ON THE FINITE CYCLICITY OF OPEN PERIOD ANNULI

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Abstract

Let Π be an open, relatively compact period annulus of real analytic vector field X_0 on an analytic surface. We prove that the maximal number of limit cycles which bifurcate from Π under a given multiparameter analytic deformation X_λ of X_0 is finite provided that X_0 is either a Hamiltonian or generic Darbouxian vector field.

1. Statement of the result

Let S be a real analytic surface without border (compact or not), and let X_0 be a real analytic vector field on it. An open period annulus of X_0 is a union of period orbits of X_0 which is bianalytic to the standard annulus $S^1 \times (0, 1)$, the image of each circle $S^1 \times \{u\}$ being a periodic orbit of X_0.

Let X_λ, $\lambda \in (\mathbb{R}^n, 0)$ be an analytic family of analytic vector fields on S, and let Π be an open period annulus of X_0. The *cyclicity* $\mathrm{Cycl}(\Pi, X_\lambda)$ of Π with respect to the deformation X_λ is the maximal number of limit cycles of X_λ which tend to Π as λ tends to zero (see Definition 2). Clearly the vector field X_0 has an analytic first integral f in the period annulus Π which has no critical points. In what follows, we suppose that the open period annulus Π is relatively compact (i.e., its closure $\bar{\Pi} \subset S$ is compact).

Definition 1

We say that X_0 is a Hamiltonian vector field provided that it has a first integral with isolated critical points in a complex neighborhood of $\bar{\Pi}$. We say that X_0 is a generic Darbouxian vector field provided that all singular points of X_0 in a neighborhood of $\bar{\Pi}$ are orbitally analytically equivalent to linear saddles $\dot{x} = \lambda x$, $\dot{y} = -y$ with $\lambda > 0$.

Remark 1

Note that if X_0 is a plane vector field with a first integral H as above, then

$$X_0 = H_1\Big(H_y \frac{\partial}{\partial x} - H_x \frac{\partial}{\partial y}\Big),$$

DUKE MATHEMATICAL JOURNAL
Vol. 152, No. 1, © 2010 DOI 10.1215/00127094-2010-005
Received 3 July 2008. Revision received 1 May 2009.
2000 *Mathematics Subject Classification.* Primary 34C07; Secondary 34C10.

where H_1 is a nonvanishing real-analytic function in some complex neighborhood of Π. In the case when X_0 is a generic Darbouxian vector field, as we see in Section 2, it can be covered by a planar Darbouxian vector field with a first integral of the Darboux type $H = \prod_{i=1}^{n} P_i^{\lambda_i}$ for some analytic functions P_i in a complex neighborhood of Π.

The main results of the article are the following.

THEOREM 1
The cyclicity $\operatorname{Cycl}(\Pi, X_\lambda)$ *of the open period annulus* Π *of a Hamiltonian vector field* X_0 *is finite.*

THEOREM 2
The cyclicity $\operatorname{Cycl}(\Pi, X_\lambda)$ *of the open period annulus* Π *of a generic Darbouxian vector field* X_0 *is finite.*

The above theorems are a particular case of Roussarie's conjecture (see [17], [19, p. 23]), which claims that the cyclicity $\operatorname{Cycl}(\Gamma, X_\lambda)$ of every compact invariant set Γ of X_0 is finite. Indeed, $\operatorname{Cycl}(\Pi, X_\lambda) \leq \operatorname{Cycl}(\bar{\Pi}, X_\lambda)$.

It is worthwhile to underline that the present article does not prove the finite cyclicity of the closed period annulus $\bar{\Pi}$. Indeed, for this we would need to prove the finite cyclicity of polycycles $\Gamma \subset \bar{\Pi} \setminus \Pi$ at the boundary of the open annulus Π, which is beyond the scope of the article. If the analytic family of analytic vector fields X_λ, $\lambda \in (\mathbb{R}^n, 0)$, is generic (in appropriate sense) and the period annulus is bounded by a homoclinic loop (or one-saddle cycle), then the Roussarie's theorem (see [16, Theorem C], [6, Theorem 4]) implies

$$\operatorname{Cycl}(\bar{\Pi}, X_\lambda) = \operatorname{Cycl}(\Pi, X_\lambda). \tag{1}$$

It seems natural to expect that things would always work out this way. A counterexample to (1) was given recently by Dumortier and Roussarie [6]. They constructed a generic analytic deformation X_λ, $\lambda \in (\mathbb{R}^4, 0)$, of a Hamiltonian vector field X_0 with a period annulus Π bounded from one side by a heteroclinic loop (two-saddle cycle) and such that

$$\operatorname{Cycl}(\Pi, X_\lambda) = 3 < 4 = \operatorname{Cycl}(\bar{\Pi}, X_\lambda).$$

The inequality is due to the presence of an *alien* limit cycle that does not correspond to a zero of the Poincaré-Pontryagin function M_k (see (2)).

To prove the finite cyclicity of the open annulus, we note first that it suffices to show its finite cyclicity with respect to a given one-parameter deformation X_ε. This argument is based on Hironaka's desingularization theorem (see [20], [8]). Consider the first return map associated to Π and a one-parameter deformation X_ε:

$$t \to t + \varepsilon^k M_k(t) + \cdots, \quad t \in (0, 1), \varepsilon \sim 0, k \geq 1. \tag{2}$$

The cyclicity of the open period annulus Π is finite if and only if the Poincaré-Pontryagin function M_k has a finite number of zeros in $(0, 1)$. Of course, we do not suppose that $k = 1$ (in which case M_1 is an Abelian integral and the result is well known). It has been shown in [7] that M_k, $k \geq 1$, allows an integral representation as a linear combination of iterated path integrals along the ovals of Π of length at most k. The finite cyclicity follows then from the nonaccumulation of zeros of such iterated integrals at 0 and 1. The proof of this fact is different in the Hamiltonian and in the generic Darbouxian case.

In the Hamiltonian case we observe that M_k satisfies a Fuchsian equation (see [9], [7]). We prove in Section 4 that the associated monodromy representation is quasi-unipotent, which implies the desired property.

In the Darbouxian case the above argument does not apply. (There is no Fuchsian equation satisfied by M_k.) We prove the nonoscillation property of an iterated integral by making use of its Mellin transformation, along the lines of [14]. It seems to be difficult to remove the genericity assumption in the Darbouxian case. (Without this the Hamiltonian case is a subcase of the Darbouxian one.)

Theorem 2 should be considered in the context of recent results dealing with generalization of the Varchenko-Khovanskii result (see [22], [13]) to integrable systems with generalized Darboux integral. The ultimate goal is to prove that for the pseudo-Abelian integrals corresponding to Darboux integrable systems with the first integral being the product of real powers of polynomials of degree at most m, the number of zeros is bounded from above by a constant depending only on m and degree of the form ω.

The main result of [14] (see also [3]) claims local boundedness of the number of zeros of pseudo-Abelian integrals for generic Darboux integrable systems, where genericity conditions are exactly the same as in Theorem 2. Generalization of this result to perturbations of various degeneracies of Darboux systems would imply global boundedness of the number of zeros of pseudo-Abelian integrals for the aforementioned class of Darboux integrable systems. In [4] this is proved for generalized Darboux integrable systems with the first integral of type $e^Q \prod P_i^{\lambda_i}$ of generic type.

Recently the existential upper bound in the Varchenko-Khovanskii theorem was replaced by a constructive double exponential one (see [2]). There is no hope at the moment of providing any effective upper bound for the number of zeros of pseudo-Abelian integrals.

The full generalization of the result of [14] to iterated pseudo-Abelian integrals should claim uniform boundedness of the number of zeros of an iterated integral corresponding to an analytic family of Darboux integrable systems. Unfortunately, it cannot be achieved by simple generalizations of the above arguments: unlike pseudo-Abelian integrals, iterated pseudo-Abelian integrals are not annihilated by a finite number of applications of Petrov operators. This can be immediately seen from the fact that poles of the Mellin transforms of iterated integrals do not form a finite union of arithmetic progressions. We strongly believe that the local boundedness holds, but it seems that the tools available at the moment do not provide this result.

The article is organized as follows. In Section 2 we recall the definition of cyclicity and the reduction of multiparameter to one-parameter deformations. In Section 3 we reduce the case of a vector field on a surface to the case of a plane vector field.

Theorems 1 and 2 are proved in Sections 4 and 5, respectively.

