y f_2(x)dx\Big)dy,$$

$$\int_0^1 \omega_1\omega_2\omega_3 = \int_0^1 f_1(z)\Big[\int_0^z f_2(y)\Big(\int_0^y f_3(x)dx\Big)dy\Big]dz.$$

Integrals of higher order and along a path on a Riemann surface are defined in a similar way, e.g., Ref. 6. The fundamental matrix of solutions of the linear system (17) takes the form

$$Y(\lambda) = I + \int_P^\lambda J + \int_P^\lambda J^2 + \int_P^\lambda J^3 = \begin{pmatrix} 1 & Y_{12}(\lambda) & Y_{13}(\lambda) & Y_{14}(\lambda) \\ 0 & 1 & Y_{23}(\lambda) & Y_{24}(\lambda) \\ 0 & 0 & 1 & Y_{34}(\lambda) \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$J = \begin{pmatrix} 0 & \omega_{12}(\lambda) & \omega_{13}(\lambda) & \omega_{14}(\lambda) \\ 0 & 0 & \omega_{23}(\lambda) & \omega_{24}(\lambda) \\ 0 & 0 & 0 & \omega_{34}(\lambda) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \ \omega_{ij} = a_{ij}(\lambda)d\lambda.$$

and hence

$$Y_{12}(\lambda) = \int_P^\lambda \omega_{12},$$

$$Y_{13}(\lambda) = \int_P^\lambda \omega_{12}\omega_{23} + \int_P^\lambda \omega_{13},$$

$$Y_{14}(\lambda) = \int_P^\lambda \omega_{12}\omega_{23}\omega_{34} + \int_P^\lambda \omega_{12}\omega_{24} + \int_P^\lambda \omega_{13}\omega_{34} + \int_P^\lambda \omega_{14},$$

$$Y_{23}(\lambda) = \int_P^\lambda \omega_{23},$$

$$Y_{24}(\lambda) = \int_P^\lambda \omega_{23}\omega_{34} + \int_P^\lambda \omega_{24},$$

$$Y_{34}(\lambda) = \int_P^\lambda \omega_{34}.$$

As for the first variational equation, the fundamental matrix $Y(\lambda)$ is a globally multivalued, but locally meromorphic matrix function on the elliptic curve S_c, (12) (after removing eventually the points $\lambda = 0, 1, c, \infty$). The monodromy matrix T_α of Y along a closed path α is given therefore by the same matrix Y, in which the integrals $\int_P^\lambda$ are replaced by $\int_\alpha$. A more careful analysis will show, however, that when α can be lifted to a closed loop on S_c, then the double and triple iterated integrals in T_α are reduced to usual complete elliptic integrals of first and second kind. This would imply the involutivity of the monodromy operators T_0, T_1, T_c (defined in Sec. III A) as well the virtual commutativity of the monodromy group of E_2.

Let α be a closed path on $D_0(c)$ which lifts to a closed path on the elliptic curve S_c, where

$$S_c \rightarrow D_0(c),$$

$$(\lambda, \mu) \mapsto \lambda,$$

are double ramified covering over $\lambda = 0, 1, c, \infty$. Denote the monodromy matrix of the second variational equation (16) along α by T_α.

Proposition 2: The entries of the monodromy matrix T_α along a closed loop α on the elliptic surface S_c are quadratic polynomials in the complete elliptic integrals of first and second kind along α.

Until the end of this subsection, we sketch the proof the Proposition 2. Note that $\omega_{23} = d\,h$ is an exact form, where

$$h(\lambda) = -\frac{\lambda(\lambda-1)}{2E(c,\lambda)} = -\frac{1}{2(\lambda-c)}.$$

This combined with the identity

$$\int_P^\lambda \omega_1\omega_2 + \int_P^\lambda \omega_2\omega_1 = \int_P^\lambda \omega_1 \int_P^\lambda \omega_2$$

allows to express the iterated integrals of length two and three via usual Riemann integrals of meromorphic one-forms. Indeed, for every differential 1-form ω, we have

$$\int_P^\lambda \omega_{23}\omega = h(\lambda) \int_P^\lambda \omega - \int_P^\lambda h(\lambda)\omega. \tag{18}$$