2. Cyclicity and nonoscillation of the Poincaré-Pontryagin-Melnikov function

Definition 2
Let X_λ be a family of analytic real vector fields on a surface S, depending analytically on a parameter $\lambda \in (\mathbb{R}^n, 0)$, and let $K \subset S$ be a compact invariant set of X_{λ_0}. We say that the pair (K, X_{λ_0}) has cyclicity $N = \mathrm{Cycl}((K, X_{\lambda_0}), X_\lambda)$ with respect to the deformation X_λ, provided that N is the smallest integer having the property; there exist $\varepsilon_0 > 0$ and a neighborhood V_K of K, such that for every λ such that $\|\lambda - \lambda_0\| < \varepsilon_0$, the vector field X_λ has no more than N limit cycles contained in V_K. If $\tilde{K}$ is an invariant set of X_{λ_0} (possibly noncompact), then the cyclicity of the pair $(\tilde{K}, X_{\lambda_0})$ with respect to the deformation X_λ is

$$\mathrm{Cycl}\big((\tilde{K}, X_{\lambda_0}), X_\lambda\big) = \sup\big\{\mathrm{Cycl}((K, X_{\lambda_0}), X_\lambda) : K \subset \tilde{K}, K \text{ is a compact}\big\}.$$

The above definition implies that when $\tilde{K}$ is an open invariant set, then its cyclicity $\mathrm{Cycl}((\tilde{K}, X_{\lambda_0}), X_\lambda)$ is the maximal number of limit cycles which tend to $\tilde{K}$ as λ tends to zero. To simplify the notation, and if there is no danger of confusion, we write $\mathrm{Cycl}(K, X_\lambda)$ in place of $\mathrm{Cycl}((K, X_{\lambda_0}), X_\lambda)$.

Example 1
Let $f_\varepsilon(t) = \varepsilon e^{-1/t}(t sin(1/t) - \varepsilon)$, $f_\varepsilon(0) = 0$. One can easily see that $f_\varepsilon(t) = 0$ has a finite number of isolated positive zeros for each ε, and this number tends to infinity as $\varepsilon \to 0$. Below we construct a germ X_ε of a vector field having a monodromic planar singular point at the origin, with a return map $x \to x + f_\varepsilon(x)$. Since isolated singular points of the return map correspond to limit cycles, we see that the vector field X_ε has a finite number of limit cycles for each ε, and this number tends to infinity as ε tends

to zero. So the cyclicity of the open period annulus $\Pi = \mathbb{R}^2 \setminus \{0\}$ is infinity. Note, however, that the vector field X_ε is not analytic at the origin.

Here is a construction. On the strip $S = [0, \delta] \times \mathbb{R}$, consider the equivalence relation $(r, \phi) \sim (r + f_\varepsilon(r), \phi - 2\pi)$. Let $p : S \to S/\sim$ be the corresponding projection, and define $\tilde{X}_\varepsilon = p_*(\partial_\phi)$. One can check that for δ small enough, a thus-defined $\tilde{X}_\varepsilon$ is a blowup of a smooth vector field X_ε defined near the origin, and the return map of X_ε is as prescribed by construction.

Let $\Delta \subset S$ be a cross-section of the period annulus Π which can be identified to the interval $(0, 1)$. Choose a local parameter u on Δ. Let $u \mapsto P(u, \lambda)$ be the first return map, and let $\delta(u, \lambda) = P(u, \lambda) - u$ be the displacement function of X_λ. For every closed interval $[a, b] \subset \Delta$, there exists $\varepsilon_0 > 0$ such that the displacement function $\delta(u, \lambda)$ is well defined and analytic in $\{(u, \lambda) : a - \varepsilon_0 < u < b + \varepsilon_0, \|\lambda\| < \varepsilon_0\}$. For every fixed λ there is a one-to-one correspondance between isolated zeros of $\delta(u, \lambda)$ and limit cycles of the vector field X_λ.

Let $u_0 \in \Delta$, and let us expand

$$\delta(u, \lambda) = \sum_{i=0}^{\infty} a_i(\lambda)(u - u_0)^i.$$

Definition 3 (Bautin ideal; [18, Section 2], [19, Section 4.3.1])
We define the Bautin ideal $\mathcal{I}$ of X_λ to be the ideal generated by the germs $\tilde{a}_i$ of a_i in the local ring $\mathcal{O}_0(\mathbb{R}^n)$ of analytic germs of functions at $0 \in \mathbb{R}^n$.

This ideal is Noetherian. Let $\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_p$ be a minimal system of its generators, where $p = \dim_{\mathbb{R}} \mathcal{I}/\mathcal{M}\mathcal{I}$ and $\mathcal{M}$ is the maximal ideal of the local ring $\mathcal{O}_0(\mathbb{R}^n)$. Let $\varphi_1, \varphi_2, \ldots, \varphi_p$ be analytic functions representing the generators of the Bautin ideal in a neighborhood of the origin in $\mathbb{R}^n$.

PROPOSITION 1 (Roussarie [19, Section 4.3.2])
The Bautin ideal does not depend on the point $u_0 \in \Delta$. For every $[a, b] \subset \Delta$, there is an open neighborhood U of $[a, b] \times \{0\}$ in $\mathbb{R} \times \mathbb{R}^n$ and analytic functions $h_i(u, \lambda)$ in U, such that

$$\delta(u, \lambda) = \sum_{i=0}^{p} \varphi_i(\lambda) h_i(u, \lambda). \tag{3}$$

The real vector space generated by the functions $h_i(u, 0)$, $u \in [a, b]$ is of dimension p.

Suppose that the Bautin ideal is principal and generated by $\varphi(\lambda)$. Then

$$\delta(u, \lambda) = \varphi(\lambda) h(u, \lambda), \tag{4}$$

where $h(u, 0) \not\equiv 0$. The maximal number of the isolated zeros of $h(u, \lambda)$ on a closed interval $[a, b] \subset (0, 1)$ for sufficiently small $|\lambda|$ is bounded by the number of the zeros of $h(u, 0)$, counted with multiplicity, on $[a, b]$. This follows from the Weierstrass preparation theorem, properly applied (see [8]). Therefore, to prove the finite cyclicity of Π it is enough to show that $h(u, 0)$ has a finite number of zeros on $(0, 1)$. Consider a germ of analytic curve $\xi : \varepsilon \mapsto \lambda(\varepsilon)$, $\lambda(0) = 0$, as well as the analytic one-parameter family of vector fields $X_{\lambda(\varepsilon)}$. The Bautin ideal is principal, $\delta(u, \varepsilon) = \varphi(\varepsilon) h(u, \varepsilon)$, and

$$\delta\big(u, \lambda(\varepsilon)\big) = \varepsilon^k M_k(u) + \dots, \; M_k(u) = c\, h(u, 0),\; c \neq 0,$$

where the dots stay for terms containing ε^i, $i > k$. M_k is the so-called kth-order higher Poincaré-Pontryagin-Melnikov function associated to the one-parameter deformation $X_{\lambda(\varepsilon)}$ of the vector field X_0. Therefore, if the cyclicity of the open period annulus is infinite, then M_k has an infinite number of zeros on the interval $(0, 1)$.

Of course, in general the Bautin ideal is not principal. However, by making use of the Hironaka's theorem, we can always principalize it. More precisely, after several blowups of the origin of the parameter space, we can replace the Bautin ideal by an ideal sheaf that is principal (see [8] for details). This proves the following.

PROPOSITION 2
If the cyclicity $\mathrm{Cycl}(\Pi, X_\lambda)$ *of the open period annulus* Π *is infinite, then there exists a one-parameter deformation* $\lambda = \lambda(\varepsilon)$*, such that the corresponding higher-order Poincaré-Pontryagin-Melnikov function* M_k *has an infinite number of zeros on the interval* $(0, 1)$.

In Sections 3 and 4 we prove the nonoscillation property of M_k in the Hamiltonian and the Darbouxian cases (under the restrictions stated in Theorem 2).

3. Reduction to the planar vector field case

Let X_0 be a real analytic vector field on a real analytic surface S. Let Π be an open period annulus of X_0 with compact closure. Let the map

$$\tau : \Pi \to S^1 \times (0, 1)$$

be a bianalytic isomorphism such that $\delta_t = \tau^{-1}(S^1 \times \{t\})$ is a closed orbit of X_0. We assume that X_0 is either Hamiltonian or generalized Darbouxian in some neighborhood of the closure $\bar{\Pi}$ of Π. Theorems 1 and 2 claim that cyclicity of Π in any family of analytic deformation X_λ of X_0 is finite.

This section is devoted to the reduction of this general situation to the case of a vector field X_0 on $\mathbb{R}^2$ of Hamiltonian or Darboux type near its polycycle. Then Theorems 1 and 2 follow from Theorems 3 and 4.

First, note that it is enough to prove finite cyclicity of $\tau^{-1}(S^1 \times (0, \varepsilon))$ only. Indeed, finite cyclicity of $\tau^{-1}(S^1 \times [\varepsilon, 1-\varepsilon])$ follows from Gabrielov's theorem, and finite cyclicity of $\tau^{-1}(S^1 \times (1-\varepsilon, 1))$ can be reduced to the above by replacing t by $1-t$.

Consider the Hausdorf limit $\Gamma = \lim_{t\to 0} \tau^{-1}(S^1 \times \{t\})$. It is a connected union of several fixed points $a_1, \dots, a_n$ of X_0 (not necessarily pairwise different) and orbits $\Gamma_1, \dots, \Gamma_n$ of X_0 such that Γ_i exits from a_i and enters a_{i+1} (where a_{n+1} denotes a_1).

From now on we consider only a sufficiently small neighborhood U of Γ. We assume that $U \cap \Pi = \tau^{-1}(S^1 \times (0, \varepsilon))$, and we denote this intersection again by Π. We consider first the Darbouxian case. Note that Γ cannot consist of just one singular point of X_0 by assumptions about linearizability of singular points of X_0 in this case.

LEMMA 1

Assume that Theorem 2 holds if U is orientable and all a_i are different. Then Theorem 2 holds in full generality.

Proof

Assume that for some real analytic surface $\tilde{U}$ there is an analytic mapping $\pi : \tilde{U} \to U$ which is a finite covering on Π. Then the cyclicity of Π for X_λ is the same as the cyclicity of $\pi^{-1}(\Pi)$ for the lifting X_λ to $\tilde{U}$. The claim of the lemma follows from this principle applied to two types of coverings.

First, taking a double covering of U as $\tilde{U}$, we can assume that U is orientable.

Second, let U be represented as a union of neighborhoods U_i of a_i together with neighborhoods V_i of Γ_i. Glue $\tilde{U}$ as $\tilde{U} = \tilde{U}_1 \cup \tilde{V}_1 \cup \cdots \cup \tilde{V}_n$, where $\tilde{U}_i$ are bianalytically equivalent to U_i and disjoint, and $\tilde{V}_i$ are bianalytically equivalent to V_i, with natural glueing of $\tilde{U}_i$ to $\tilde{V}_i$, of $\tilde{V}_i$ to $\tilde{U}_{i+1}$, and of $\tilde{U}_1$ to $\tilde{V}_n$ (see Figure 1). In other words, $\pi : \tilde{U} \to U$ is one-to-one away from a_i and k_i-to-one in a neighborhood of a_i if a_i appears k_i times in the list $\{a_1, \dots, a_n\}$. Evidently, π is one-to-one on Π and so is bianalytic. □

We now define a first integral H of X_0 in U. Take any smooth point a on some side γ_1 of Γ, and let H be a local first integral of X_0 in a neighborhood U_a of a such that $H(a) = 0$ and $dH(a) \neq 0$. Since U is orientable, Π lies from one side of Γ, and we can assume that intersection of U_a with each cycle δ_t is connected. This allows us to extend H to a first integral of X_0 defined on $\Pi \cap U$. Changing the sign of H if necessary, we can assume that $H > 0$ on $\Pi \cap U_a$. We define $H(\Gamma) = 0$ by continuity.

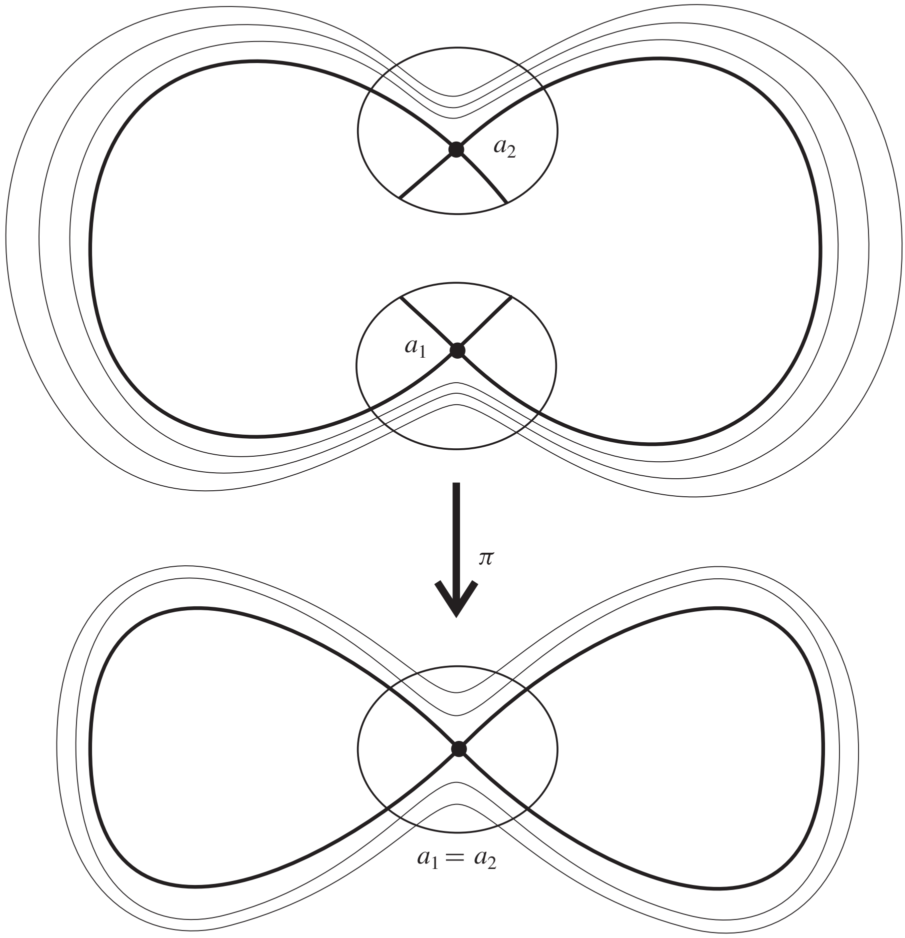

Figure 1. Proof of Lemma 1

LEMMA 2
Extension of H to $\Pi \cap U$ by flow of X_0 can be extended to a multivalued holomorphic function defined in a neighborhood of Γ in a complexification of U.

Proof
First, H is analytic in some neighborhood of Γ_1, as it is an analytic function extended by analytic flow of X_0. Choose local linearizing coordinates (x, y) near a_2 in such a way that $\Gamma_1 = \{y = 0\}$. By assumption, yx^{μ} is the local first integral of X_0 near a_2. Therefore $H = f(yx^{\mu})$, and restricting to a transversal $x = x_0 \ll 1$, one can see that f is analytic and invertible. Therefore H can be extended to a neighborhood of a_2.

Moreover, $(f^{-1}(H))^{1/\mu}$ is an analytic local first integral near the point $y = 1$ of Γ_2. Therefore it can be extended to a neighborhood of Γ_2 (here we use that U is orientable, so Γ_2 is different from Γ_1) and, as above, to a neighborhood of a_3 (here we use that $a_2 \neq a_3$), and so on. □

Note that from the above construction it follows that near each Γ_i the first integral H is equal, up to an invertible function, to x^{λ_i}, where $\{x = 0\}$ is a local equation of γ_i. Also, near any singular point of Γ the first integral H is equal, up to an invertible function, to $x^\lambda y^\mu$.

COROLLARY 1
The one-form $\frac{dH}{H}$ is a meromorphic one-form in U with logarithmic singularities only.

Assume that $n \geq 3$. One can easily construct a C^∞-isomorphism of a sufficiently small neighborhood U of Γ with a neighborhood of a regular n-gon in R^2 in such a way that the image of $\Pi \cap U$ lies inside the n-gon and image of Γ coincides with the n-gon. Due to [10], some neighborhood $U^{\mathbb{C}}$ of U in its complexification is a Stein manifold. This implies that this isomorphism can be chosen to be bianalytic. Similarly, for $n = 2$ one can map bianalytically a neighborhood of U to a union of two arcs $\{x^2 + (|y| + 1)^2 = 2\} \subset \mathbb{R}^2$ (which, for the rest of the article, is called a *regular* 2-gon).

We transfer everything to the plane using this isomorphism and denote the images on the plane of the previously defined objects by the same letters. The first integral H takes the form $H = H_1 \prod_{i=1}^n P_i^{\lambda_i}$, where P_i are analytic functions in U with $\{P_i = 0\} = \Gamma_i$, H_1 is an analytic function nonvanishing in its neighborhood U, and $\lambda_i > 0$. Note that $H > 0$ in the part of U lying inside the n-gon. Further, we assume that $H_1 \equiv 1$, so $H = \prod P_i^{\lambda_i}$. (One can achieve this by, e.g., taking $P_1 H_1^{1/\lambda_1}$ instead of P_1.)

The family X_λ becomes a family of planar analytic vector fields defined in a neighborhood U of a regular n-gon $\Gamma \subset \mathbb{R}^2$, and X_0 has a first integral H of Darboux type in U. Let $X_\varepsilon = X_{\lambda(\varepsilon)}$ be a one-parametric deformation of X_0 as in Proposition 2. Define meromorphic forms ω^2, ω_ε as

$$\omega^2(X_0, \cdot) = \frac{dH}{H}, \qquad \omega^2(X_\varepsilon, \cdot) = X_0 + \omega_\varepsilon. \tag{5}$$

According to [7, Theorem 2.1], M_k can be represented as a linear combination of iterated integrals over $\{H = t\}$ of forms which are combinations of Gauss-Manin derivatives of ω_ε.