It follows that $Y_{13}(\lambda)$ and $Y_{24}(\lambda)$ are in fact Riemann integrals along meromorphic differential forms. The reader may check that these one-forms have no residues on S_c. It remains to analyze $Y_{14}(\lambda)$. Using (18), we obtain

$$\int_P^\lambda \omega_{12}\omega_{23}\omega_{12} = \int_P^\lambda (\omega_{12}h(\lambda))\,\omega_{12} - \int_P^\lambda \omega_{12}(h(\lambda)\omega_{12}), \tag{19}$$

which implies

$$\int_P^\lambda \omega_{12}\omega_{23}\omega_{12} + \int_P^\lambda \omega_{12}\omega_{24} + \int_P^\lambda \omega_{13}\omega_{12} = \int_P^\lambda \omega_{12}\overline{\omega}$$
$$+ (\int_P^\lambda \omega_{12})(\int_P^\lambda h\,\omega_{12} + \int_P^\lambda \omega_{13}),$$

where $\overline{\omega} := (-2h(\lambda)b(\lambda) + d(\lambda) - g(\lambda))\sqrt{E(c, \lambda)}\,d\lambda$. The two forms ω_{12} and $\overline{\omega}$ have dependent cohomology classes in $H^1_{DR}(\Gamma)$, where S_c is the elliptic curve(12). In fact,

$$\omega_{12} = \frac{c(c-1)d\lambda}{2\sqrt{E(c, \lambda)}} \quad \text{and} \quad \overline{\omega} = \left(\frac{1-2c}{2}\right)\frac{d\lambda}{\sqrt{E(c, \lambda)}} + \frac{1}{2}d\left(\frac{F(c, \lambda)}{\sqrt{E(c, \lambda)}}\right).$$

Thus, everything is reduced to quadratic expressions in suitable Riemann integrals along meromorphic differential form without residues on S_c. (the latter claim is straightforward to check). From this the Proposition follows. □

Corollary 1: Let α, β be closed loops on S_c starting at the same point, so they can be composed. Then $T_\alpha T_\beta = T_\beta T_\alpha$ and the monodromy group of E_2 is virtually commutative.

Indeed, as the homology class of the loop $\alpha\beta\alpha^{-1}\beta^{-1}$ is zero, then

$$T_\alpha T_\beta T_\alpha^{-1} T_\beta^{-1} = T_{\alpha\beta\alpha^{-1}\beta^{-1}}$$

is the unit matrix. Note that the monodromy operators along closed loops on S_c generate a subgroup of the monodromy group of the second variational equation (16) of finite index. Therefore, as expected, the monodromy group is virtually commutative.

IV. THE HOLONOMY AT INFINITY

Let Γ be a leaf of the Painlevé VI foliation and $(\mathbb{C}^2, 0)$ be a germ of a cross-section to Γ at some regular point P. Each homotopy class of closed loops $\gamma \subset \Gamma$, starting at P, defines a germ of a diffeomorphism

$$h_\gamma : \mathbb{C}^2, 0 \to \mathbb{C}^2, 0$$

and a homomorphism (the holonomy representation of the fundamental group of Γ)

$$\pi_1(\Gamma, P) \to Diff(\mathbb{C}^2, 0) : \gamma \mapsto h_\gamma.$$

The holonomy group of the foliation along Γ is the image of this map (which will be confounded with the representation itself). Different points in the leaf and different cross-sections give rise to representations conjugated by germs of holomorphic diffeomorphisms.

The holonomy group of the Painlevé VI foliation at infinity is, by definition, the holonomy group along the vertical leaf $D_0 = D_0(c)$, $c \neq 0, 1, \infty$, shown on Fig. 1. The leaf $D_0(c)$ is a four-punctured Riemann sphere, the punctures corresponding to a^0, a^1, a^c, a^∞. The holonomy group of the Painlevé VI foliation along $D_0(c)$ is generated by three germs of analytic diffeomorphisms

$$h_0, h_1, h_c \tag{20}$$

corresponding to loops on $D_0(c)$ making one turn around a^0, a^1, a^c respectively.