Recall that the Gauss-Manin derivative of a form η is defined as a form η' such that $d\eta = d(\log H) \wedge \eta'$. In general, η' cannot be uniquely defined from this equation, though its restrictions to $\{H = t\}$ are defined unambiguously. However, since $U^{\mathbb{C}}$ is Stein, in our situation one can choose a meromorphic in U representative of η', with poles on $\check{\Gamma}$ only (where $\check{\Gamma}$ is the union of lines containing sides of Γ).

Therefore Theorem 2 follows from the next claim.

THEOREM 3
Let $H = \prod_{i=1}^{n} P_i^{\lambda_i}$ be as above, and let $\gamma(t) \subset \{H = t\}$ be the connected component of its level set lying inside Γ. Zeros of polynomials in iterated integrals $I(t) = \int_{\gamma(t)} \omega_1 \cdots \omega_k$ corresponding to meromorphic one-forms $\omega_1, \ldots, \omega_k$ with poles in $\check{\Gamma}$ cannot accumulate to zero.

From the above discussion it is clear that Theorem 1 follows in its turn from the following.

THEOREM 4
Let

$$X_0 = H_y \frac{\partial}{\partial x} - H_x \frac{\partial}{\partial y},$$

where H is a real analytic function with isolated singularities in some complex neighborhood of the closed period annulus $\bar{\Pi} = \{\gamma(t) : 0 \le t \le 1\}$, where $\gamma(t) \subset \{H = t\}$ is the connected component of the level set of H lying inside Γ. Zeros of the first nonvanishing Poincaré-Pontryagin function M_k corresponding to a one-parameter analytic deformation X_ε of X_0 cannot accumulate to zero.

4. Nonoscillation in the Hamiltonian case

Theorem 4 follows from the next two results.

THEOREM 5 ([7], [9, Theorem 2])
The Poincaré-Pontryagin function M_k satisfies a linear differential equation of Fuchs type in a suitable complex neighborhood of $0 \in \mathbb{C}$.

THEOREM 6
The local monodromy operator at the origin of the Fuchs equation mentioned in Theorem 5 is quasi-unipotent.

Let us recall that an endomorphism is called unipotent if all its eigenvalues are equal to 1 and called quasi-unipotent if all of them are roots of the unity. The above theorems imply that the Poincaré-Pontryagin-Melnikov function has a representation in a neighborhood of $u = 0$:

$$M_k(u) = \sum_{i=0}^{N} \sum_{j=0}^{N} u^{\mu_i} \big(\log(u)\big)^j f_{ij}(u),$$

where $N \in \mathbb{N}$, $\mu_j \in \mathbb{Q}$, and f_{ij} are functions analytic in a neighborhood of $u = 0$. This shows that the zeros of $M_k|_{(0,1)}$ do not accumulate to zero. Of course, similar arguments hold in a neighborhood of $u = 1$, so M_k has a finite number of zeros on $(0, 1)$. This completes the proof of Theorem 2 in the Hamiltonian case. To the end of the section we prove Theorem 6. The open real surface S is analytic and hence possesses a canonical complexification. Similarly, any analytic family of analytic vector fields X_λ is extended to a complex family of vector fields, depending on a complex parameter. In this section, by abuse of notation, the base field is $\mathbb{C}$. A real object and its complexification are denoted by the same letter.

Let $U \supset \bar{\Pi}$ be an open complex neighborhood of $\bar{\Pi}$ in which the complexified vector field X_0 has an analytic first integral f with isolated critical points. The restriction of f on the interval $(0, 1)$ (after identifying Π to $S^1 \times (0, 1)$) is a local variable with finite limits at 0 and 1. Therefore we may suppose that $f(0) = 0$, $f(1) = 1$, and the restriction of f to $(0, 1)$ is the canonical local variable on $(0, 1) \subset \mathbb{R}$. The function f defines a locally trivial Milnor fibration in a neighborhood of every isolated critical point. There exists a complex neighborhood U of $\bar{\Pi}$ in which F has only isolated critical points. Moreover, the compactness of $\bar{\Pi}$ implies that there exists a complex neighborhood $D \subset \mathbb{C}$ of the origin, homeomorphic to a disk, such that the fibration

$$U \cap \{f^{-1}(D \setminus \{0\})\} \xrightarrow{f} D \setminus \{0\} \tag{6}$$

is locally trivial, and the fibers $f^{-1}(t) \cap U$ are open Riemann surfaces homotopy equivalent to a bouquet of a finite number of circles. Consider a one-parameter analytic deformation X_ε of the vector field X_0. As f is a first integral of X_0, there exists a unique symplectic two-form ω^2, such that

$$\omega^2(X_0, \cdot) = df.$$

Indeed, if in local coordinates

$$X_0 = a\,\frac{\partial}{\partial x} + b\,\frac{\partial}{\partial y},$$

then $X_0 \cdot df = 0$ implies $(a, b) = \lambda(f_y, -f_x)$, where λ is analytic in U and nonvanishing in Π. It follows that

$$\omega^2 = \frac{dx \wedge dy}{\lambda}.$$

Define a unique meromorphic one-form ω_ε by the formula

$$\omega^2(X_\varepsilon, \cdot) = df + \omega_\varepsilon.$$

The one-form ω_ε is meromorphic in U, depends analytically on ε, and $\omega_0 = 0$. Its pole divisor does not depend on ε. Indeed, in the local variables above, it is defined by $\lambda = 0$. Therefore $\omega_\varepsilon = \sum_{i \geq 1} \varepsilon^i \omega_i$, where ω_i are given meromorphic one-forms in U with a common pole divisor which does not intersect the period annulus Π. In the complement of the singular locus of X_ε, the vector field X_ε and the one-form $df + \omega_\varepsilon$ define the same foliation and therefore define the same first return map associated to Π. Denote this map by $P(t, \varepsilon)$, where $t \in (0, 1)$ is the restriction of f to a cross-section of the period annulus Π. (This does not depend on the choice of the cross-section.) We have

$$P(t, \varepsilon) = t + \sum_{k \geq 1} \varepsilon^k M_k(t).$$

On each leaf of the foliation defined by X_ε, we have $df = -\omega_\varepsilon$, which implies

$$M_1(t) = \int_{\gamma_t} \omega_1,$$

where $\{\gamma_t\}_t$ is the family of periodic orbits (with appropriate orientation) of X_0, $\Pi = \cup_{t \in (0,1)} \gamma_t$ (see [15]). Thus the first Poincaré-Pontryagin-Melnikov function is an Abelian integral, and its monodromy representation is straightforward. Namely, the meromorphic one-form ω_1 restricts to a meromorphic one-form on the fibers of the Minlor fibration (6). We may also suppose that $\omega_1|_{f^{-1}(t)}$ has a finite number of poles $\{P_i(t)\}_i$ (after choosing appropriately the domain U). Denote

$$\Gamma_t = U \cap \left\{ f^{-1}(t) \setminus \{P_i(t)\}_i \right\}.$$

The Milnor fibration (6) induces a representation

$$\mathbb{Z} = \pi_1(D \setminus \{0\}, *) \to \mathrm{Aut}\big(H_1(\Gamma_t, \mathbb{Z})\big), \tag{7}$$

which implies the monodromy representation of M_1. Suppose first that ω_1 is analytic in U. It is well known that the operator of the classical monodromy of an isolated critical point of an analytic function is quasi-unipotent (see, e.g., [12]). Therefore the representation in $\mathrm{Aut}(H_1(U\cap\{f^{-1}(t)\}, \mathbb{Z}))$ of a small loop about zero in $\pi_1(D\setminus\{0\}, *)$ is quasi-unipotent. More generally, let ω_1 be meromorphic one-form with a finite number of poles on the fibers $U \cap \{f^{-1}(t)\}$. A monodromy operator permutes the poles, and hence an appropriate power of it leaves the poles fixed. Therefore this operator is quasi-unipotent too, and Theorem 5 is proved in the case $M_1 \neq 0$. Of course, it is well known that an Abelian integral has a finite number of zeros (see [13], [22]).

Let M_k be the first nonzero Poincaré-Pontryagin-Melnikov function. Its universal monodromy representation was constructed in [9, Section 2.1]. For the convenience of the reader, we reproduce it here. Recall first that $M_k(t)$ does not depend on the choice of cross-section of the period annulus and that the first return map $P(t, \varepsilon)$ does not have this property. Equivalently, $M_k(t)$ depends on the *free* homotopy class of the loop γ_t in $\pi_1(\Gamma_t)$ and the fact that this property does not hold true for the first return map $P(t, \varepsilon)$ (which depends on the homotopy class of γ_t in $\pi_1(\Gamma_t, *)$, with a fixed initial point). Let $F = \pi_1(\Gamma_t, *)$ be the fundamental group of Γ_t. It is a finitely generated free group. Let $\mathcal{O} \subset \pi_1(\Gamma_t)$ be the orbit of the loop γ_t under the action of $\mathbb{Z} = \pi_1(D \setminus \{0\}, *)$ induced by (6). The preimage of the set $\mathcal{O}$ under the canonical map

$$F = \pi_1(\Gamma_t, *) \to \pi_1(\Gamma_t)$$

is a normal subgroup of F which we denote by G. The commutator subgroup $(G, F) \subset F$ is the normal subgroup of F generated by commutators $(g, f) = g^{-1}f^{-1}gf$. The Milnor fibration (6) induces a representation

$$\mathbb{Z} = \pi_1(D \setminus \{0\}, *) \to \mathrm{Aut}\big(G/(G, F)\big). \tag{8}$$

According to [9, Theorem 1], the monodromy representation of M_k is a subrepresentation of the monodromy representation dual to (8). Unfortunately the free Abelian group $F/(G, F)$ is not necessarily of finite dimension. To obtain a finite-dimensional representation, we use the fundamental fact that M_k has an integral representation as an iterated path integral of length k (see [7, Theorem 2.1]).