The main result of the paper, Theorem 1, follows from Lemma 1 and Lemma 2 formulated bellow.

Let G be a group with three generators a, b, c and the following defining relations

$$a^2 = b^2 = c^2 = (abc)^2 = 1.$$

An element $g \in G$ can be therefore represented by a word formed by the letters a, b, c. The length $l(g)$ of a word $g \in G$ is the number of letters in g, and only the equivalence class of $l(g)$ in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$

is well defined. We get a homomorphism

$$G \to \mathbb{Z}_2 : g \mapsto l(g)$$

and let G^0 be its kernel.

Lemma 1: The group G^0 is commutative.

Proof: The above lemma has a transparent geometric meaning : G^0 is identified to the fundamental group of an elliptic curve which, as well known, is commutative. To see this, consider the compact Riemann surface S with affine model

$$S = \{(x, y) : y^2 = x(x - 1)(x - t)\}, t \neq 0, 1$$

as well the natural projection

$$\pi : S \to \mathbb{P} : (x, y) \to x.$$

Let $\tilde{S} = S \setminus \{(x, 0) : x = 0, 1, t, \infty\}$ and $\tilde{\mathbb{P}} = \mathbb{P} \setminus \{0, 1, t, \infty\}$. Let $P \in \tilde{S}$ and, by abuse of notation, $p = \pi(P)$. The fundamental group $\pi_1(\tilde{\mathbb{P}}, p)$ is the free group generated by a, b, c, where a, b, c are represented by closed loops making one turn around $0, 1, t$. With orientations appropriately chosen, abc is represented by a loop around ∞. The fundamental group $\pi_1(\tilde{S}, p)$ is free with five generators. The projection

$$\pi : \tilde{S} \to \mathbb{P} : (x, y) \to x$$

is a two sheeted covering which induces a monomorphism

$$\pi_* : \pi_1(\tilde{S}, P) \to \pi_1(\tilde{\mathbb{P}}, p),$$

such that $\pi_1(\tilde{\mathbb{P}}, p)/\pi_*(\pi_1(\tilde{S}, P)) = \mathbb{Z}_2$. This can be resumed in the following exact sequence of homomorphisms

$$1 \to \pi_1(\tilde{S}, P) \to \pi_1(\tilde{\mathbb{P}}, p) \to \mathbb{Z}_2 \to 1.$$

An element of the fundamental group $\pi_1(\tilde{S}, P)$ represented by a closed loop which makes one turn around one of the ramification points on S is mapped by π to a^2, b^2, c^2 or $(abc)^2$. It follows that the induced homomorphism

$$\pi_1(S, P) \to G$$

is well defined, where G is the group defined above. The image of $\pi_1(S, P)$ in G consistes of words of even length and each word of even length has a unique pre-image (lift of a closed loop with respect to the projection). Therefore, the following sequence of homomorphisms is exact

$$1 \to \pi_1(S, P) \to G \to \mathbb{Z}_2 \to 1$$

and $Im(\pi_1(S, P)) = G^0$ is commutative. The lemma is proved.   □

Next, we apply the above Lemma to the holonomy group along $D_0(c)$. This group has three generators h_0, h_1, h_c, see (20), and let h_∞ be the holonomy map associated to a closed loop making one turn around $\lambda = \infty$. If the orientations of the underlying closed loops are appropriately chosen, then $h_\infty = h_0 h_1 h_c$.

Lemma 2:

$$h_0^2 = h_1^2 = h_c^2 = (h_0 h_1 h_c)^2 = id.$$

Proof: Recall that, according to Sec. II the Okamoto surface $\bar{E}$ is obtained from the Hirzebruch surface $\Sigma_{(\varepsilon)}^{(2)}$ after $8 = 4 \times 2$ blow up's at the four singular points a^0, a^1, a^t, a^∞. It follows that the holonomy group along the leaf $D_0 = D_0(t) \subset \bar{E}$ coincides with the holonomy group of the divisor leaf $D_0 = D_0(t) \subset \Sigma_{(\varepsilon)}^{(2)}$, see Figs. 1 and 2. Further, because of the symmetry of Painlevé VI (Ref. 15, Proposition 2.2) it suffices to show that $h_0^2 = id$.
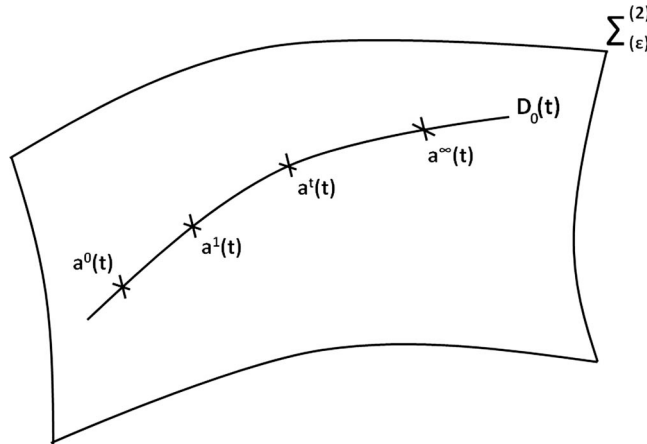
FIG. 2. The divisor $D_0(t)$ in the chart W_2.

The idea of the proof of Lemma 2 is as follows. By Theorem 3 consider the family of solutions $\lambda = \lambda(t)$, $\mu = \mu(t)$, with initial conditions

$$(\lambda(t_0), \mu(t_0)) \in D_2^0(t_0).$$

Here λ, μ are coordinates in appropriate chart on the variety $\bar{E}_c$. Clearly, these solutions are analytic for $t \sim t_0$. Upon a successive contraction of the divisors D_2^0 and D_1^0 we get an infinite family of analytic solutions $\lambda = \lambda(t)$, $\mu = \mu(t)$ which tend to the point a^0. The corresponding leaves are in fact holomorphic curves at a^0 (Ref. 15, Lemme 2_{VI}) which are therefore parameterized by suitable convergent Puiseux series

$$\lambda \rightarrow (t(\lambda), \mu(\lambda)).$$

The monodromy of these series when λ makes one turn around the origin is readily computed to be an involution. We claim that all leaves "sufficiently close" to $D_0(c)$ are obtained in such a way, with some $t_0 \sim c$. If true, this would imply that $h_0^2 = id$ as this holds true for the monodromy map of the holomorphic curves through a_0.

To make these considerations rigorous, consider the chart W_2 on the Hirzebruch surface, with coordinates (λ_2, μ_2), see Sec. II. The Painlevé foliation along the leaf $D_0(c) = \{t = c, \mu_2 = 0\}$ on $\bar{E}$, after $8 = 4 \times 2$ blow downs, is defined by (10). Until the end of this section we replace, as in Sec. III (λ_2, μ_2) by (λ, μ).

Consider an open neighborhood $\tilde{U}_c$ of the divisor $D_2^0 = D_2^0(c) = \mathbb{P}^1$ shown on Fig. 1. in the three-dimensional space $\bar{E}$. As the Painlevé foliation is transversal to $D_2^0(c)$, then we shall suppose that $\tilde{U}_c$ intersects any leaf of the foliation into an open disc, and that $\tilde{U}_c$ is a union of such discs. After a contraction of $D_2^0(t)$ and $D_1^0(t)$, $\forall t$, the neighborhood $\tilde{U}_c$ is transformed to a cone-like domain $U_c \subset \Sigma_{(\epsilon)}^{(2)} \times B$. The processus of blowing down the divisors $D_2^0(t)$ and $D_1^0(t)$, and the effect on the domain $U_c \cap \{t = const.\}$ is shown on Fig. 3. As the neighborhood $\tilde{U}_c$ is a union of regular leaves, then the contracted domain U_c is an union of leaves of (10) intersecting at $a^0(t)$. Each leaf of (10) is therefore a holomorphic curve at $a^0(t)$, tangent to the plane $\{\lambda + \kappa_0\mu = 0\}$ there. Another important feature of the contracted neighborhood U_c is that it contains the domain