To use this, define by induction $F_{i+1} = (F_i, F)$, $F_1 = F$. We later consider the associated graded group

$$\mathrm{gr}F = \bigoplus_{i=1}^{\infty} \mathrm{gr}^i F, \quad \mathrm{gr}^i F = F_i/F_{i+1}. \tag{9}$$

It is well known that an iterated integral of length k along a loop contained in F_{k+1} vanishes identically. Therefore, to study the monodromy representation of M_k, we truncate with respect to F_{k+1} and obtain a finite-dimensional representation. Namely, for every subgroup $H \subset F$ we denote

$$\tilde{H} = (H \cup F_{k+1})/F_{k+1}.$$

The representation (8) induces a homomorphism

$$\pi_1(\mathbb{C} \setminus D, *) \to \operatorname{Aut}\big(\tilde{G}/(\tilde{G}, \tilde{F})\big), \tag{10}$$

and the monodromy representation of M_k is a subrepresentation of the representation dual to (10) (see [7]). The Abelian group $\tilde{G}/(\tilde{G}, \tilde{F})$ is, however, finitely generated. Indeed, the lower central series of $\tilde{F} = \tilde{F}_1$ is

$$\tilde{F}_1 \supseteq \tilde{F}_2 \supseteq \cdots \tilde{F}_k \supseteq \{\mathrm{id}\},$$

and hence $\tilde{F}$ is a finitely generated nilpotent group. Each subgroup of such a group is finitely generated too (see, e.g., [11]).

The central result of this section is the following proposition, from which Theorem 6 follows immediately.

PROPOSITION 3
The monodromy representation (10) is quasi-unipotent.

Indeed, M_k satisfies a Fuchsian equation on D, whose monodromy representation is a subrepresentation of the representation dual to (10) (see [7, Theorem 1.1], [9, Theorem 1]). To prove Proposition 3 we recall first some basic facts from the theory of free groups (see, e.g., Serre [21], Hall [11]). The graded group $\mathrm{gr}F$ (9) associated to the free finitely generated group F is a Lie algebra with a bracket induced by the commutator $(\cdot, \cdot)$ on F. The Milnor fibration (6) induces a representation

$$\mathbb{Z} = \pi_1(D \setminus \{0\}, *) \to \operatorname{Aut}_{\mathrm{Lie}}(\mathrm{gr}F), \tag{11}$$

where $\operatorname{Aut}_{\mathrm{Lie}}(\mathrm{gr}F)$ is the group of Lie algebra automorphisms of $\mathrm{gr}F$. Let l be a generator of $\pi_1(D \setminus \{0\}, *)$. It induces automorphisms $l_* \in \operatorname{Aut}_{\mathrm{Lie}}(\mathrm{gr}F)$ and $l_*|_{\mathrm{gr}^k F} \in \operatorname{Aut}(\mathrm{gr}^k F)$. We note that $\mathrm{gr}^1 F = H_1(\Gamma_t, \mathbb{Z})$, and hence $l_*|_{\mathrm{gr}^1 F}$ is quasi-unipotent.

PROPOSITION 4
Let $l_* \in \operatorname{Aut}_{\mathrm{Lie}}(\mathrm{gr}F)$ be such that $l_*|_{\mathrm{gr}^1 F}$ is quasi-unipotent. Then for every $k \geq 1$, the automorphism $l_*|_{\mathrm{gr}^k F}$ is quasi-unipotent.

Let $X = \{x_1, x_2, \ldots, x_\mu\}$ be the free generators of F, and consider the free Lie algebra L_X on X. It is a Lie subalgebra of the associative noncommutative algebra of polynomials in the variables x_i with a Lie bracket $[x, y] = xy - yx$. The canonical map $(x, y) \mapsto [x, y]$ induces an isomorphism of Lie algebras $\mathrm{gr}F \to L_X$ (see [21, Theorem 6.1]). Let $L_X^k \subset L_X$ be the graded piece of degree k. We show that $l_*|_{L_X^k}$ is quasi-unipotent. The proof is by induction. Suppose that the restriction of l_* on $\mathrm{gr}^1 F = L_X^1 = H_1(\Gamma_t, \mathbb{Z})$ is quasi-unipotent; that is, for some p, q, the restriction of $(l_*^p - \mathrm{id})^q$ on $\mathrm{gr}^1 F$ is zero. The operator $\mathrm{Var}_* = l_*^p - \mathrm{id}$ is a linear automorphism but not a Lie algebra automorphism. The identity

$$\begin{aligned}\mathrm{Var}_*[x, y] &= (l_*^p - \mathrm{id})(xy - yx) = l_*^p x l_*^p y - l_*^p y l_*^p x - xy + yx \\ &= [\mathrm{Var}_* x, \mathrm{Var}_* y] + [\mathrm{Var}_* x, y] + [x, \mathrm{Var}_* y]\end{aligned}$$

shows that the restriction of Var^{2q} on L_X^2 vanishes identically. Therefore the automorphism l_* restricted to L_X^2 or $\mathrm{gr}^2 F$ is quasi-unipotent. The case $k \geq 3$ is similar. Proposition 4 is proved. □

According to the above proposition, for every $k \in \mathbb{N}$ there are integers m_k, n_k, such that the polynomial $p_k(z) = (z^{m_k} - 1)^{n_k}$ annihilates $l_*|_{\mathrm{gr}^k F}$. Proposition 3 follows in turn from the following.

PROPOSITION 5
The polynomial $p = \prod_{i=1}^k p_i$ *annihilates* $l_* \in \mathrm{Aut}(\tilde{G}/(\tilde{G}, \tilde{F}))$.

Proof
Let $l \in \pi_1(D \setminus \{0\}, *)$. It induces an automorphism of the Abelian groups $G/(G, F)$, $G \cap F_i/(G \cap F_i, F)$, F_i/F_{i+1} denoted, by abuse of notation, by l_*. We denote by $p_i(l_*) = (l_*^{m_i} - \mathrm{id})^{n_i}$ the corresponding homomorphisms. It follows from the definitions that the diagram (12) of Abelian groups is commutative. (The vertical arrows are induced by the canonical projections.) Therefore if an equivalence class $[\gamma] \in G/(G, F)$ can be represented by a closed loop $\gamma \in F_i$, then $p_i(l_*)[\gamma]$ can be represented by a closed loop in F_{i+1}. Therefore, for every $[\gamma] \in G/(G, F)$, the equivalence class $p(l_*)$ can be represented by a closed loop in F_{k+1}. In other words $p(l_*)$ induces the zero automorphism of $\mathrm{Aut}(\tilde{G}/(\tilde{G}, \tilde{F}))$:

$$\begin{tikzcd} F_i/(F_i, F) \arrow[r, "p_i(l_*)"] & F_i/(F_i, F) \\ G \cap F_i/(G \cap F_i, F) \arrow[r, "p_i(l_*)"] \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] & G \cap F_i/(G \cap F_i, F) \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] \\ G/(G, F) \arrow[r, "p_i(l_*)"] & G/(G, F) \end{tikzcd} \tag{12}$$

□

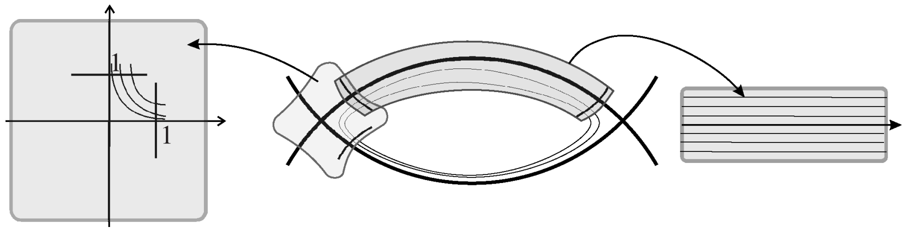

Figure 2. Covering polycycle by linearizing charts

5. Nonoscillation in the Darboux case

In this section we prove Theorem 3. First, we consider *elementary iterated integrals*, that is, the iterated integrals over the piece of the cycle lying in the linearizing charts of the saddles. We give a representation of the Mellin transform of the elementary iterated integral as a converging multiple series. This representation provides an asymptotic series for the elementary iterated integral with some explicit estimate of the error (see Theorem 7).

The general iterated integral of length k turns out to be a polynomial (depending on X_0 and k only) in elementary iterated integrals by Lemma 3. We give an analogue of the estimates of Theorem 7 for such polynomials. This allows us to prove a quasi-analyticity property: if the asymptotic series corresponding to the iterated integral is zero, then the integral itself is zero. This implies Theorem 3 since the zeros of the partial sums of the asymptotic series do not accumulate to zero (see Corollary 3).

5.1. Iterated integral as a polynomial in elementary iterated integrals

Take a neighborhood of the polycycle Γ, and suppose that it is so small that it can be covered by linearizing charts of saddles and linearizing charts of saddle connections. We fix linearizing coordinates near each saddle, and for each saddle we take two transversals given by $\{x = 1\}$ and $\{y = 1\}$ in linearizing coordinates (see Figure 2).

Let $\gamma(u)$, $u \in [0, 1]$, be a parameterization of a cycle $\gamma = \gamma_t \subset \{H = t\}$. (We fix some sufficiently small $t > 0$ for a moment.) The above transversals cut γ into several pieces γ_j: those lying near the sides of the polycycle and those near the vertices. We call these pieces elementary. Let us call the iterated integral over an elementary piece γ_j an *elementary iterated integral*. Our immediate goal is to prove the following lemma.

LEMMA 3

An iterated integral is a polynomial with integer coefficients in elementary iterated integrals. The polynomial depends on the length of the iterated integral and on the number of vertices of the polycycle only.

Proof

Let $0 = v_0 < v_1 < \cdots < v_m < 1$ be the parameterization of the ends of these pieces, that is, of points of intersections of γ_t with the transversals. The iterated integral in the parameterized form is equal to

$$\int_\Delta g_1(u_1)\cdots g_k(u_k)\,du_1\cdots du_k,$$

where $\Delta = \{0 \le u_1 \le \cdots \le u_k \le 1\} \subset \mathbb{R}^k$ is a simplex.

Consider connected components of the complement of Δ to the union of hyperplanes $\bigcup_{i,j}\{u_j = v_i\}$. Each connected component can be defined as

$$\{0 \le u_1 \le \cdots \le u_{i_1} < v_1 < u_{i_1+1} \le \cdots < v_m < u_{i_m+1} \le \cdots \le u_k \le 1\},$$

that is, a product $\Delta_1 \times \cdots \times \Delta_m$ of several simplices of smaller dimension of the form $\Delta_j = \{v_j < u_{i_{j+1}} \le \cdots \le u_{i_{j+1}} < v_{j+1}\}$. Therefore, by the Fubini theorem, the integral of $g_1(u_1)\cdots g_k(u_k)$ over this connected component is equal to the product of integrals $\int_{\Delta_j} g_{i_{j+1}}\cdots g_{i_{j+1}}\,du_{i_{j+1}}\cdots du_{i_{j+1}}$, that is, to the product of iterated elementary integrals $\int_{\gamma_j}\omega_{i_{j+1}}\cdots\omega_{i_{j+1}}$. □

5.2. Mellin transform of elementary iterated integrals

There are two types of elementary pieces: those lying in linearizing charts covering sides of the polycycle and those lying in linearizing charts covering saddles.

Evidently, the elementary iterated integrals corresponding to the pieces of the first type are just meromorphic functions of the parameter on the transversal, that is, of t^{1/λ_i}. We can represent these elementary integrals as a converging sum $\sum_{m\in\mathbb{Z}_{>-M}} c_m t^{m/\lambda_i}$, and, eventually, after rescaling t, we can assume that $|c_m| \le 2^{-m}C$, where $C > 0$ is some constant.

From this moment we assume that the elementary piece $\gamma(t)$ lies near the saddle $\{P_1 = P_2 = 0\}$. Our goal is to describe the Mellin transforms of iterated integrals $\int_{\gamma(t)}\omega_1\cdots\omega_l$.

Recall that the Mellin transform of a function $f(t)$ on the interval $[0, 1]$ is defined as

$$\mathcal{M}f(s) = \int_0^1 t^{s-1} f(t)\,dt. \tag{13}$$

To describe the Mellin transform of the elementary iterated integrals over $\gamma(t)$, let us introduce a generalized compensator. For $l \in \mathbb{N}$ and $\alpha = (m_1, n_1, \ldots, m_l, n_l) \in \mathbb{Z}^{2l}$, we define $\ell_\alpha^l(s;\lambda_1,\lambda_2)$ as

$$\ell_\alpha^l(s;\lambda_1,\lambda_2)=\prod_{j=0}^{l}\Big(s+\lambda_1^{-1}\sum_{i=1}^{j}m_i+\lambda_2^{-1}\sum_{i=j+1}^{l}n_i\Big)^{-1}. \tag{14}$$

We call $\mathcal{M}^{-1}\ell_\alpha^l(s;\lambda_1,\lambda_2)$ a generalized compensator. The particular case of $l=1$ corresponds to the Ecalle-Roussarie compensator. The generalized compensator is a finite linear combination of monomials of type $t^\mu(\log t)^{l'}$ for $l'\le l$.

The λ_1,λ_2 are the same until the end of this section, so for brevity we omit them from the notation of a generalized compensator.

LEMMA 4
After some rescaling of t, the Mellin transform of an elementary iterated integral is defined for $\Re s$ big enough and is given by the formula

$$\mathcal{M}\int\omega_1\cdots\omega_l=\sum_\alpha c_\alpha\ell_\alpha^l,\quad \alpha\in(\mathbb{Z}_{>-M})^{2l}, \tag{15}$$

where M is an upper bound for the order of poles of ω_i on Γ. Moreover, $|c_\alpha|\le 2^{-|\alpha|}C$.

This is a straightforward generalization of the construction of [14], which corresponds to $l=1$.

Proof
In the linearizing coordinates the first integral is written as $H=x^{\lambda_1}y^{\lambda_2}$, and $\gamma(t)=\{x^{\lambda_1}y^{\lambda_2}=t\}\cap\{0\le x,y\le 1\}$. The Mellin transform of the iterated integral can be computed explicitly for monomial forms $\omega_i=x^{m_i-1}y^{n_i}\,dx$:

$$\begin{aligned}
&\mathcal{M}\int\omega_1\cdots\omega_l\\
&=\int_0^1 t^{s-1}\int_{t^{1/\lambda_1}}^1 x_1^{m_1-1}y_1^{n_1}\int_{x_1}^1 x_2^{m_2-1}y_2^{n_2}\int_{x_2}^1\cdots\int_{x_{l-1}}^1 x_l^{m_l-1}y^{n_l}\,dx_l\cdots dx_1dt\\
&=\int_0^1 t^{(n_1+\cdots+n_l)/\lambda_2}t^{s-1}\int_{t^{1/\lambda_1}}^1 x_1^{m_1-1-n_1\mu}\int_{x_1}^1\cdots\int_{x_{l-1}}^1 x_l^{m_l-1-n_l\mu}\,dx_l\cdots dx_1dt\\
&=\int_0^1 x_l^{m_l-1-n_l\mu}\int_0^{x_l}x_{l-1}^{m_{l-1}-1-n_{l-1}\mu}\cdots\int_0^{x_1^{\lambda_1}}t^{(n_1+\cdots+n_l)/\lambda_2+s-1}\,dt\cdots dx_l\\
&=\lambda_1^{-l}\prod_{j=0}^{l}\Big(s+\lambda_1^{-1}\sum_{i=1}^{j}m_i+\lambda_2^{-1}\sum_{i=j+1}^{l}n_i\Big)^{-1}=\lambda_1^{-l}\ell_\alpha^l.
\end{aligned} \tag{16}$$

A similar formula holds for other choices of monomial forms ω_i.

After rescaling H, we can assume that the linearizing chart covers the bidisk $\{0 \le |x|, |y| \le 2\}$. Then the coefficients of the forms ω_i are meromorphic in the bidisk, with poles on $\{xy = 0\}$ of order at most M. So ω_i can be represented as a convergent power series

$$\omega_i = \sum_{m,n \in \mathbb{Z}_{>-M}} \left(c'_{i,m,n} x^{m-1} y^n \, dx + c''_{i,m,n} x^m y^{n-1} \, dy \right) \tag{17}$$

with coefficients $c'_{i,m,n}$, $c''_{i,m,n}$ decreasing as $O(2^{-m-n})$. Therefore the elementary iterated integral can be represented as 2^l sums (according to 2^l choices of monomial forms $x^{m-1} y^n \, dx$ or $x^m y^{n-1} \, dy$ in (17)) of type

$$\sum_{m_1,n_1,\ldots,m_l,n_l \in \mathbb{Z}^{2l}_{>-M}} (c'_{1,m_1,n_1} \cdots c'_{l,m_l,n_l} \int x^{m_1-1} y^{n_1} \, dx \ \cdots \ x^{m_l-1} y^{n_l} \, dx, \tag{18}$$

which is a converging multiple sum of elementary iterated integrals of monomial forms as in (16), with coefficients being products of $c'_{i,m,n}$, $c''_{i,m,n}$, $i = 1, \ldots, l$ and $m, n \in \mathbb{Z}_{>-M}$. From (16) one gets upper bounds for the elementary iterated integrals of monomial forms, which implies that for $\Re s$ big enough one can perform the Mellin transform termwise (due to uniform convergence of the series under the integral sign in (13)), and we get the required formula. □

The formula (15) defines the analytic continuation of the Mellin transform of I to the whole complex plane as a meromorphic function with poles in $\Sigma = \Sigma(I) = \left(-\lambda_1^{-1}\right) \mathbb{Z}_{>-M} + \left(-\lambda_2^{-1}\right) \mathbb{Z}_{>-M}$. We denote it by $\mathcal{M} I$ as well.