$$\{|\frac{\lambda}{\mu} + \kappa_0| < \varepsilon, |t - c| < \delta, |\lambda| < \varepsilon, |\mu| < \varepsilon\}, \tag{21}$$

for all sufficiently small $\varepsilon, \delta > 0$. The domain (21) is a direct product of the cone

$$\{(\lambda, \mu) \in \mathbb{C}^2 : |\frac{\lambda}{\mu} + \kappa_0| < \varepsilon, |\lambda| < \varepsilon, |\mu| < \varepsilon\}$$
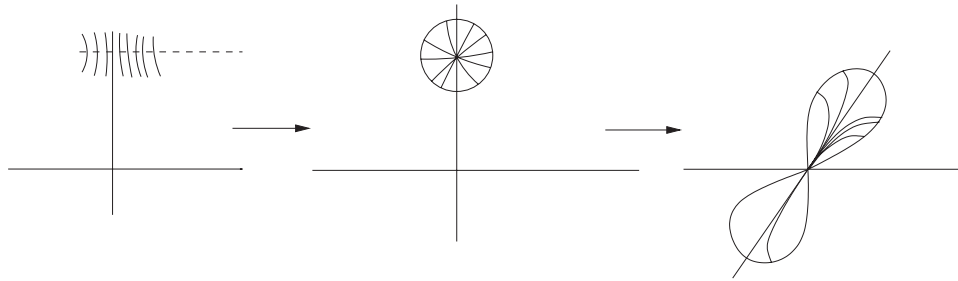
FIG. 3. Blowing down the divisors $D_2^0(c)$ and $D_1^0(c)$, and the domain U_c.

with vertex $(0, 0, c)$ and axis $\lambda + \kappa_0 \mu = 0$, and the disc $\{t \in \mathbb{C} : |t - c| < \delta\}$. Consider the following projection map

$$\pi : (\lambda, \mu, t) \mapsto (\lambda + \kappa_0 \mu, 0, c) \tag{22}$$

from a neighborhood of the point $a^0(c) = (0, 0, c)$ to the line $D_0(c)$. The pre-image of $\pi^{-1}(a^0(c))$ is the complex two-plane $\{\lambda + \kappa_0 \mu = 0\}$.

Consider a cross-section $\sigma = (\mathbb{C}^2, 0)$ to $D_0(c)$, contained in the plane $\lambda = \varepsilon > 0$. It is a (germ of a) complex two-dimensional disc centered at the origin on which the holonomy map is defined

$$h_0 : \sigma \to \sigma.$$

To define geometrically h_0, consider the path $\gamma = [0, \varepsilon] \subset D_0(c)$ (a real interval) connecting $\lambda = \varepsilon$ to the origin on the λ-plane $D_0(c)$. We claim that for any initial condition on σ sufficiently close to $(\varepsilon, 0, c)$ there is a lift Γ of γ along π, to a path contained in a leaf of the Painlevé foliation, starting at the above initial point. Moreover, we claim that when λ tends to zero along γ, then the corresponding point of Γ tends to $(0, 0, t)$ for some t, $|t - c| \leq \varepsilon$. For this purpose, we prove first that γ can be lifted at least until it intersects the cone-like domain U_c. As the leaves in U_c are curves holomorphic at $a^0(c)$, then the result will follow.

To lift γ until it intersects U_c we construct a suitable compact set, in which γ can be lifted. Namely, let K be the closure of the following set:

$$\{(\lambda, \mu, t) : |\frac{\lambda}{\mu} + \kappa_0| \geq \delta, |t - c| \leq \delta, |\lambda + \kappa_0 \mu| \leq \delta\}.$$

It is easily seen that K is compact. The foliation (10) is transverse to the fibers of the map π at a point (λ, μ, t) if

$$2E(t, \lambda) + F(t, \lambda)\mu + \kappa_0 \left[E_\lambda(t, \lambda) + F_\lambda(t, \lambda)\mu + G\mu^2 \right] \mu \neq 0.$$

In a suitable neighborhood of the point $(0, 0, c)$ we have

$$2E(t, \lambda) + F(t, \lambda)\mu + \kappa_0 \left[E_\lambda(t, \lambda) + F_\lambda(t, \lambda)\mu + G\mu^2 \right] \mu = 2t(\lambda + \kappa_0 \mu) + \cdots ,$$

where the dots stand for $O(|\lambda|^2 + |\mu|^2)$ uniformly in $t \sim c$. We conclude that when $(\lambda, \mu, t) \in K$ and belongs to a suitable neighborhood of the point $(0, 0, c)$, then the foliation (10) is transverse to the fibers of the map π, and hence the path γ can be lifted until its lift reaches the border of K. In the case when $|t - c| < \delta$, this means that the path can be lifted until the cone-like domain U_c which is filled up by holomorphic curves (leaves of the foliation). Therefore, the path can be further lifted until the origin and the claim is proved.

It remains to show that in the course of the lifting $|t - c| < \delta$ holds true. This follows after integrating the differential

$$\frac{dt}{t(t-1)} = \frac{1}{2E(t, \lambda)/\mu + F(t, \lambda)} d\lambda,$$

along the path γ. Indeed, when $(\lambda, \mu, t) \in K$ and belongs to a suitable neighborhood of the point $(0, 0, c)$, then

$$\frac{2E(t, \lambda)}{\mu} \sim -2\kappa_0 c, \, F(t, \lambda) \sim \kappa_0 c,$$

and hence

$$\frac{2E(t, \lambda)}{\mu} + F(t, \lambda) \sim -\kappa_0 c$$

is bounded from zero, provided that $\kappa_0 \neq 0$.

We conclude that every path can be lifted until it crosses the domain U_c in which the leaves of the foliation are holomorphic curves. The monodromy of the Puiseux series $t = t(\lambda')$, $\mu = \mu(\lambda')$, when $\lambda' = \lambda + \kappa_0 \mu$ makes one turn around the origin is easily described : it is an involution. This follows geometrically from the fact, that the divisor $D_2^0(t)$ is obtained after two blow up's from $a^0(t)$. Analytically, this means that if z is a local coordinate on the projective line $D_2^0(t)$, then U_c is an union of holomorphic curves (leaves) parameterized by z and t, such that for fixed z, t we have

$$\lambda' = \lambda + \kappa_0 \mu = z\mu^2 + O(\mu^3),$$

where $-\kappa_0$ is the coordinate of $D_1^0(t) \cap D_2^0(t)$ and $z \in D_2^0(t)$ is the intersection point of the leaf and $D_2^0(t)$. Therefore, μ is an analytic function in $\sqrt{\lambda'}$ and the result follows. Finally, we note that the holonomy map h_0 depends analytically on the parameters of the Painlevé foliation. As $h_0^2 = id$ for $\kappa_0 \neq 0$ then this holds true for all κ_0. To resume, we proved

Proposition 3: There exists a neighborhood of the point $a^0(c) = (0, 0, c)$, such that every leaf of the Painlevé foliation is a holomorphic curve, which is a ramified two-sheeted covering of the divisor $D_0(c) = \{\mu = 0\}$ along the projection map π (22), with ramification point $(\lambda = \mu = 0)$.

The above Proposition generalizes (Ref. 15, Lemma 2_{VI}) and implies Lemma 1.   □

Proof of Theorem 1: According to Lemma 2, each element of the holonomy group is a word made with the letters h_0, h_1, h_c. By Lemma 1, the subgroup of the holonomy group formed by words of even length is commutative.

ACKNOWLEDGMENTS

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