For any $s \in \mathbb{C}$, denote by $\rho(s)$ the minimal distance from S to the set of poles $\Sigma(I)$ of $\mathcal{M} I$. The following estimate is the keystone of the proof since it allows us to estimate the difference between the elementary iterated integral and the partial sum of its asymptotic series.

LEMMA 5
Let $I = \int \omega_1 \cdots \omega_l$ be an elementary iterated integral, and let C be defined as in Lemma 4. Then $|\mathcal{M} I(s)| \le C \rho(s)^{-l}$.

Proof
Indeed, the absolute value of each term in the sum in (15) can be estimated from above as $|c_\alpha| \rho(s)^{-l}$, and the estimate follows from $|c_\alpha| < 2^{-|\alpha|} C$. □

Usually the inverse Mellin transform is defined as

$$\mathcal{M}^{-1} g = \frac{1}{2\pi i} \int_{M+i\mathbb{R}} t^{-s} g(s) \, ds.$$

However, for Mellin transforms of iterated integrals one can choose the contour of integration as the boundary of $\{\Re s \leq M < +\infty, |\Im s| \leq 1\}$, where the fast decreasing of t^{-s} allows us to circumvent all analytic difficulties.

LEMMA 6
The inverse Mellin transform of Mellin transforms of elementary iterated integrals can be defined for $t \in (0, 1)$ as

$$\mathcal{M}^{-1} g = \frac{1}{2\pi i} \int_{\partial\Pi} t^{-s} g(s)\, ds, \qquad \Pi = \{\Re s \leq M < +\infty, |\Im s| \leq 1\}, \tag{19}$$

where M is sufficiently big.

Proof
Indeed, $|\ell_\alpha^l(s)| \leq 1$ on Π, so (15) converges uniformly on this contour, and one can apply the transformation (19) termwise due to exponential decreasing of t^{-s} on $\partial\Pi$. However, for each generalized compensator the transformation (19) does define the inverse Mellin transform, as each generalized compensator is just a rational function in s. □

COROLLARY 2
An elementary iterated integral can be represented as a convergent sum

$$\int \omega_1 \cdots \omega_l = \sum_\alpha c_\alpha \mathcal{M}^{-1} \ell_\alpha^l, \tag{20}$$

where α, c_α are as in Lemma 4.

5.3. Asymptotic series of elementary iterated integrals
The inverse Mellin transform of ℓ_α^l is a linear combination of monomials of the type $t^\mu (\log t)^j$, where $\mu \in \lambda_1^{-1}\mathbb{Z} + \lambda_2^{-1}\mathbb{Z}$ and $0 \leq j \leq l$. Collecting together similar terms in the expression for the elementary iterated integral I, we get a formal series $\hat{I}$:

$$\hat{I} = \sum_{\mu, j} \hat{c}_{\mu,j} t^\mu (\log t)^j, \quad \text{where } \mu \in \lambda_1 \mathbb{Z}_{>-M} + \lambda_2 \mathbb{Z}_{>-M}, 0 \leq j \leq l. \tag{21}$$

Convergence of this series depends on arithmetic properties of the tuple $\{\lambda_i\}$ (see [3] for a discussion of this phenomena in the case of pseudo-Abelian integrals).

We prove below that $\hat{I}$ is an asymptotic series of I. While it is not true that *all* partial sums of $\hat{I}$ approximate well the elementary iterated integral, *some* partial sums do, and this is the crucial observation.

THEOREM 7
$\hat{I}$ is an asymptotic series of I. Moreover, for each $p \in \mathbb{N}$ there exists $s_p \in [p, p+1]$ such that the partial sums $\hat{I}_p = \sum_{j,\mu<s_p} \hat{c}_{\mu,j} t^{\mu} (\log t)^j$ of $\hat{I}$ satisfy the following:

$$|I(t) - \hat{I}_p(t)| \leq C s_p^l t^{s_p}, \quad t \in [0, 1], \tag{22}$$

where C depends on I but not on p.

Proof
Poles of $\mathcal{M}I$ are of the form $-\lambda_1^{-1} m - \lambda_2^{-1} n$, where $m_i, n_i \in \mathbb{Z}_{>-M}$. Since $\lambda_1, \lambda_2 > 0$, there are $O(p)$ poles on the interval $J_p = [-p-1, -p]$, $p \in \mathbb{N}$. Therefore on each interval J_p one can find a point $-s_p$ such that $\rho(-s_p) > O(p^{-1}) = O(s_p^{-1})$.

For each $p \in \mathbb{N}$, let us split the contour of integration $\partial\Pi$ into two parts: the boundary of $\Pi'_p = \{-s_p \leq \Re s \leq M, |\Im s| \leq 1\}$ and the boundary of $\Pi_p = \{\Re s \leq -s_p, |\Im s| \leq 1\}$. The inverse Mellin transform (19) is then split into two integrals:

$$\mathcal{M}^{-1} g = \frac{1}{2\pi i} \int_{\partial\Pi'_p} t^{-s} g(s)\, ds + \frac{1}{2\pi i} \int_{\partial\Pi_p} t^{-s} g(s)\, ds. \tag{23}$$

In the compact domain Π'_p, the function $\mathcal{M}I$ has finitely many poles, so the integral $(1/2\pi i) \int_{\partial\Pi'_p} t^{-s} \mathcal{M}I\, ds$ depends only on the Laurent parts of $\mathcal{M}I$ at these poles. Now, any Laurent coefficient of $\mathcal{M}I$ is a multiple sum of corresponding Laurent coefficients of terms of (15). (Convergence is guaranteed due to the fast decreasing of c_α.) This implies that the Mellin transform of the partial sum $\hat{I}_p(t)$ is a rational function of s with the same poles and the same Laurent parts at these poles as $\mathcal{M}I$ in Π'_p. Therefore the first integral in (23) is exactly the partial sum $\hat{I}_p(t)$ of $\hat{I}$ as defined above.

Therefore $I(t) - \hat{I}_p(t) = (1/2\pi i) \int_{\partial\Pi_p} t^{-s} \mathcal{M}I\, ds$. By Lemma 5, $|\mathcal{M}I(s)| \leq O(p^l)$ on $\partial\Pi_p$, and (22) follows. □

5.4. Iterated integrals
Here we extend Theorem 7 to the algebra $\mathcal{A}$ generated by elementary iterated integrals.

Let $f = P(I_1, \ldots, I_k) \in \mathcal{A}$ be an element in $\mathcal{A}$, where $P \in \mathbb{C}[u_1, \ldots, u_k]$ and $I_1, \ldots, I_k$ are elementary integrals. Substitution of convergent series from (20) instead of $I_1, \ldots, I_k$ gives a representation of f as a converging multiple sum of products (of length at most k) of generalized compensators. Collecting similar terms, we obtain a formal series $\hat{f}$ similar to (21), probably divergent.

THEOREM 8
For any $p \in \mathbb{N}$, there exists $s_p \in [p, p+1]$ such that the partial sum $\hat{f}_p$ of $\hat{f}$ satisfies the following:

$$|f - \hat{f}_p| \leq C s_p^d t^{s_p} \tag{24}$$

for some C, d independent of p.

Before proof of Theorem 8, let us show that it implies Theorem 3.

COROLLARY 3
Let $f \in \mathcal{A}$. If $\hat{f} = 0$, then $f \equiv 0$ on $[0, 1)$. Also, isolated zeros of f cannot accumulate to zero.

Proof
To prove the first claim, take a limit as $s_p \to +\infty$ in (24).

Now, if $f \not\equiv 0$, then for some μ we have $|f - t^\mu P(\log t)| = o(t^\mu)$ with some nonzero polynomial P (where $-\mu$ is the rightmost pole of $\mathcal{M} f$). This implies the second claim. □

The proof of Theorem 8 occupies the rest of the article.

5.4.1. Mellin transform of a product of several generalized compensators
For $V = (v_1, \dots, v_n) \in \mathbb{R}^n$, define $\ell_v(s) = \prod_{i=1}^n (s+v_i)^{-1}$. Let $V^j = (v_1^j, \dots, v_{n_j}^j) \in \mathbb{R}^{n_j}$, $j = 1, \dots, k$, and define $\Phi(V^1, \dots, V^k)(s) = \mathcal{M}\left[\prod_{j=1}^k \left(\mathcal{M}^{-1}\ell_{V^j}\right)\right]$.

This is a rational function of s. We want to show that it depends polynomially on $\{V^j\}$. Let K denote the set of functions $\kappa : \{1, \dots, k\} \to \mathbb{Z}$ with the condition $\kappa(j) \in \{1, \dots, n_j\}$, and define $w_\kappa = v_{\kappa(1)}^1 + \cdots + v_{\kappa(k)}^k$.

LEMMA 7
Let $S = S(V^1, \dots, V^k) = \prod_{\kappa \in \mathrm{K}} (s + w_\kappa)$ be a polynomial in $\mathbb{R}[V^1, \dots, V^k; s]$. There exists a polynomial $R = R_{n_1, \dots, n_k} \in \mathbb{R}[V^1, \dots, V^k; s]$ such that $\Phi(V^1, \dots, V^k)(s) = RS^{-1}$, $\deg_s R < \deg_s S$.

Proof
By continuity of both sides, it is enough to prove this for a dense subset of $\bigoplus_{j=1}^k \mathbb{R}^{n_j}$ consisting of nonresonant tuples $(V^1, \dots, V^k)$, namely, for those tuples for which all w_κ are different.

Let $\mathbb{C}_\infty(s)$ be the ring of rational functions in s vanishing at infinity, and define the convolution $f_1 * f_2$ for $f_1, f_2 \in \mathbb{C}_\infty(s)$ by extending the rule

$$\frac{1}{s+a} * \frac{1}{s+b} = \frac{1}{s+a+b}$$

by linearity and continuity to the whole $\mathbb{C}_\infty(s)$ (in particular, $(s+a)^{-k} * (s+b)^{-l} = (s+a+b)^{-k-l+1}$). The thus-defined convolution is Mellin dual to the usual product. Therefore $\Phi(V^1, \ldots, V^k)(s) = \ell_{V^1} * \cdots * \ell_{V^k}$. Decomposing each factor into simple fractions

$$\ell_{V^j} = \sum_i \frac{\Re s_{v_i^j} \ell_V^j}{s + v_i^j}, \quad \Re s_{v_i^j} \ell_V^j = \Big(\prod_{i' \neq i} (v_i^j - v_{i'}^j)\Big)^{-1},$$

and opening brackets, we see that

$$\Phi(V^1, \ldots, V^k)(s) = \sum_{\kappa \in \mathrm{K}} \frac{\prod_{j=1}^k \Re s_{v^j_{\kappa(j)}} \ell_V^j}{s + w_\kappa}.$$

Reducing to a common denominator, we see that $\Phi(V^1, \ldots, V^k)(s)$ is a rational function in v_i^j, s, with denominator dividing $S \prod_{i,i',j} (v_i^j - v_{i'}^j)$.

We claim that the factors $(v_i^j - v_{i'}^j)$ do not enter the denominator of $\Phi(V^1, \ldots, V^k)(s)$. Indeed, the presence of such factor would mean that for each fixed $s \in \mathbb{C}$, $\Phi(V^1, \ldots, V^k)(s)$ becomes unbounded as v_i^j tends to $v_{i'}^j$. This is evidently not true; for any tuple $(V^1, \ldots, V^k)$ and every sufficiently big $s \in \mathbb{R}$, the function $\Phi(V^1, \ldots, V^k)(s)$ is locally bounded near $(V^1, \ldots, V^k, s)$. □

5.5. Mellin transform of a product of elementary iterated integrals

Let $I = I_1 \ldots I_k$ be a product of several elementary iterated integrals, and let the order of I_j be l_j. Then using the representation (20) for I_j and opening brackets, we see that

$$\mathcal{M} I = \sum_{\alpha_1, \ldots, \alpha_k} c_{\alpha_1} \cdots c_{\alpha_k} \ell_{\alpha_1}^{l_1} * \cdots * \ell_{\alpha_k}^{l_k}), \tag{25}$$

where $\alpha_j \in (\mathbb{Z}_{>-M})^{2l_j}$ and $|c_{\alpha_j}| \leq 2^{-|\alpha_j|} C_j$. We do not assume that the pairs $(\lambda_{j1}, \lambda_{j2})$, $j = 1, \ldots, k$, are the same for all iterated integrals I_j.

LEMMA 8

The Mellin transform of a product of elementary iterated integrals can be continued to the whole complex plane $\mathbb{C}$ as a meromorphic function with poles in $-\sum_{i=1}^k \sum_{j=1,2} \lambda_{ij}^{-1} \mathbb{Z}_{>-M}$.

Let $\rho(s)$ be the distance from s to the set of poles of $\mathcal{M}I$. Then

$$|\mathcal{M}I(s)| \leq C\big(\rho(s)\big)^{-\prod l_j}(|s|+1)^d \tag{26}$$

for some $d > 0$.

Moreover, the inverse Mellin transform defined in Lemma 6 gives the inverse Mellin transform of $\mathcal{M}I$.

Proof

Let us estimate from above the terms $\ell^{l_1}_{\alpha_1} * \cdots * \ell^{l_k}_{\alpha_k}$ from (25). By Lemma 7, it is equal to $R(V^1,\ldots,V^k;s)/S(V^1,\ldots,V^k;s)$, where $V^j = (v^j_1,\ldots,v^j_{l_j})$ is defined by

$$v^j_i = -\lambda^{-1}_{j1}\sum_{p=1}^{i} m^j_p - \lambda^{-1}_{j2}\sum_{p=i+1}^{l_j} n^j_p, \quad \alpha_j = (m^j_1,\ldots,n^j_{l_j}) \in \mathbb{Z}^{2l_j}_{>-M},$$

as in (14). This means that $V^j = L_j\alpha_j$ for some linear map $L_j : \mathbb{R}^{l_j} \to \mathbb{R}^{l_j}$. Therefore R is a polynomial in $(s;\alpha_1,\ldots,\alpha_k)$, and

$$|R(s)| \leq \mathrm{const}(1+s)^d\Big(1+\sum|\alpha_j|\Big)^d \quad \text{for } d = \deg R \geq 0.$$

From the other side, S is a monic polynomial in s of degree $\prod l_j$ with roots in the poles of $\mathcal{M}I$, so $|S(s)| \geq (\rho(s))^{\prod l_j}$. Taken together, this means that

$$\Big|\ell^{l_1}_{\alpha_1} * \cdots * \ell^{l_k}_{\alpha_k})\Big| \leq \mathrm{const}\big(\rho(s)\big)^{-\prod l_j}(1+s)^d\Big(1+\sum|\alpha_j|\Big)^d. \tag{27}$$

Using $|c_{\alpha_j}| \leq 2^{-|\alpha|}C$ by Lemma 4, we estimate $|\mathcal{M}I(s)|$ from above as

$$|\mathcal{M}I(s)| \leq \mathrm{const}\big(\rho(s)\big)^{-\prod l_j}(1+s)^d \sum_{\alpha_1,\ldots,\alpha_k} 2^{-\sum|\alpha_j|}\Big(1+\sum|\alpha_j|\Big)^d, \tag{28}$$

which, by convergence of the last series, proves (26).

Moreover, for $t \in (0,1)$ the function t^{-s} decreases exponentially on $\partial\Pi$, so (27) implies that in the integral

$$\frac{1}{2\pi i}\int_{\partial\Pi} t^{-s} \sum_{\alpha_1,\ldots,\alpha_k} c_{\alpha_1}\cdots c_{\alpha_k}\ell^{l_1}_{\alpha_1} * \cdots * \ell^{l_k}_{\alpha_k}),$$

one can perform integration termwise. However, for each term the above integral gives the inverse Mellin transform, as each term is just a rational function of s. This finishes proof of the lemma. □

5.5.1. Proof of Theorem 8

Let I now be a polynomial in several elementary iterated integrals, $I = P(I_1, \dots, I_k)$. The set of poles of the Mellin transform $\mathcal{M}I$ of I is the union of sets of poles of Mellin transforms of each monomial of P, so the number of poles of $\mathcal{M}I$ on an interval $J_p = [-p-1, -p]$ grows as some power of p.

This means that for each $p \in \mathbb{N}$, one can find $s_p \in J_p$ such that the distance $\rho(s_p)$ from p to the set of poles of $\mathcal{M}I$ is bigger than $|s_p|^{-d'}$ for some $d' > 0$. Then, splitting the contour of integration of the inverse Mellin transform as in Theorem 7, we conclude from Lemma 8 that $|\mathcal{M}I| < C|s_p|^{d''}$ on the $\partial\Pi_p$ for some fixed $d'' > 0$, and the claim follows. $\square$

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